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ON SOME BOUNDS FOR POLYNOMIAL ROOTS OBTAINED WHEN DETERMINING THE R-ORDER OF ITERATIVE PROCESSES[†]

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1. Introduction. Let an iterative method I in a Banach space B produce sequences of iterates $\{x^{(k)}\}$ with $\lim\limits_{k\to\infty} x^{(k)} = x^*$. In many cases, one can show for the corresponding sequences of errors $e^{(k)} = ||x^{(k)} - x^*||$ the recursion

$$e^{(k+1)} \leq \gamma \prod_{i=0}^{n} \left(e^{(k-i)}\right)^{q^i(p+1)}, \quad n \leq k, \quad k \geq 0,$$

where γ, p, q are positive and independent of k. In order to calculate the R-order of convergence of I $O_R(I, x^*)$ (see Ortega and Rheinboldt [2]) one has to compute the unique positive root $\sigma_{p,q}^{(n)}$ of the polynomial

$$P_n(x) = x^n - (p+1)\sum_{k=1}^{n} q^k x^{n-k}, \quad p \geq 0, \quad q > 0. \tag{1}$$

J.W.Schmidt [1] has shown that

$$O_R(I, x^*) \geq \sigma_{p,q}^{(n)}$$

is valid. The following estimates for $\sigma_{p,q}^{(n)}$ are known in the literature (see [3]-[5]):

$$\frac{n}{n+1}(p+q+1) < \sigma_{p,q}^{(n)} < p+q+1, \quad n > q/(p+1) \tag{2}$$

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$$\sigma_{p,q}^{(n)} > \frac{(p+1)(1+S_1+S_2)}{1+(p+1)S_1}, \quad n \geq 2, \text{where} \tag{3}$$

$$S_1 = \begin{cases} n(n-1)/2, & q = 1 \\ q((n-1)q^n - nq^{n-1}+1)/(q-1)^2, & q \neq 1 \end{cases}$$

$$S_2 = \begin{cases} n-1, & q = 1 \\ (q^n - q)/(q-1), & q \neq 1 \end{cases}$$

$$p+q+1-\frac{(p+1)q^n}{(p+q+1)^n}(1+1/n)^n < \sigma_{p,q}^{(n)} < p+q+1-\frac{(p+1)q^n}{(p+q+1)^n}, \quad n > q/(p+1). \tag{4}$$

In general, the lower bound (4) is very accurate. It was proved by Traub [9] for the case $p \geq 0$, $q = 1$. The purpose of this note is to show some new estimates for $\sigma_{p,q}^{(n)}$.

2. Some estimation formulas. In order to prove our results, we make use of some well-known estimations in the literature. The following theorems are used so as to obtain these estimations.

Theorem A (E.Deutsch [6]). *Let $A = (a_{ij})$ be a nonnegative and irreducible $n \times n$-matrix and let the positive vectors x, y be defined by*

$$Ax = Dx, \quad A^T y = Dy,$$

where $D = diag\,(d_1, \ldots, d_n) > 0$. If x is not an eigenvector of A, then it follows for the spectral radius $\rho(A)$ of A

$$\rho(A) > \frac{y^T Dx}{y^T x}. \tag{5}$$

In [4] M.Petkovic and Lj.Petkovic derive the estimation (3) by means of (5).

Theorem B (E.Deutsch [6]). *Under the same assumptions as in Theorem A we get the inequalities*

$$\rho(A) > t - \prod_{i=1}^{n}(t-d_i)^{x_i y_i / y^T x} > \frac{y^T Dx}{y^T x} \tag{6}$$

for all $t > \rho(A) + \max\limits_{1 \leq i \leq n}(d_i - a_{ii})$.

Theorem C (Westerfield [7]). *Let z be the unique positive root of the equation*

$$x^n = \sum_{k=1}^{n} q_k^k x^{n-k}, \quad q_k \geq 0, \quad 1 \leq k \leq n \tag{7}$$

and let positive quantities q_k, $1 \leq k \leq n$ after being arranged in order of decreasing magnitudes, form a sequence

$$q_1 \geq q_2 \geq \dots \geq q_n.$$

Then z satisfies the inequality

$$z \leq \sum_{r=1}^{n} q_r g_r,$$

where

$$g_1 = y_1, \quad g_r = y_r - y_{r-1}, \quad r = 2, 3, \dots, n$$

and y_k is the positive root of the polynomial

$$y^k = \sum_{r=1}^{k} y^{k-r}, \quad k = 1, \dots, n.$$

Theorem D (Bojanov [8]). *Let*

$$q_k = \sum_{j=1}^{m} a_{jk}, \quad k = 1, \dots, m; \quad m \geq 1, \quad a_{jk} \geq 0.$$

Let x_j be the positive zeros of the polynomials

$$x^n = \sum_{k=1}^{n} a_{jk}^k x^{n-k}, \quad j = 1, \dots, m$$

then for the positive root z of the polynomial

$$x^n = \sum_{k=1}^{n} q_k^k x^{n-k}$$

the following estimation holds:

$$z < x_1 + \dots + x_m.$$

Another application of Theorem D gives a lower bound:

$$z > \sum_{r=1}^{n} q_r g_{n+1-r}.$$

3. Further results. The estimations of Section 2 may help us to find out the following bounds.

Theorem 1. *Under the assumptions of Theorem C we get the bound*

$$z > \frac{\lambda_1 + \mu_1}{1 + \mu_1}, \tag{8}$$

where

$$\lambda_j = \sum_{k=j}^{n} q_k^k, \quad \mu_j = \sum_{k=j}^{n} (k-1) q_k^k, \quad j = 1, \dots, n$$

P r o o f. Polynomial (7) if of the form

$$P(x) = x^n - \sum_{k=1}^{n} I_k x^{n-k} \quad (I_k = q_k^k).$$

Let us associate the following matrix to $P(x)$ [4]

$$A = \begin{pmatrix} I_1 & I_2 & I_3 & \dots\dots & I_{n-1} & I_n \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ \dots & \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & 0 & & 1 & 0 \end{pmatrix}$$

where $\det(xE - A) = P(x)$. The matrix A is non-negative and irreducible. Perron-Frobenius theorem implies that A has a positive eigenvalue λ equals to its spectral radius $\rho(A)$. It is obvious that $\rho(A) = z$. Moreover, let x be chosen so that $x = (1, \dots, 1)^T$. Then from $Ax = Dx$ we get $D = \mathrm{diag}\,(\lambda_1, 1, 1, \dots, 1)$.

Similarly, one can derive

$$y = \alpha(1, \lambda_2, \lambda_3, \dots, \lambda_n)^T, \quad \alpha > 0$$

$$y^T D x = \alpha(\lambda_1 + \mu_1)$$

$$y^T x = \alpha(1 + \mu_1).$$

Now, from (5) we get (8). □

We can derive another lower bound as follows:

Theorem 2. *Under the assumption of Theorem 1 we get the lower bound*

$$z > I_1 + 2\lambda_2 - \sqrt[1+\mu_1]{\lambda_2 (I_1 - 1 + 2\lambda_2)^{\mu_1}}. \tag{9}$$

P r o o f. From Theorem B, the values of

$$H(t) = t - \prod_{i=1}^{n} (t - d_i)^{x_i y_i / y^T x}$$

for

$$t > \rho(A) + \max_{1 \le i \le n} (d_i - a_{ii})$$

are strict lower bounds for $\rho(A)$. However, since $\rho(A)$ is not known, one should evaluate $H(t)$ for

$$t = \beta + \max_i (d_i - a_{ii})$$

where β is an upper bound of $\rho(A)$. Some of the possible values of β are: the largest row sum of A, the largest column sum of A, $\max_i d_i$.

In view of (6) and since

$$t = \max_i d_i + \max(d_1 - I_1, d_2 - 0, \dots, d_n - 0) = I_1 + 2\lambda_2$$

it follows that

$$z > t - \prod_{i=1}^{n} (t - d_i)^{x_i y_i / y^T x} \ge$$

$$I_1 + 2\lambda_2 - \left(\lambda_2 (I_1 - 1 + 2\lambda_2)^{\lambda_2 + \lambda_3 + \cdots + \lambda_n} \right)^{\frac{1}{1+\mu_1}}$$

which completes the proof of the theorem. □

Remark. From the bounds thus obtained we get a new bound for $\sigma_{p,q}^{(n)}$ (see (3)):

$$\sigma_{p,q}^{(n)} > t - \sqrt[1+\gamma]{(p+1)(q + q^2 + \cdots + q^{n-1})(t-1)^{\gamma}} \tag{3'}$$

$$t = (p+1)(1 + 2(q + q^2 + \cdots + q^{n-1}))$$

$$\gamma = (p+1) \sum_{k=1}^{n-1} k q^k.$$

A numerical example follows. For zero $\sigma_{p,q}^{(n)} \approx 5.1$ of the polynomial

$$x^6 = 5x^5 + 0.5x^4 + 0.05x^3 + 0.005x^2 + 0.0005x + 0.00005$$

$$(n = 6, \quad p = 4, \quad q = 0.1)$$

we get the bounds

from (2): $\sigma_{p,q}^{(6)} > 4.371$,
from (3): $\sigma_{p,q}^{(6)} > 3.817$,
from (3′): $\sigma_{p,q}^{(6)} > 4.815$.

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