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A THEOREM FOR THE IRREDUCIBLE MATRICES WITH A SLIGHTLY DOMINANT PRINCIPAL DIAGONAL

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Abstract. This paper considers irreducible matrices with a slightly dominant principal diagonal. The theorem of O. Taussky giving a sufficient condition for non–singularity of such matrices is generalized. A new sufficient condition for convergence of Jackoby's method for solving systems of linear equations for which the matrix of coefficients has a slightly dominant principal diagonal is proved.

Theorem of O. Taussky [1]. *Let matrix D of n-th order satisfy the following conditions:*

(i) *D is irreducible;*

(ii) *D has a slightly dominant principal diagonal , i.e.*

$$H_i = |d_{ii}| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |d_{ij}| \geq 0, \;\; i = 1, \dots, n;$$

(iii) *There is a strict inequality at least for one i in above inequalities, i.e. there exists i_0, $1 \leq i_0 \leq n$, for which $H_i > 0$.*

Then the matrix D is non-singular.

Corollary. *Let $A = \{a_{ij}\}$ be an irreducible matrix for which the following conditions are satisfied:*

a) $a_{ii} = 0, \;\; i = 1, \dots, n;$

b) $\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq 1,$

c) *There exists i_0, for which* $\sum\limits_{\substack{j=1 \\ j\neq i_0}}^{n} |a_{i_0 j}| < 1.$

Then the matrix A has no eigenvalue λ, with $|\lambda| = 1$.

Assertion 1. *In the inequalities of the triangle $|x - y| \geq \big||x| - |y|\big|$ and $|x + y| \leq |x| + |y|$ an equality is obtained iff $\overline{x}y = |\overline{x}y|$.*

Lemma. *Consider the matrix $A = \{a_{ij}\}$ with a slightly dominant principal diagonal and $a_{11} \neq 0$. We denote*

$$H_i = |a_{ii}| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \;\; i = 1, \ldots, n;$$

$$H_i^1 = |a_{ii}^1| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}^1|, \;\; i = 1, \ldots, n;$$

where $a_{i1}^1 = 0$, $(i = 2, \ldots, n)$; $a_{1j}^1 = a_{1j}$, $(j = 1, \ldots, n)$; and a_{ij}^1 $(i = 2, \ldots, n,\; j = 2, \ldots, n)$ are obtained after the first step of reducing the matrix A to the upper – triangular form by Gauss method. Then

$$H_i^1 - H_i \geq \left|\frac{a_{i1}}{a_{11}}\right| H_1 \geq 0, \;\; i = 2, \ldots, n.$$

Proof.

$$H_i^1 - H_i = |a_{ii}^1| - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}^1| - |a_{ii}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

$$= \left|a_{ii} - \frac{a_{1i}a_{i1}}{a_{11}}\right| - \sum_{\substack{j=2 \\ j\neq i}}^{n} \left|a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}}\right| - |a_{ii}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

$$\geq |a_{ii}| - \left|\frac{a_{1i}a_{i1}}{a_{11}}\right| - \sum_{\substack{j=2 \\ j\neq i}}^{n} \left(|a_{ij}| + \left|\frac{a_{1j}a_{i1}}{a_{11}}\right|\right) - |a_{ii}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

$$= -\left|\frac{a_{1i}a_{i1}}{a_{11}}\right| + |a_{i1}| - \left|\frac{a_{i1}}{a_{11}}\right| \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{1j}|$$

$$= \left|\frac{a_{i1}}{a_{11}}\right| \left(|a_{11}| - \sum_{j=2}^{n} |a_{1j}|\right) = \left|\frac{a_{i1}}{a_{11}}\right| H_1 \geq 0.$$

Note. It follows from the proof above and from Assertion 1 that the equality $H_i^1 - H_i = 0$ is obtained for

$$\frac{\overline{a}_{ij}a_{1j}a_{i1}}{a_{11}} = -\left|\frac{\overline{a}_{ij}a_{1j}a_{i1}}{a_{11}}\right| \text{ for } i,j = 2,\dots,n;\; i \neq j \text{ and}$$
$$\frac{\overline{a}_{ii}a_{1i}a_{i1}}{a_{11}} = \left|\frac{\overline{a}_{ii}a_{1i}a_{i1}}{a_{11}}\right| \text{ for } i = 2,\dots,n.$$

Consider a matrix A, which satisfies the following conditions :

(1) $$a_{ii} = 0,\; i = 2,\dots,n;$$

(2) $$\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = 1,\; i = 2,\dots,n;$$

(3) Matrix A is irreducible.

Theorem 1. *If matrix A satisfies the conditions* (1), (2), (3) *and has an eigenvalue $\lambda = 1$, then*

(4) $$\overline{a}_{ij}a_{i1}a_{1j} = |\overline{a}_{ij}a_{i1}.a_{1j}|$$

$$\begin{pmatrix} i,j = 2,\dots,n \\ i \neq j \end{pmatrix}.$$

(5) $$a_{i1}a_{1i} = |a_{i1}a_{1i}| = |a_{i1}a_{1i}|$$

Proof. Let $\Delta = |A - E| = 0$. Let a step be carried out by the Gauss method for reducing the matrix $A - E$ to an upper triangular form, i.e. for $k = 1,\dots,n$ we multiply the first row of Δ by a_{k1} and add it to the k-th row. We obtain $\Delta = \det\{S_{ij}\}$, for which :

$$S_{11} = -1,$$

$$S_{i1} = 0,\; S_{1i} = a_{i1}\; (i = 2,\dots,n);$$

$$S_{ii} = -1 + a_{1i}a_{i1}\; (i = 2,\dots,n);$$

$$S_{ki} = a_{ki} + a_{k1}a_{1i}\; (i,k = 2,\dots,n;\; i \neq k).$$

Consider matrix $S = \{S_{ij}\}_{i,j=2}^{n}$. According to the Lemma matrix S has a slightly dominant principal diagonal, i.e $H_i \geq 0$, for $i = 2,\dots,n$. We are going to prove that S is an irreducible matrix.

Let us assume that S is a reducible matrix. This assumption presupposes the existence of irreducible matrices S_{11}, S_{22},..., S_{pp} with dimensions $l_1, l_2, \dots, l_p$ respectively so that after some permutations of rows and columns, if necessary, we obtain matrix S

$$S = \left\| \begin{matrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{21} & S_{22} & \dots & S_{2n} \\ \dots & \dots & .. & \dots \\ S_{n1} & S_{n2} & \dots & S_{nn} \end{matrix} \right\|$$

with determinant $0 = \Delta = \det S = \det S_{11} \times \det S_{22} \times \dots \times \det S_{pp}$.

We assume that $\det S_{kk} = 0$, for any k, $1 \leq k \leq p$. We denote $N_k = l_1 + l_2 + \dots + l_{k-1}$ for $k > 1$, $N_1 = 0$. The matrix S_{kk} is irreducible. If it is singular in terms of O Taussky's theorem, the following conditions are satisfied:

$$|a^1_{ii}| - \sum_{\substack{j=N_k+1 \\ j\neq 1}}^{N_k+l_k} |a^1_{ij}| = 0 \text{ for } i = N_k + 1, \dots, N_k + l_k.$$

However, from the Lemma and conditions (1), (2) it follows that

$$0 = |a^1_{ii}| - \sum_{\substack{j=N_k+1 \\ j\neq 1}}^{N_k+l_k} |a^1_{ij}| \geq |a^1_{ii}| - \sum_{\substack{j=N_k+1 \\ j\neq i}}^{N_k+l_k} |a_{ij}| \geq 1 - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = 0.$$

Therefore $a_{ij} = 0$ for $i = N_k+1, \dots, N_k+l_k,\ j = 1, \dots, N_k$ and $j = N_k+1, \dots, n$ and matrix A is of the following form

$$A = \left\| \begin{matrix} A_{11} & A_{12} & \dots & A_{1k} & \dots & \dots & A_{1p} \\ \dots & \dots & .. & \dots & .. & .. & \dots \\ 0 & 0 & \dots & A_{kk} & 0 & \dots & 0 \\ \dots & \dots & .. & \dots & .. & .. & \dots \\ \dots & \dots & .. & \dots & .. & .. & \dots \\ A_{p1} & A_{p2} & \dots & A_{pk} & \dots & \dots & A_{pp} \end{matrix} \right\|$$

i.e. A is reducible. The derived contradiction with (3) shows that matrix S_{kk} is non-singular and therefore, if matrix S is reducible, then it is non-singular. This contradiction proves that matrix S is irreducible. However, S has a slightly dominant principal diagonal and therefore $H^1_i = H_i = 0$ for $i = 2, \dots, n$ should be satisfied (from Taussky's Theorem). According to the Lemma

$$\begin{aligned} \frac{\overline{a}_{ij} a_{1j} a_{i1}}{-1} &= -\left|\frac{\overline{a}_{ij} a_{1j} a_{i1}}{-1}\right| \text{ for } i, j = 2, \dots, n;\ i \neq j \\ \text{and } \frac{-\overline{1} a_{1i} a_{i1}}{-1} &= \left|\frac{-\overline{1} a_{1i} a_{i1}}{-1}\right|. \end{aligned}$$

Evidently conditions (4), (5) follow from this. Thus the theorem has been proved.

We shall call a complex square matrix A ε–matrix if $A = \varepsilon\alpha\overline{\varepsilon}$, where the real matrix $\alpha = (\alpha_{ij})$ has only nonnegative elements and $\varepsilon = \mathrm{diag}(1, \varepsilon_2, \dots, \varepsilon_n)$ with $|\varepsilon_i| = 1$, $i = 2, \dots, n$. It is evident that every ε–matrix satisfies conditions (4), (5). So, the following theorem is to a certain extent opposite to Theorem 1.

Theorem 2. *If A is an ε–matrix and A satisfies conditions* (1), (2) *and* (3), *it has an eigenvalue $\lambda = 1$.*

Consider the determinant $\Delta = |A - E|$. By multiplying the i-th column of the determinant Δ by ε_i and adding it to the first one, we obtain $\Delta = \det\{S_{ij}\}$, for which, according to the ε–matrix definition and condition (2), we obtain the following equations:

$$\begin{aligned} S_{11} &= -1 + \sum_{j=2}^{n} a_{1j}\varepsilon_j = -1 + \sum_{j=2}^{n} \alpha_{1j}\varepsilon_j\overline{\varepsilon}_j = -1 + \sum_{j=2}^{n} \alpha_{1j} = 0; \\ S_{i1} &= a_{i1} + \sum_{\substack{j=2 \\ j\neq i}}^{n} a_{ij}\varepsilon_j - \varepsilon_i = \varepsilon_i\alpha_{i1} + \sum_{\substack{j=2 \\ j\neq i}}^{n} \alpha_{ij}\varepsilon_i\overline{\varepsilon}_j\varepsilon_j - \varepsilon_i \\ &= \varepsilon_i\Big(-1 + \alpha_{i1} + \sum_{\substack{j=2 \\ j\neq i}}^{n} \alpha_{ij}\Big) = \varepsilon_i\Big(-1 + \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_{ij}\Big) = 0 \end{aligned}$$

for $i = 2, \dots, n$. Thus $S_{i1} = 0$ for $i = 1, \dots, n$. Hence $\Delta = 0$.

Corollary 1. *If matrix A satisfies conditions (1), (2), (3) and has an eigenvalue λ by module equal to* 1, *then there exists θ, $\theta \in [0, 2\pi]$, so that for every $i, j, p = 1, \dots, n$, $i \neq j$, $i \neq p$, $p \neq j$ the following equations are satisfied:*

$$\overline{a}_{ij}.a_{ip}.a_{pj} = e^{i\theta}.|\overline{a}_{ij}.a_{ip}.a_{pj}| \tag{6}$$

$$a_{ip}.a_{pi} = e^{2i\theta}.|a_{ip}.a_{pi}| \tag{7}$$

Conversely, if matrix A satisfies conditions (1), (2), (3) and $A = \varepsilon.\alpha.\overline{\alpha}.e^{i\theta}$, then it has an eigenvalue $\lambda = e^{i\theta}$ by module equal to 1.

The simple corollary from Theorem 1 is a generalization of Taussky's theorem. A number of variants of such generalizations are known (e.g. see [2], [3], [4], [5], [6]). Let us consider the case when all H_i are equal to zero. In other words let us eliminate condition (iii) from Taussky's theorem.

Corollary 2. *Let matrix $B = \{b_{ij}\}$ of n order satisfy the conditions:*

1. *B is irreducible.*
2. *For each $i = 1, \dots, n$ it is true that*

$$H_i = b_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}| = 0.$$

3. *At least one of the conditions*

$$\begin{aligned} &(8) \qquad \frac{\overline{b}_{ij} b_{pj} b_{ip}}{b_{pp}} = -\left|\frac{\overline{b}_{ij} b_{pj} b_{ip}}{b_{pp}}\right| \\ & \qquad\qquad\qquad\qquad\qquad\qquad \left(\begin{array}{c} i, j, p = 1, \dots, n \\ i \neq j,\ i \neq p,\ j \neq p \end{array}\right). \\ &(9) \qquad \frac{b_{ip} b_{pi}}{b_{ii} b_{pp}} = \left|\frac{b_{ip} b_{pi}}{b_{ii} b_{pp}}\right| \end{aligned}$$

is not satisfied.

Then matrix B is non-singular.

Consider three diagonal matrices

$$T = \begin{pmatrix} 1 & a_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_1 & 1 & a_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & b_2 & 1 & a_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b_{n-1} & 1 & a_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & b_n & 1 \end{pmatrix}$$

Such matrices are used for example in some numerical methods for differential equations (see [7], 6.3.5). Matrix T has a slightly dominant principal diagonal when $|a_0| \leq 1$, $|b_n| \leq 1$ and $|a_i| + |b_i| \leq 1$. $(i = 1, 2, \dots, n-1)$. The conditions of Corollary 2 in this case are

(a) for irreducibility $a_i.b_{i+1} \neq 0$, $(i = 0, 1, \dots, n-1)$;

(b) for 2.: $|a_0| = |b_n| = 1$, $|a_i| + |b_i| = 1$, $(i = 1, \dots, n-1)$.

Conditions (8) are satisfied for every matrix T and the conditions (9) in this case are equivalent to $a_i b_{i+1} > 0$, $(i = 0, 1, \dots, n-1)$. Therefore, if conditions (a) and (b) are satisfied, matrix T is non singular, iff $a_i b_{i+1} < 0$ at least for one i $(i = 0, 1, \dots, n-1)$.

Corollary 3. *The method of Jacoby for solving the system of linear equations $Bx = f$ by the formula $x^k = Ax^{k-1} + g$ is convergent if matrix $A = (\mathrm{diag}B)^{-1}B - E$ satisfies the conditions (2) and (3) and does not satisfy conditions (6) and (7). For matrix B these conditions are:*

$$\begin{aligned} \frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}} &= e^{i\theta}\left|\frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}}\right| \\ & \qquad\qquad \left(\begin{array}{c} i,j,p=1,\dots,n \\ i\neq j,\ i\neq p,\ j\neq p \end{array}\right). \\ \frac{b_{ip}b_{pi}}{b_{ii}b_{pp}} &= e^{2i\theta}\left|\frac{b_{ip}b_{pi}}{b_{ii}b_{pp}}\right| \end{aligned}$$

Corollary 4. *Consider the method of Gauss–Zeidel for solving the system of linear equations $Bx = f$ by the formula*

$$(D+L)x^k = -Ux^{k-1} + f,$$

where L, D and U are strict lower triangular, diagonal, and strict upper triangular matrices respectively, so that $B = D + L + U$. It is known that a necessary and sufficient condition for convergence of this method is that all roots λ of the equation $\det(U + \lambda(D + L)) = 0$ *must be in the unit circle. It holds if matrix B satisfies both the first and the second condition of Corollary* 2 *and there exist no $\theta \in [0, \pi]$ for which the following conditions are satisfied:*

$$\begin{aligned} \frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}} &= -e^{i\theta}\left|\frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}}\right| \quad \textit{at } (i-j)(j-p)(i-p) > 0 \\ \frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}} &= -\left|\frac{\overline{b}_{ij}b_{pj}b_{ip}}{b_{pp}}\right| \quad \textit{at } (i-j)(j-p)(i-p) < 0 \\ \frac{b_{ip}b_{pi}}{b_{i1}b_{pp}} &= e^{2i\theta}\left|\frac{b_{ip}b_{pi}}{b_{ii}b_{pp}}\right| \quad \left(\begin{array}{c} i,j,p=1,\dots,n \\ i\neq j,\ i\neq p,\ j\neq p \end{array}\right). \end{aligned}$$

It is necessary to evaluate the checking algorithm for the practical elucidation of the singularity of matrix B or the convergence of the methods of Jacoby and Gauss–Zeidel. It is evident that apart from the n conditions $H_i = 0$ $(i = 1, \dots, n)$, one has to check the validity of other $(n-1)+(n-1).(n-2) = (n-1)^2$ equations, i.e. to carry out no more than $(n-1)^2$ number of multiplications, but in general, considerably less than $(n-1)^2$. In comparison, for one step of the Jacoby method $(n-1)^2$ multiplications are necessary.

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