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# APPROXIMATE DETERMINATION OF THE RATIONAL FUNCTION OF THE BEST RESTRICTED APPROXIMATION IN HAUSDORFF METRIC†

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**1. Definitions and notations.** The problem of numerical determination of the polynomial of the best restricted uniform approximation for a given continuous function $f$ on a finite interval $\Omega = [a,b]$ is considered in [1]. More precisely, let functions $l, f, u$ satisfy the inequalities $l(x) < f(x) < u(x)$, $x \in \Omega$. Let $K = K(H_n; l, u) = \{p : p(x) = \sum_{i=0}^{n} a_i x^i\,;\, l(x) \le p(x) \le u(x),\ x \in \Omega\}$ and $E(K;f) = \inf_{p\in K}\ \max_{x\in\Omega} |f(x) - p(x)| = \inf_{p\in K} \|f - p\|$.

The element $q \in K$ which satisfies $E(K;f) = \|f - q\|$ is called polynomial of best restricted approximation ( note that $q$ does not exist always). In practice (digital filters synthesis, [2]) it is important to approximate discontinuous functions by polynomials or rational functions under the restrictions that the approximate element lies between two given functions. For this purpose we use a Hausdorff metric [3]. Using the notations from [4],[5] let $R = \{x : -\infty < x < \infty\}$, $S^M(R) = \{[a,b] : a,b \in R,\ -M \le a \le b \le M,\ M > 0\}$, $A_\Omega^M = \{f : \Omega \to S^M(R),\ \Omega = [a,b]\}$ be the set of all segment-valued bounded functions in the interval $\Omega$. If $f \in A_\Omega^M$, then $F(f) \in A_\Omega^M$, $F(f;x) = [I(f;x), S(f;x)]$, where:

$$I(f;x) = \lim_{\delta\to 0} I(\delta, f; x) = \lim_{\delta\to 0} \inf\{y \in f(t) \,:\, t \in [x-\delta, x+\delta] \cap \Omega\}\,,$$

$$S(f;x) = \lim_{\delta\to 0} S(\delta, f; x) = \lim_{\delta\to 0} \sup\{y \in f(t) \,:\, t \in [x-\delta, x+\delta] \cap \Omega\}$$

is called completed graph of $f$. Let $f \in A_\Omega^M$. The functions $l$ and $u$ satisfy:

1. $l(x) < +\infty\,,\ u(x) > -\infty$, for $x \in \Omega$;

2. the sets $\Omega_{-\infty} = \{x \in \Omega : l(x) = -\infty\}$ and $\Omega_{+\infty} = \{x \in \Omega : u(x) = +\infty\}$ are open;

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3. $l \in A^M_{\Omega-}$, $u \in A^M_{\Omega+}$, where $\Omega- = \Omega \backslash \Omega_{-\infty}$, $\Omega+ = \Omega \backslash \Omega_{+\infty}$;

4. $l(x) < f(x) < u(x)$, for $x \in \Omega$.

If the functions $f, g \in A^M_\Omega$ the one-sided Hausdorff distance in $\Omega$, with parameters $\alpha(x) > 0$, $\beta(x) > 0$ from the function $g$ to the function $f$ is defined as in [5]:

$$h(d,f) = h(\Omega,\alpha,\beta;g,f) = \max_{(x,y)\in F(g)} \min_{(\xi,\eta)\in F(f)} \max\left\{\frac{|x-\xi|}{\alpha(x)}, \frac{|y-\eta|}{\beta(x)}\right\}$$

and the Hausdorff distance in $\Omega$, with parameters $\alpha(x) > 0$, $\beta(x) > 0$ between $g$ and $f$ is the number

$$H(f,g) = H(g,f) = H(\Omega,\alpha,\beta;f,g) = max\{h(\Omega,\alpha,\beta;f,g), h(\Omega,\alpha,\beta;g,f)\}.$$

For a given function $f \in A^M_\Omega$ we denote

$$E_H(f) = E_H(K,\Omega,\alpha,\beta;f,l,u) = \inf\{H(\Omega,\alpha,\beta;q,f) : q \in K(H_n;l,u)\}$$

$$E_h(f) = E_h(K,\Omega,\alpha,\beta;f,l,u) = \inf\{h(\Omega,\alpha,\beta;q,f) : q \in K(H_n;l,u)\}$$

The elements $p, q \in K$ are called elements of the best $h-$ and $H-$approximation of $f$ if they satisfy $E_h(f) = h(p,f)$ and $E_H(f) = H(q,f)$ respectively.

**2. The algorithm.**

Let $d_h(\Omega,\alpha,\beta;p,f;x) = \min_{(\xi,\eta)\in F(f)} \max\left\{\frac{|x-\xi|}{\alpha(x)}, \frac{|p(x)-\eta|}{\beta(x)}\right\}$,

$$D_h(p,f;x) = D_h(\Omega,\alpha,\beta;p,f;x) = \operatorname{sgn}[f(x)-p(x)]\, d_h(\Omega,\alpha,\beta;p,f;x),$$

$$\operatorname{sgn}[(f-p)(x)] = \operatorname{sgn}[f(x)-p(x)] = \begin{cases} +1, & p(x) < \min\{z : z \in f(x)\} \\ -1, & p(x) > \max\{z : z \in f(x)\} \\ 0, & p(x) \in f(x) \end{cases},$$

$$h(p,f) = h(\Omega,\alpha,\beta;p,f) = \max\{d_h(\Omega,\alpha,\beta;p,f;x) : x \in \Omega\},$$

$$D_r(p;f,l,u;x) = \begin{cases} \max\{D_h(p,f;x),\ D_h(p,l;x) + h(p,f)\}, & \operatorname{sgn}[(f-p)(x)] = +1, \\ \min\{D_h(p,f;x),\ D_h(p,u;x) - h(p,f)\}, & \operatorname{sgn}[(f-p)(x)] = -1, \end{cases}$$

$$D_r(p;f,l,u;x) = \operatorname{sgn}[(f-p)(x)]\, d_r(p;f,l,u;x),$$

$$r(p;f,l,u) = r(\Omega,\alpha,\beta;p;f,l,u) = \max\{d_r(p;f,l,u;x) : x \in \Omega\},$$

Note, that if $p \in K(H_n;l,u)$, then $h(p,f) = r(p;f,l,u)$.

Modifying the algorithm in [1] we suggest the following numerical method of determining the polynomial $p \in K(H_n;l,u)$, for which $E_h(K,\Omega,\alpha,\beta;f,l,u) = h(\Omega,\alpha,\beta;p,f)$.

Step 0 Set $s = 1$, $X^s = \{x_i^s\}_{i=0}^{n+1}$, $x_i^s < x_{i+1}^s$, $x_i^s \in \Omega$;
$X_f^s = X^s$, $X_l^s = \emptyset$, $X_u^s = \emptyset$, $X^s = X_f^s + X_l^s + X_u^s$, $f^0 = f$, $p^0 \equiv 0$ .

Step 1 Find $p^s \in K$ - element of the best $h$-approximation on discrete set $X^s$ for $\hat{f}$ with restrictions $\hat{l}, \hat{u}$, where:
for $x \in X^s$, $\hat{f}(x) = f(x)$, $\hat{l}(x) = l(x)$, $\hat{u}(x) = u(x)$.

Step 1.1 Put $t = 1$, $\overline{p}(x) = p^s(x)$, $\overline{X}_f = X_f^s$, $\overline{X}_l = X_l^s$, $\overline{X}_u = X_u^s$, $\overline{X} = X^s$.

Step 1.2 Define the discrete functions $\overline{f}, \overline{l}, \overline{u}$ on the discrete set $\overline{X}$
$\overline{l}(x) = \hat{l}(x) - \overline{p}(x)$ for $x \in \overline{X}_l$ ; $\overline{l}(x) = -\infty$ for $x \in \overline{X} \backslash \overline{X}_l$ ;
$\overline{u}(x) = \hat{u}(x) - \overline{p}(x)$ for $x \in \overline{X}_u$ ; $\overline{u}(x) = +\infty$ for $x \in \overline{X} \backslash \overline{X}_u$ ;
$\overline{f}(x) = D_h(\Omega, \alpha, \beta; \overline{p}, f; x)$ for $x \in \overline{X}_f$.

Step 1.3 Solve the linear system

$$\left| \begin{array}{lll} \sum_{j=0}^{n} c_j\,(x_i)^j \, + \, (-1)^i\, c_{n+1} & = \overline{f}(x_i), & x_i \in \overline{X}_f \\ \sum_{j=0}^{n} c_j\,(x_i)^j & = \overline{l}(x_i), & x_i \in \overline{X}_l \\ \sum_{j=0}^{n} c_j\,(x_i)^j & = \overline{u}(x_i), & x_i \in \overline{X}_u \end{array} \right.$$

put $p^s(x) = \overline{p}(x) + q(x)$, $e^s = |c_{n+1}|$ and define $D_h(p^s, f; x)$,
$e_{max} = \max_{x \in \overline{X}_f}\{d_h(p^s, f; x)\}$, $e_{min} = \min_{x \in \overline{X}_f}\{d_h(p^s, f; x)\}$
If $\{e_{max} - e_{min}\}/e_{max} < \varepsilon$ or $t > T_{max}$, then go to Step 2
otherwise put $\overline{p}(x) = p^s(x)$, $t = t + 1$ and go to Step 1.2

Step 2 Find set of points $X_f^{s+1}$, $X_l^{s+1}$, $X_u^{s+1}$ and
$X^{s+1} = X_f^{s+1} + X_l^{s+1} + X_u^{s+1} = \{x_i^{s+1}\}_{i=0}^{n+1}$ such that
$D_r(p^s; f, l, u; x_i^{s+1}) = (-1)^i \sigma d_r(p^s; f, l, u; x_i^{s+1})$ where
$\sigma = \mathrm{sgn}[D_r(p^s; f, l, u; x_0^{s+1})] = \pm 1$, $d_i^s = d_r(p^s; f, l, u; x_i^{s+1})$
there exists $\mu$, $0 \le \mu \le n$ for which $d_\mu^s = r(p^s; f, l, u)$
and $\mathrm{sgn} D_r(p^s; f, l, u; x_i^{s+1}) = (-1)^i \mathrm{sgn} D_r(p^s; f, l, u; x_0^{s+1})$.

Step 3 Define sets of points $X_f^{s+1}$, $X_l^{s+1}$, $X_u^{s+1}$ as follows:
$X_f^{s+1} = \{x_i^{s+1} : D_r(p^s; f, l, u; x_i^{s+1}) = D_h(p^s, f; x_i^{s+1})\}$
$X_l^{s+1} = \{x_i^{s+1} : D_r(p^s; f, l, u; x_i^{s+1}) = D_h(p^s, l; x_i^{s+1}) + h(p^s, f)\}$
$X_u^{s+1} = \{x_i^{s+1} : D_r(p^s; f, l, u; x_i^{s+1}) = D_h(p^s, u; x_i^{s+1}) - h(p^s, f)\}$

Step 4 Find the numbers $d_{max}^s = \max\{d_i^s\}_{i=0}^n$; $d_{min}^s = \min\{d_i^s\}_{i=0}^n$.
If $d_{max}^s - d_{min}^s < \varepsilon$ or $s > S_{max}$, then go to Step 5
else put $s = s + 1$ and go to Step 1.

Step 5 If $s \le S_{max}$, then we accept that $p^s$ is the required element of the best approximation; stop the computation.

**3. Convergence of the algorithm.** Let $\Omega$, $\Omega-$, $\Omega+$, $f \in A^M_{\Omega}$, $l \in A^M_{\Omega-}$, $u \in A^M_{\Omega+}$, be given and let us define for $\delta > 0$ the following functions:

$$\bar{l}(\delta;\Omega,\alpha,\beta;f,l;x) = \max\{l(x),\ I(\delta\alpha,\Omega,f;x) - \delta\,\beta(x)\},$$

$$\overline{u}(\delta;\Omega,\alpha,\beta;f,u;x) = \min\{u(x),\ S(\delta\alpha,\Omega,f;x) + \delta\,\beta(x)\},$$

where $I(\delta\alpha,\Omega,f;x)$ and $S(\delta\alpha,\Omega,f;x)$ are given by

$$I(\delta\alpha,\Omega,f;x) = \inf\{y \in f(t) \,:\, t \in [x-\delta\alpha(x), x+\delta\alpha(x)] \cap \Omega\},$$

$$S(\delta\alpha,\Omega,f;x) = \sup\{y \in f(t) \,:\, t \in [x-\delta\alpha(x), x+\delta\alpha(x)] \cap \Omega\}.$$

Let for $\delta > 0$ the functions $\alpha$, $\beta$, $f$, $l$, $u$ satisfy

$$H(\bar{l}(\delta;\Omega,\alpha,\beta;f,l)\,,\ f) = H(\overline{u}(\delta;\Omega,\alpha,\beta;f,u)\,,\ f) = \delta.$$

Let the best $H$-approximation satisfy $E_H(K,\Omega,\alpha,\beta;f,l,u) = \lambda \le \delta$, then:

1) $H(\bar{l}(\delta;\Omega,\alpha,\beta;f,l)\,,\ f) = H(\overline{u}(\delta;\Omega,\alpha,\beta;f,l)\,,\ f) = \lambda$.

2) There exists only one element of best $H$-approximation $p^* \in K(H_n;l,u)$ such that $E_h(K,\Omega,\alpha,\beta;f,l,u) = h(\Omega,\alpha,\beta;p^*,f) = \lambda = H(\Omega,\alpha,\beta;p^*,f) = E_H(K,\Omega,\alpha,\beta;f,l,u)$.

3) The sequence of polynomials from $H_m$, which is generated by the algorithm, converges uniformly to the polynomial of the best restricted approximation, i.e.
$\lim\limits_{s\to\infty} = p^*$, $p^* \in K(H_n;l,u)$, $\{e^s\}$ increases and
$\lim\limits_{s\to\infty} = e^* = E_h(K,\Omega,\alpha,\beta;f,l,u) = h(\Omega,\alpha,\beta;p^*,f)$.

**4. Numerical Experiments.** The suggested algorithm was used for calculating polynomials and the rational functions of the best restricted Hausdorff approximation of the following two functions:

$$f_1(x) = \begin{cases} 1\,, & 0 \le x < 0.25 \\ [0,1]\,, & x = 0.25 \\ 0\,, & 0.25 < x \le 0.50 \\ 2x-1\,, & 0.50 < x \le 0.75 \\ 0.5\,, & 0.75 < x \le 1.00 \end{cases} ; \qquad f_2(x) = \begin{cases} +1\,, & 0 \le x < 0.5 \\ [0,1]\,, & x = 0.5 \\ 0\,, & 0.5 < x \le 1.0 \end{cases}$$

The restrictions $l$ and $u$ are given by the lines connecting the points:

| $l_1$ | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $x$ | 0.00 | 0.20 | 0.20 | 0.55 | 0.80 | 1.00 |
| $y$ | 0.98 | 0.90 | $0-\varepsilon$ | $0-\varepsilon$ | 0.40 | 0.48 |

| $l_2$ | | | | |
|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 |
| $x$ | 0.00 | 0.40 | 0.40 | 1.00 |
| $y$ | 0.98 | 0.91 | $0-\varepsilon$ | $0-\varepsilon$ |

| $u_1$ | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $x$ | 0.00 | 0.30 | 0.30 | 0.45 | 0.70 | 1.00 |
| $y$ | $1+\varepsilon$ | $1+\varepsilon$ | 0.05 | 0.05 | 0.60 | 0.53 |

| $u_2$ | | | | |
|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 |
| $x$ | 0.00 | 0.60 | 0.60 | 1.00 |
| $y$ | $1+\varepsilon$ | $1+\varepsilon$ | 0.10 | 0.03 |

where $\varepsilon = 10^{-7}$. Parameters of $h-$distance are $\alpha(x) = 1,\ \beta(x) = 1,\ \Omega = [0,1]$. A modification of the above algorithm works in the case of rational functions $R_{m,k} = \{f : f = p/q,\ p \in H_m,\ q \in H_k,\ q(x) > 0, x \in \Omega\}$ (see [4]). The next table shows the order of the $h-$approximation (denoted by $E_h$ ) of the functions $f_1$ and $f_2$ for different $m$ and $k$. The non-restricted approximation $E_h^o$ is given for comparison.

| for $f_1,\ l_1,\ u_1$ | | | for $f_2,\ l_2,\ u_2$ | | |
|---|---|---|---|---|---|
| $m,k$ | $E_h$ | $E_h^o$ | $m,k$ | $E_h$ | $E_h^o$ |
| 17,0 | NONE | 0.0445 | 9,0 | NONE | 0.0735 |
| 18,0 | 0.0549 | 0.0436 | 10,0 | 0.0980 | 0.0735 |
| 19,0 | 0.0485 | 0.0410 | 11,0 | 0.0845 | 0.0605 |
| 20,0 | 0.0485 | 0.0400 | 12,0 | 0.0835 | 0.0650 |
| 21,0 | 0.0465 | 0.0390 | 13,0 | 0.0740 | 0.0585 |
| 22,0 | 0.0437 | 0.0370 | 14,0 | 0.0735 | 0.0585 |
| 23,0 | 0.0435 | 0.0363 | 15,0 | 0.0665 | 0.0530 |
| 25,0 | 0.0400 | 0.0340 | 16,0 | 0.0660 | 0.0530 |
| 29,0 | 0.0367 | 0.0308 | 17,0 | 0.0605 | 0.0490 |
| 30,0 | 0.0360 | 0.0305 | 49,0 | 0.0275 | 0.0231 |
| 49,0 | 0.0246 | 0.0208 | 50,0 | 0.0275 | 0.0231 |
| 50,0 | 0.0245 | 0.0208 | 3,3 | 0.0585 | 0.0425 |
| 5,4 | 0.0292 | 0.0227 | 4,4 | 0.0333 | 0.0235 |

Fig.1 gives the functions $f_1,\ l_1,\ u_1$ and its elements of the best restricted Hausdorff approximation $q \in R_{18,0}$ and $p \in R_{5,4}$. If we denote $q(x) = \sum_{i=0}^{18} c_i\, x^i$ and $p(x) = \left\{\sum_{i=0}^{5} a_i\, x^i\right\} / \left\{\sum_{i=0}^{4} b_i\, x^i\right\}$, then:

| i | $a_i$ | i | $b_i$ |
|---|---|---|---|
| 0 | 1.0000001000000 | 0 | 1.0000000000000 |
| 1 | -11.880330908410 | 1 | -11.543377875903 |
| 2 | 53.256837933185 | 2 | 49.182439602652 |
| 3 | -109.14616431720 | 3 | -92.720877302972 |
| 4 | 94.784810736638 | 4 | 68.305617786789 |
| 5 | -20.487944559694 | | |

| i | $c_i$ | i | $c_i$ |
|---|---|---|---|
| 0 | .98000000000001 | 10 | 15736242931.872 |
| 1 | 6.8044288938890 | 11 | -40145396289.375 |
| 2 | -738.02146726530 | 12 | 72226815298.382 |
| 3 | 28275.728517794 | 13 | -93484846045.936 |
| 4 | -520265.05951906 | 14 | 86833169289.904 |
| 5 | 5098036.7323643 | 15 | -56585921954.300 |
| 6 | -25246425.638918 | 16 | 24591273522.966 |
| 7 | 18915802.713044 | 17 | -6404956380.0384 |
| 8 | 565670393.04459 | 18 | 756625300.02572 |
| 9 | -4086950760.2534 | | |

Fig.2 gives the functions $f_2,\ l_2,\ u_2$ and its elements of the best restricted Hausdorff approximation $q \in R_{14,0}$ and $p \in R_{3,3}$. If we denote $q(x) = \sum_{i=0}^{14} c_i\, x^i$ and $p(x) = \left\{\sum_{i=0}^{3} a_i\, x^i\right\} / \left\{\sum_{i=0}^{3} b_i\, x^i\right\}$, then:

| i | $a_i$ | i | $b_i$ |
|---|---|---|---|
| 0 | 1.0000001000000 | 0 | 1.0000000000000 |
| 1 | -4.3124280439352 | 1 | -3.9080972037703 |
| 2 | 6.0554721464579 | 2 | 4.0044972838375 |
| 3 | -2.7430442979484 | 3 | -.14214262116408 |

| i | $c_i$ | i | $c_i$ |
|---|---|---|---|
| 0 | 1.0000001000000 | 8 | 20307978.516452 |
| 1 | -4.5028466279729 | 9 | -26845924.827509 |
| 2 | 302.27917674644 | 10 | 22648284.222115 |
| 3 | -7747.1574390266 | 11 | -10907155.663855 |
| 4 | 99333.842307524 | 12 | 1843156.8388665 |
| 5 | -734243.53278209 | 13 | 634691.40575051 |
| 6 | 3385065.7916338 | 14 | -253030.70197841 |
| 7 | -10170707.509891 | | |

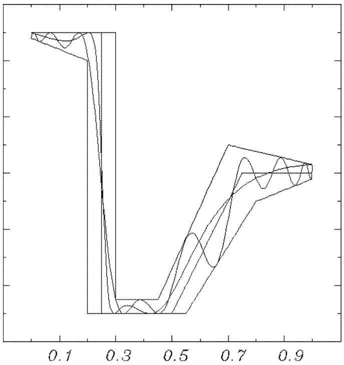


Fig.1

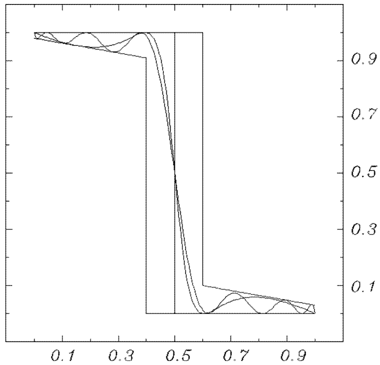


Fig.2

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