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GEODESIC DOUBLE DIFFERENTIAL FORMS, EULER-POISSON-DARBOUX EQUATIONS AND THE SELBERG TRACE FORMULA FOR HYPERBOLIC SPACE FORMS

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Abstract. We derive Selberg's trace formula for the p-spectrum of compact hyperbolic space forms by using orthogonal representations. We consider twisted double differential forms in connection with Euler-Poisson-Darboux equations. This paper is a completion and generalization of our paper [15].

1. Introduction. Selberg's trace formula states a duality relation between the eigenvalue spectrum of the Laplace operator on a compact hyperbolic space form and the geometric spectrum with certain weights. Selberg was the first to reach to the trace formula about 1950/51. The idea of taking the trace seemed quite natural, since it seemed that to obtain individual eigenfunctions or eigenforms of the Laplace operator would be very difficult. The trace formulas bear a very striking resemblance to the so-called explicit formulas of prime number theory.

There are many papers on the two-dimensional case. We refer to [4,13] and the references given there. Papers [5,7,8,9,14,19] comprise results on the p-spectrum in higher dimensional cases. The proofs of the versions of the trace formula used there are only sketched. The present paper is a generalization of [15]; it involves more information on our basic tools, especially on geodesic double differential forms in connection with Euler-Poisson-Darboux equations, but we omit the calculations given in [15].

One can apply Selberg's trace formula and our methods used for its proof to make spectral estimates (cf. [17]) and to solve lattice problems (cf. [16]).

Let $\mathcal{G}$ be a properly discontionuous group of isometries of n-dimensional hyperbolic space $\mathbb{H}_n$ of constant curvature -1 without fixed points (with the exception of the identity map id) with compact fundamental domain and let us consider the related Killing-Hopf space form $V = \mathbb{H}_n/\mathcal{G}$. Let ρ be an orthogonal representation of $\mathcal{G}$ on $\mathbb{R}^m$. We consider $\mathbb{R}^m$-valued differential p-forms on the space from V. The isometries $b \in \mathcal{G}$ induce mappings b^* for the p-differential forms on $\mathbb{H}_n$ (cf.[15]). The $\mathbb{R}^m$-valued differential form $\alpha = \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^m \end{pmatrix}$ with the differential forms $\alpha^1, \dots, \alpha^m$ as components is called $\mathcal{G}$-automorphic or twisted, if we have

$$\begin{pmatrix} b^*\alpha^1 \\ \vdots \\ b^*\alpha^m \end{pmatrix} = \rho(g) \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^m \end{pmatrix}$$

for all $b \in \mathcal{G}$. As pointed out in [5], Hodge theory is used. Let us denote by S_p the set of eigenvalues of the Laplace operator $\Delta = d\delta + \delta d$ for $\mathbb{R}^m$-valued differential forms on V. Let us denote by d and δ the differential and codifferential operator, respectively. We denote by $d_p(\mu)$ and $\delta_p(\mu)$ the dimension of the eigenspaces of closed ($d\alpha = 0$) eigenforms and that of coclosed twisted p-eigenforms ($\delta\alpha = 0$) for an eigenvalue $\mu \in S_p$, respectively. The dimension of harmonic twisted p-forms is denoted by B_p. By the telescopage theorem of Mc Kean and Singer we have $d_p^\delta(\mu) = d_{p+1}^d(\mu)$ for $\mu \in S_p \backslash \{0\}$ and $p = 0, 1, \dots, n-1$. Using the Hodge star operator one also gets $d_p^\delta(\mu) = d_{n-p}^d(\mu)$. Let Ω be the set of nontrivial free homotopy classes of V. In every class $\omega \in \Omega$ there lies exactly one closed geodesic line. We denote by $l(\omega)$ and $\nu(\omega)$ its length and multiplicity, respectively. The parallel displacement along the closed geodesic line induces an isometry of the tangent space at every point of the geodesic line with the eigenvalues $\beta_1(\omega), \dots, \beta_{n-1}(\omega)$, 1 with $|\beta_i(\omega)| = 1$ $(i = 1, \dots, n-1)$. Let $e_p(\omega)$ be p-th elementary symmetric function of $\beta_i(\omega)$ $(i = 1, \dots, n-1)$ and put $e_0(\omega) = 1$. Further on we introduce the weight

$$\sigma(\omega) := \frac{1}{\nu(\omega)} e^{Nl(\omega)} \prod_{j=1}^{n-1} \frac{1}{e^{l(\omega)} - \beta_j(\omega)} \quad \text{with} \quad N = \frac{n-1}{2}.$$

There is an one-to-one relation between the set of free homotopy classes and the classes of conjugate elements of $\mathcal{G}$. So we can define the trace tr $\rho(\omega)$ to be the trace tr $\rho(b)$ for elements of the related class of conjugate elements.

We will prove the

Theorem (Selberg's trace formula): *Let $h(r)$ be an analytic function in the strip $|\operatorname{Im} r| < N + \delta$ with $N = \frac{n-1}{2}$, $0 < \delta < \frac{1}{2}$, which is even, $h(r) = h(-r)$, and satisfies*

$$|h(r)| \leq A(1 + |r|)^{-n-\delta}.$$

From the Fourier transform

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru} dr$$

of $h(r)$ we can state the trace formula

$$\sum_{\mu\in S_P} d_p^*(\mu)\, h(r_p(\mu)) = \mathrm{vol}\ \ V\ \langle S_p^n, g\rangle + \sum_{\omega\in\Omega} l(\omega)\,\sigma(\omega)\,e_p(\omega)\,\mathrm{tr}\ \rho(\omega)\,g(l(\omega))$$

for $p = 0,\ldots,n-1$ *with* $r_p(\mu) = \sqrt{\mu - (p-N)^2}$,

$$\langle S_p^n, g\rangle = \frac{2\binom{n-1}{p}(4\pi)^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})}\begin{cases} \int_0^\infty \Big(\prod_{u=0,u\neq|p-N|}^{N}(r^2+u^2)\Big)\,h(r)\,dr & \textit{for } n \textit{ odd} \\ \int_0^\infty \Big(\prod_{u=\frac{1}{2},u\neq|p-N|}^{N}(r^2+u^2)\Big)\,r\,h(r)\tanh(\pi r)\,dr & \textit{for } n \textit{ even}\end{cases}$$

$$d_p^*(0) = (-1)^p(B_0 - B_1 + \ldots + (-1)^p B_p) + K_0, \quad N = \tfrac{n-1}{2},$$

$$K_0 = \begin{cases} 0 & \textit{for } n \textit{ odd and for } (p \le \frac{n-2}{2},\ \ n\ \ \textit{even}) \\ (-1)^{p+1-\frac{n}{2}}\,\pi^{\frac{-(n+1)}{2}}\,\Gamma(\frac{n+1}{2})m\ \mathrm{vol}\ V & \textit{for } (p \ge \frac{n}{2},\ \ n\ \textit{even}),\end{cases}$$

$d_p^*(\mu) = d_p^\delta(\mu)$ *for* $\mu > 0$. vol V *thereby denotes the volume of the space form* V.

2. Geodesic double differential forms and Euler-Poisson-Darboux equations. Using the geodesic distance $r(x,y)$ of the points $x, y \in H_n$ P. Günther [10] introduced the double differential forms

$$\begin{aligned} \sigma_0(x,y) &= 1, & \tau_0(x,y) &= 0, \\ \sigma_1(x,y) &= \sinh r(x,y) d\hat{d}r(x,y), & \tau_1(x,y) &= dr(x,y)\hat{d}r(x,y), \\ \sigma_p &= \tfrac{1}{p}\sigma_{p-1} {}_\wedge \curlywedge \sigma_1 & \tau_p &= \sigma_{p-1} {}_\wedge \curlywedge \tau_1 \end{aligned}$$

d and $\hat{d}$ thereby denote the differentials with respect to x and y, respectively. For double differential p-form $\phi_p(x,y)$ and a function $h(r)$ of the geodesic distance $r = r(x,y)$ we have (C^2-differentiability is supposed)

$$\begin{aligned} \triangle(h(r)\phi_p(x,y)) = \ & -(h''(r) + (n-1)\coth r h'(r))\phi_p(x,y) \\ & -2h'(r)(q^i\nabla_i)\phi_p(x,y) + h(r)\triangle\phi_p(x,y). \end{aligned}$$

The Laplace operator is used with respect to x and by $(q^i\nabla_i)$ the derivation in the direction of the geodesic line connecting x and y (using Einstein summation convention), cf. [10, 12] is denoted. The following equation holds:

$$(q^i\nabla_i)\sigma_p(x,y) = 0, \quad (q^i\nabla_i)\tau_p(x,y) = 0.$$

It is useful to introduce

$$\alpha_p = \sigma_p + \cosh r \tau_p \quad \text{and} \quad \beta_p = \cosh r \sigma_p + \tau_p.$$

Next we have

$\triangle \alpha_p = -p(n-p+1)\alpha_p, \quad d\alpha_p = 0,$

$\triangle \beta_p = -(p+1)(n-p)\beta_p, \quad \delta\beta_p = 0.$

In terms of σ_p and τ_p we have (cf. [10])

$$\triangle \sigma_p = \Big(-p(n-p-1) + 2p\frac{1}{\sinh^2 r}\Big)\sigma_p - 2(n-p)\frac{\cosh r}{\sinh^2 r}\tau_p,$$

$$\triangle \tau_p = \Big(-(p-1)(n-p) + 2(n-p)\frac{1}{\sinh^2 r}\Big)\tau_p - 2p\frac{\cosh r}{\sinh^2 r}\sigma_p.$$

For $\lambda > n+5$ we define for $r(x,y) < t$

$$\begin{aligned} A(t,\lambda,x,y,n,p) :=\ & c^*(\lambda,n,p)\Big((\lambda-n-3)\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-5}{2}} \sinh^2 r \sigma_p \\ & -(n-p)\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-3}{2}}(\cosh r\sigma_p + \tau_p)\Big)(x,y), \\ B(t,\lambda,x,y,n,p) :=\ & c^*(\lambda,n,p)\Big((\lambda-n-3)\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-5}{2}} \sinh^2 r \tau_p \\ & -p\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-3}{2}}(\cosh r\tau_p + \sigma_p)\Big)(x,y) \end{aligned}$$

with $c^*(\lambda,n,p) = (-1)^p \dfrac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})}\pi^{-\frac{n}{2}}$ and $A(t,\lambda,x,y,n,p) = 0, \;\; B(t,\lambda,x,y,n,p) = 0$ for $r(x,y) \geq t$.

We also define

$M^\alpha(t,\lambda,x,y,n,p) := c^*(\lambda,n,p)\Big(\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-3}{2}}\alpha_p\Big)(x,y),$

$M^\beta(t,\lambda,x,y,n,p) := c^*(\lambda,n,p)\Big(\{2(\cosh t - \cosh r)\}^{\frac{\lambda-n-3}{2}}\beta_p\Big)(x,y)$

for $r(x,y) < t$ and $M^\alpha(t,\lambda,x,y,n,p) = 0, \;\; M^\beta(t,\lambda,x,y,n,p) = 0$ for $r(x,y) \geq t$. We immediately obtain

$$\begin{aligned} & A(t,\lambda,x,y,n,p) = (\lambda-3)\cosh t M^\beta(t,\lambda-2,x,y,n,p) \\ & -(\lambda-3)M^\alpha(t,\lambda-2,x,y,n,p) - (\tfrac{\lambda-3}{2} - p + \tfrac{n}{2})M^\beta(t,\lambda-2,x,y,n,p), \\ & B(t,\lambda,x,y,n,p) = (\lambda-3)\cosh t M^\alpha(t,\lambda-2,x,y,n,p) \\ & -(\lambda-3)M^\beta(t,\lambda-2,x,y,n,p) - (\tfrac{\lambda-3}{2} + p - \tfrac{n}{2})M^\alpha(t,\lambda-2,x,y,n,p). \end{aligned}$$

Using the above formulas, we get by direct calculation

Proposition 1. *For $\lambda > n+9$ the equalities holds*

$$\Big(\frac{d^2}{dt^2}+\lambda\frac{d}{dt}+\triangle+((p+1)(n-p)+\frac{\lambda^2-(n+1)^2}{4})\Big)\Big(\sinh^{1-\lambda} tA(t,\lambda,x,y,n,p)\Big)=0,$$

$$\Big(\frac{d^2}{dt^2}+\lambda\frac{d}{dt}+\triangle+(p(n+1-p)+\frac{\lambda^2-(n+1)^2}{4})\Big)\Big(\sinh^{1-\lambda} tB(t,\lambda,x,y,n,p)\Big)=0.$$

This motivates us to consider the following Euler-Poisson-Darboux equation. Let us denote by $z(t,\lambda,\mu,n)$ the uniquely determined solution of the differential equation

$$\Big(\frac{d^2}{dt^2}+\lambda\coth t\frac{d}{dt}+(\mu+\frac{\lambda^2-(n-1)^2}{4})\Big)(z(t,\lambda,\mu,n))=0$$

with the initial conditions

$$z(0,\lambda,\mu,n)=1,\quad \frac{d}{dt}z(t,\lambda,\mu,n)|_{t=0}=0.$$

We can express $z(t,\lambda,\mu,n)$ in terms of Gauß hypergeometric function $F(.,.,.,.)$:

$$z(t,\lambda,\mu,n)=\Big(\frac{\cosh t+1}{2}\Big)^{\frac{1-\lambda}{2}}F(\frac{1}{2}-\chi(\mu),\frac{1}{2}+\chi(\mu),\frac{\lambda+1}{2},\frac{1-\cosh t}{2})$$

with

$$\chi(\mu)=\begin{cases}\sqrt{N^2-\mu} & \text{for } \mu\le N^2\\ i\sqrt{\mu-N^2} & \text{for } \mu>N^2\end{cases}\qquad \text{and}\quad N=\frac{n-1}{2}.$$

According to [11, 15] we have

$$z(t,\lambda,\mu,n)=(\frac{1}{\lambda+1}\sinh t\frac{d}{dt}+\cosh t)\,z(t,\lambda+2,\mu,n)$$

and

$$z(t,\lambda_2,\mu,n)=\frac{2\sinh^{1-\lambda_2}t}{B\Big(\frac{\lambda_1+1}{2},\frac{\lambda_2-\lambda_1}{2}\Big)}\int_0^t\{2(\cosh t-\cosh s)\}^{\frac{\lambda_2-\lambda_1-2}{2}}\sinh^{\lambda_1}s\ z(s,\lambda_1,\mu,n)ds$$

for $\lambda_2\ge\lambda_1+2$. We use the Euler-Poisson-Darboux parameter λ to get better convergence properties of eigenform expansions of certain twisted double differential forms. For this purpose we need the estimation

$$|z(t,\lambda,\mu,n)|\le c\sinh^{-\frac{\lambda}{2}}t(\mu-N^2)^{-\frac{\lambda}{4}} \tag{1}$$

for $\mu>\mu^*>N^2$, $N=\frac{n+1}{2}$, $t>0$.

Next we introduce

$$\mathfrak{x}(t,\lambda,\mu,n,p) := z(t,\lambda,\mu+(p+1)(n-p)-n,n),$$

$$\mathfrak{y}(t,\lambda,\mu,n,p) := z(t,\lambda,\mu+p(n+1-p)-n,n),$$

$$\mathfrak{u}(t,\lambda,\mu,n,p) := -\frac{1}{\lambda+1}(\frac{\lambda+1}{2}-p+\frac{n}{2})\sinh^2 t\,\mathfrak{y}(t,\lambda+2,\mu,n,p)+\cosh t\mathfrak{y}(t,\lambda,\mu,n,p),$$

$$\mathfrak{v}(t,\lambda,\mu,n,p) := -\frac{1}{\lambda+1}(\frac{\lambda+1}{2}+p-\frac{n}{2})\sinh^2 t\mathfrak{x}(t,\lambda+2,\mu,n,p)+\cosh t\,\mathfrak{x}(t,\lambda,\mu,n,p).$$

It follows

$$\mathfrak{x}(t,\lambda,\mu,n,p) = (\frac{1}{\lambda+1}\sinh t\frac{d}{dt}+\cosh t)\,\mathfrak{x}(t,\lambda+2,\mu,n,p),$$

$$\mathfrak{y}(t,\lambda,\mu,n,p) = (\frac{1}{\lambda+1}\sinh t\frac{d}{dt}+\cosh t)\,\mathfrak{y}(t,\lambda+2,\mu,n,p),$$

$$\mathfrak{u}(t,\lambda,\mu,n,p) = (\frac{1}{\lambda+1}\sinh t\frac{d}{dt}+\cosh t)\,\mathfrak{u}(t,\lambda+2,\mu,n,p),$$

$$\mathfrak{v}(t,\lambda,\mu,n,p) = (\frac{1}{\lambda+1}\sinh t\frac{d}{dt}+\cosh t)\,\mathfrak{v}(t,\lambda+2,\mu,n,p).$$

3. Twisted double differential forms and eigenform expansions. We define the $\mathbb{R}^{m^2}$ — valued double differential forms

$$\mathcal{A}(t,\lambda,x,y,n,p) = \sum_{b\in\mathcal{G}}\rho(b)b^*A(t,\lambda,x,by,n,p),$$

$$\mathcal{B}(t,\lambda,x,y,n,p) = \sum_{b\in\mathcal{G}}\rho(b)b^*B(t,\lambda,x,by,n,p),$$

$$\mathcal{M}^\alpha(t,\lambda,x,y,n,p) = \sum_{b\in\mathcal{G}}\rho(b)b^*M^\alpha(t,\lambda,x,by,n,p),$$

$$\mathcal{M}^\beta(t,\lambda,x,y,n,p) = \sum_{b\in\mathcal{G}}\rho(b)b^*M^\beta(t,\lambda,x,by,n,p).$$

If we use b^* in connection with a double differential form it shall be taken with respect to the variable y. The fact that group $\mathcal{G}$ is properly discontinuous implies that these sums are finite. We identify twisted differential forms on $\mathbb{H}_n$ with differential forms on the space form V, and we do the same for the double differential forms. Using the spherical mean value operators defined by the kernels $A(t,\lambda,x,y,n,p)$, $B(t,\lambda,x,y,n,p)$, $M^\alpha(t,\lambda,x,y,n,p)$ and $M^\beta(t,\lambda,x,y,n,p)$ we can proceed as in [15] to get the eigenform expansions of the related twisted kernels. We want to point out that the essential information for that procedure is given by Proposition 1 on the one hand and by the knowledge of initial values on the other hand. For the initial values cf. [10, 15].

For the definition of the pointwise norm of a differential form and of a double differential form we refer to [6, 12, 15], we also need the norms obtained by integration of the pointwise norms over a fundamental domain of $\mathcal{G}$. We get norms of the $\mathbb{R}^m$-valued differential forms and of the $\mathbb{R}^{m^2}$-valued double differential forms if we take the maximum of the norms of the components. In the space of quadratic integrable $\mathbb{R}^m$-valued p-forms φ over V there exists a complete orthonormal system E_p of $\mathbb{R}^m$-valued p-eigenforms of the Laplace operator $\triangle = d\delta + \delta d$, which we can suppose to be closed ($d\varphi = 0$) or coclosed ($\delta\varphi = 0$). We decompose E_p into a set of harmonic eigenforms E_p^h, closed but not coclosed eigenforms E_p^d and coclosed but not closed eigenforms E_p^δ. We get

Proposition 2. *Let us denote by $\mu(\varphi)$ the eigenvalue of $\varphi \in E_p$: $\triangle\varphi = \mu(\varphi)\varphi$. Then we have the following eigenform expansions:*

$$\mathcal{A}(t, \lambda, x, y, n, p) = -\sum_{\varphi \in E_p^\delta} \frac{\mu(\varphi)}{\lambda - 1} \mathfrak{x}(t, \lambda, \mu(\varphi), n, p) \sinh^{\lambda-1} t\varphi(x)\varphi(y),$$

$$\mathcal{B}(t, \lambda, x, y, n, p) = -\sum_{\varphi \in E_p^d} \frac{\mu(\varphi)}{\lambda - 1} \mathfrak{y}(t, \lambda, \mu(\varphi), n, p) \sinh^{\lambda-1} t\varphi(x)\varphi(y),$$

$$\begin{aligned}\mathcal{M}^\alpha(t, \lambda, x, y, n, p) = &\sum_{\varphi \in E_p^\delta \cup E_p^h} \mathfrak{x}(t, \lambda - 2, \mu(\varphi), n, p) \sinh^{\lambda-3} t\varphi(x)\varphi(y)\\ &+ \sum_{\varphi \in E_p^d} \mathfrak{u}(t, \lambda - 2, \mu(\varphi), n, p) \sinh^{\lambda-3} t\varphi(x)\varphi(y),\\ \mathcal{M}^\beta(t, \lambda, x, y, n, p) = &\sum_{\varphi \in E_p^d \cup E_p^h} \mathfrak{y}(t, \lambda - 2, \mu(\varphi), n, p) \sinh^{\lambda-3} t\varphi(x)\varphi(y)\\ &+ \sum_{\varphi \in E_p^\delta} \mathfrak{v}(t, \lambda - 2, \mu(\varphi), n, p) \sinh^{\lambda-3} t\varphi(x)\varphi(y).\end{aligned}$$

The right hand sides are convergent with respect to x in L^2-sense uniformly with respect to y. For $\lambda > 2n + 2$ the series are pointwise convergent with respect to x and y.

The improved convergence properties for $\lambda > 2n + 2$ are a consequence of the well-known asymptotic behaviour of the eigenforms (see [6,15])

$$\sum_{\varphi \in E_p^d \cup E_p^\delta, \mu(\varphi) < T} \|\varphi(x)\|^2 = O(T^{\frac{n}{2}})$$

and estimation (1).

4. The Selberg's trace formula. The trace of the double p-form

$$\varphi(x,y)=\varphi_{i_1\dots i_p j_1\dots j_p}(x,y)dx^{i_1}\wedge\dots\wedge dx^{i_p}dy^{j_1}\mathbin{\overline{\wedge}}\dots dy^{j_p}$$

written in local coordinates in a Riemannian space is given by

$$\mathrm{tr}\ \varphi(x,x)=p!g^{i_1j_1}(x)\dots g^{i_pj_p}(x)\varphi_{i_1\dots i_pj_1\dots j_p}(x,x)$$

using the contravariant metric tensor g^{ij} and the Einstein summation convention. If we use a $\mathbb{R}^{m^2}$-valued double differential form we first take the trace of the components using the formulas above and then take the trace of the $(m\times m)$-matrix.

We will use the Poincaré coordinate system of the upper half space for $\mathbb{H}_n$: $\mathbb{H}_n=\{x=(x^1,\dots,x^n):x^n>0\}$, the metric tensor is given by $g\Big(\dfrac{\partial}{\partial x^i},\dfrac{\partial}{\partial x^j}\Big)=\dfrac{\delta_{ij}}{(x^n)^2}$ with the Kronecker symbol δ_{ij}.

In order to calculate traces we need some well-known facts about the structure of $\mathcal{G}$. There is a one-to-one relation between the conjugacy classes of elements of $\mathcal{G}$ and the free homotopy classes of the space form $V=\mathbb{H}_n/\mathcal{G}$. In this way, we can use the length $l(\cdot)$, multiplicity $\nu(\cdot)$ and the weights $\sigma(\cdot)$, $e_p(\cdot)$ in terms of free homotopy classes or in terms of the elements $b\in\mathcal{G}, b\neq\mathrm{id}$. Let $\{b_r\}_{r\in R}$ be a set of representatives of the conjugacy classes of primitive elements with an at most countable index set R. If $\mathcal{G}_r$ denotes the cyclic group generated by $b_r, r\in R$, we consider the right coset decomposition of G by $\mathcal{G}_r:\mathcal{G}_r=\bigcup_{k\in K}\mathcal{G}_rc_{rk}$. Thereby K is at most countable. $\{b_r^j\}$ with $r\in R, j\in\mathbb{Z}\backslash\{0\}$ is a set of representatives of the conjugacy classes of $\mathcal{G}$ without the identity class. Using the decomposition $\mathcal{G}=\mathrm{id}\ \cup\{c_{rk}^{-1}b_r^jc_{rk}:r\in R,k\in K,j\in\mathbb{Z}\backslash\{0\}\}$ it is not difficult to check that $F_r=\displaystyle\bigcup_{k\in K}c_{rk}F$ is a fundamental domain of the group $\mathcal{G}_r$ if F is a fundamental domain of $\mathcal{G}$. From [15] we have

$$\mathrm{tr}\ (c^{-1}bc)^*\alpha_p(x,c^{-1}bc)=\mathrm{tr}\ b^*\alpha_p(cx,bcx).$$

We consider

$$\mathcal{M}^\alpha_\#(t,\lambda,x,y,n,p)=\sum_{b\in\mathcal{G},b\neq\mathrm{id}}\rho(b)\,b^*M^\alpha(t,\lambda,x,by,n,p).$$

Using the structure of $\mathcal{G}$, we get

$$\mathrm{tr}\ \mathcal{M}^\alpha_\#(t,\lambda,x,x,n,p)$$

$$=\sum_{j\in\mathbb{Z}\backslash\{0\},r\in R}\rho(b_r^j)\int_{F_r}\{2(\cosh t-\cosh r(x,b_r^jx))\}^{\frac{\lambda-n-3}{2}}\ \mathrm{tr}\ (b_r^j)^*\alpha_p(x,b_r^jx)dx.$$

We recall that the geodesic distance $r(x,y)$ is given in the used Poincaré coordinate system by

$$\cosh r(x,y) = 1 + \frac{(x^1-y^1)^2 + \ldots + (x^n-y^n)^2}{2x^n y^n}$$

for $x=(x^1,\ldots,x^n), y=(y^1,\ldots,y^n)$. For every $b\in\mathcal{G}$ there exists such a Poincaré coordinate system that we get

$$y^i = e^{l(b)}\sum_{k=1}^{n-1}\alpha^i_k x^k \quad (i=1,\ldots,n-1), \quad y^n = e^{l(b)}x^n$$

for $x=(x^1,\ldots,x^n), bx=(y^1,\ldots,y^n)$ with an orthogonal matrix α^i_k. Now one is able to perform all calculations explicitly. Following [15] without any changes, we get

$$\begin{aligned}&\mathrm{tr}\ \mathcal{M}^\alpha_\#(t,\lambda,x,x,n,p)\\ &= \sum_{j\in\mathbb{Z}\setminus\{0\},r\in R,l(b^j_r)<t} \rho(b^j_r)\,\sigma(b^j_r)\,l(b^j_r)\Big(\{2(\cosh t-\cosh l(b^j_r)\}^{\frac{\lambda-2}{2}}\frac{2p-n-\lambda+1}{2(\lambda-2)}e_{p-1}(b^j_r)\\ &+\{2(\cosh t-\cosh l(b^j_r)\}^{\frac{\lambda-4}{2}}(e_{p-1}(b^j_r)\cosh t+e_p(b^j_r))\Big).\end{aligned}$$

We express the right hand side in terms of the free homotopy classes:

$$\begin{aligned}&\mathrm{tr}\ \mathcal{M}^\alpha_\#(t,\lambda,x,x,n,p)\\ &= \sum_{\omega\in\Omega,l(\omega)<t}\rho(\omega)\sigma(\omega)l(\omega)\Big(\{2(\cosh t-\cosh l(\omega)\}^{\frac{\lambda-2}{2}}\frac{2p-n-\lambda+1}{2(\lambda-2)}e_{p-1}(\omega)\\ &+\{2(\cosh t-\cosh l(\omega)\}^{\frac{\lambda-4}{2}}(e_{p-1}(\omega)\cosh t+e_p(\omega)\Big).\end{aligned}$$

We integrate over F:

$$\begin{aligned}&\int_F \mathrm{tr}\ \mathcal{M}^\alpha_\#(t,\lambda,x,x,n,p)\\ &= \sum_{\varphi\in E^\delta_p\cup E^h_p}\mathfrak{x}(t,\lambda-2,\mu(\varphi),n,p)\sinh^{\lambda-3}t + \sum_{\varphi\in E^d_p}\mathfrak{u}(t,\lambda-2,\mu(\varphi),n,p)\sinh^{\lambda-3}t.\end{aligned}$$

If we use the last two equations and include the term corresponding to id $\in\mathcal{G}$ with tr $\rho(\mathrm{id}\ ) = m$, we get

$$\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})}\pi^{-\frac{n}{2}}\binom{n}{p}\ m\ \mathrm{vol}\ \ V\{2(\cosh t-1)\}^{\frac{\lambda-n-3}{2}}+\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-2}{2})}2^{\frac{1-n}{2}}\pi^{-\frac{1}{2}}$$

$$\Bigg(\sum_{\omega\in\Omega,l(\omega)<t}\mathrm{tr}\ \rho(\omega)\sigma(\omega)l(\omega)\Big(\{2(\cosh t-\cosh l(\omega)\}^{\frac{\lambda-2}{2}}\frac{2p-n-\lambda+1}{2(\lambda-2)}e_{p-1}(\omega)$$

$$+\{2(\cosh t-\cosh l(\omega)\}^{\frac{\lambda-4}{2}}(e_{p-1}(\omega)\cosh t+e_p(\omega))\Big)\Bigg)$$

$$=\sum_{\varphi\in E_p^\delta\cup E_p^h}\mathfrak{x}(t,\lambda-2,\mu(\varphi),n,p)\sinh^{\lambda-3}t+\sum_{\varphi\in E_p^d}\mathfrak{u}(t,\lambda-2,\mu(\omega),n,p)\sinh^{\lambda-3}t.$$

If we compare this equation with that of the theorem we want to prove we shall see that the results for the Euler-Poisson-Darboux parameters $\lambda-2$ and λ and for the degrees $p-1$ and p are mixed in the last equation given above. By induction with respect to p we can separate the "mixed results" into the desired "pure results". This procedure is given in details in [15]. We get

$$\sum_{\mu\in S_p}d_p^*(\mu)\ \mathfrak{x}(t,\lambda,\mu,n,p)\ \sinh^{\lambda-1}t$$

$$=\frac{\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})}\binom{n-1}{p}\ m\ \mathrm{vol}\ V\sum_{u=0}^{N-|N-p|}2^{-2u}\frac{\Gamma(\frac{n}{2}-u)\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda-n+1}{2}+u)}$$

$$\cdot\prod_{v=1}^{u}((N+1-v)^2-(N-p)^2)\{2(\cosh t-1)\}^{\frac{\lambda-n-1}{2}+u}$$

$$+\frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda}{2})}2^{\frac{1-n}{2}}\pi^{-\frac{1}{2}}\sum_{\omega\in\Omega,l(\omega)<t}\rho(\omega)\sigma(\omega)l(\omega)\{2(\cosh t-\cosh l(\omega)\}^{\frac{\lambda-2}{2}}e_p(\omega).$$

Now we can proceed as in [15] to complete the proof. So, we use the last equation for $\lambda = 2n+2$ and apply the differential operator

$$\frac{d}{dt}\Big(\frac{1}{\sinh t}\ \frac{d}{dt}\Big)^n,$$

which gives an equation in the function space $\mathcal{D}'(\mathbb{R})$. We first use an even test function $g \in \mathcal{D}(\mathbb{R})$ and then we apply a standard approximation argument.

REFERENCES

[1] L. Berard-Bergéry, Laplacien et géodésiques fermées sur les formes d'espace hyperbolique compactes. Seminaire Bourbaki (1971/72), Lecture Notes in Mathematics, vol. 317, Berlin, 1973, pp. 107-120.

[2] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété Riemannienne, Lecture Notes in Mathematics, vol. 194, Berlin, 1971.

[3] J. B. Diaz and H. F. Weinberger, A solution of the singular initial value problem for the Euler-Poisson-Darboux equation, *Proc. Amer. Math. Soc.* **4** (1953), 703-715.

[4] J. Elstrodt, Die Selbergsche Spurformel für kompakte Riemannsche Flächen, *Jber. Dt. Math. Ver.* **83** (1981), 45-77.

[5] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, *Invent. Math.* **84** (1986), 523-540.

[6] A. Friedman, The wave equation for differential forms, *Pacific J. Math.* **11** (1961), 1267-1279.

[7] R. Gangolli, Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one, *Illinois J. Math.* **21** (1977), 1-41.

[8] R. Gangolli, The length spectra of some compact manifolds of negative curvature, *J. Differential Geom.* **22** (1977), 403-424.

[9] R. Gangolli and G. Warner, On Selberg's trace formula, *J. Math. Soc. Japan* **27** (1975), 328-343.

[10] P. Günther, Harmonische geodätische p-Formen in nichteuklidischen Räumen, *Math. Nachr.* **28** (1965), 291-304.

[11] P. Günther, Poisson formula and estimations for the length spectrum of compact hyperbolic space forms, *Studia Sci. Math. Hung.* **14** (1979), 49-59.

[12] P. Günther and R. Schimming, Curvature and spectrum of compact Riemannian manifolds, *J. Differential Geom.* **12** (1977), 599-618.

[13] D. A. Hejhal, The Selberg trace formula for $PSL(2, R)$, Lecture Notes in Mathematics, vol. 548, Berlin, 1976.

[14] J. Kolk, The Selberg trace formula and asymptotic behaviour of spectra, Thesis, Rijksuniversiteit Utrecht, 1977.

[15] R. Schuster, The Selberg trace formula and a Poisson formula for the p-spectrum of compact hyperbolic space forms, *Serdica* **12** (1986), 288-299.

[16] R. Schuster, On a lattice problem for geodesic double differential forms in the n-dimensional hyperbolic space, *Z. Anal. Anwendungen*, (to appear).

[17] R. Schuster, Spectral estimates for compact hyperbolic space forms and the Selberg zeta function for p-spectra (in preparation).

[18] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.* **20** (1956), 47-87.

[19] N. Subia, Formule de Selberg et formes d'espaces hyperboliques compactes, Lecture Notes in Mathematics, vol. 497, Berlin, 1975.

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