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# EQUIVARIANT UNFOLDINGS IN THE CASE OF SYMMETRY OF ORDER 4

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Abstract. In this paper we prove that at most four limit cycles not surrounding the origin in equivariant unfoldings occur in case of symmetry of order 4 and if they appear, they are hyperbolic. The result is obtained by transforming system (1) to a Liénard equation and applying a uniqueness theorem by Zhang Zhifen. It also follows that cubic systems with rotational invariance over $2\pi/4$ rad have at most four limit cycles not surrounding the origin.

**1. Introduction.** In [1], [6] vector fields which occur in unfoldings of diffeomorphisms with a non-hyperbolic fixed point after application of center manifold methods were discussed. These vector fields have a rotational invariance of $2\pi/q$ rad ($q \in \mathbb{N}, q \geq 2$). All cases for $q < 4$ were solved in [6]. The cases $q > 4$ were discussed in [2]. The case $q = 4$ remains unsolved except for some partial results obtained in [3] and [7]. The main problem is to determine the number of limit cycles occurring in the phase portrait of the vector field. This has been done by using Pontryagin integral techniques for the case $q < 4$. However, if $q = 4$, the nonexistence of a small parameter makes this method unapplicable.

In this paper we prove that the maximum number of limit cycles not surrounding the origin (next we will refer to them as "outside the origin") for $q = 4$ is four and if they exist they are hyperbolic. It follows from [3] that the remaining problem is to investigate the number of limit cycles surrounding 9 critical points in (1). In [7] it was shown that in this situation at least two limit cycles can occur. With the results of this paper it follows that this is the only limit cycle problem left for $q = 4$.

**2. Transformation to a Liénard equation.** We start with the vector field for $q = 4$ in Cartesian coordinates:

$$\begin{aligned}\frac{dx}{dt} &= \lambda_1 x - y + (\lambda_2 x - \lambda_3 y)(x^2 + y^2) + 3xy^2 - x^3 \equiv P,\\ \frac{dy}{dt} &= x + \lambda_1 y + (\lambda_3 x + \lambda_2 y)(x^2 + y^2) + 3x^2 y - y^3 \equiv Q.\end{aligned} \tag{1}$$

where $(x,y) \in \mathbb{R}^2$, $(\lambda_1,\ \lambda_2,\ \lambda_3) \in \mathbb{R}^3$, $t \in \mathbb{R}$.

First we give some necessary conditions for the existence of limit cycles outside the origin in (1). The divergence of the vector field (1) is

$$\operatorname{div}(P,Q) = 2\lambda_1 + 4\lambda_2(x^2+y^2). \tag{2}$$

Therefore a necessary condition for the existence of limit cycles is $\lambda_1\lambda_2 < 0$. Without loss of generality we can assume that:

$$\lambda_1 < 0, \quad \lambda_2 > 0. \tag{3}$$

If $\lambda_2 \geq 1$, then the vector field on the circle $\operatorname{div}(P,Q) = 0$ is directed inwards, i.e. $\operatorname{div}(P,Q) = 0$ is a closed curve without contact. Since limit cycles outside the origin have to intersect this curve, it follows that

$$0 < \lambda_2 < 1 \tag{4}$$

is a necessary condition for the existence of limit cycles outside origin. Finally it was proved in [3] that for $\lambda_3 \leq 1$ no limit cycles occur outside the origin. Therefore we can assume that:

$$\lambda_3 > 1. \tag{5}$$

We consider the polar coordinate form of (1):

$$\begin{aligned} \frac{dR}{dt} &= R[\lambda_1 + R^2(\lambda_2 - \cos(4\theta))], \\ \frac{d\theta}{dt} &= -1 + R^2(\lambda_3 + \sin(4\theta)), \end{aligned} \tag{6}$$

where $R \geq 0$, $\theta \in [0, 2\pi)$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda_1 < 0,\ 0 < \lambda_2 < 1,\ \lambda_3 > 1$.

Limit cycles surrounding the origin in (1) are destroyed in the sense that they appear as $2\pi$-periodic functions in $\theta$ in (6) and not as closed curves in a $(R,\theta)$ plane. However, the limit cycles we are dealing with (outside the origin) are still limit cycles in (6). Due to the rotational invariance over $2\pi/4$ rad in (1), the phase plane of (6) is periodic with period $2\pi/4$ rad in $\theta$. Therefore limit cycles in (6) will appear in multiples of four. We restrict ourselves to the interval $[0, 2\pi/4)$ in (6) by transforming $\varphi = 4\theta$:

$$\begin{aligned} \frac{dR}{dt} &= R[\lambda_1 + R^2(\lambda_2 - \cos\varphi)], \\ \frac{d\varphi}{dt} &= 4[-1 + R^2(\lambda_3 + \sin\varphi)], \end{aligned} \tag{7}$$

where $R \geq 0$, $\varphi \in [0, 2\pi)$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda_1 < 0, 0 < \lambda_2 < 1, \lambda_3 > 1$.

For system (7) we will prove the uniqueness of the limit cycle in the region $R \geq 0$. This implies immediately that at most four limit cycles appear outside the origin in (1). The method of proving this result is to transform (7) into a generalized Liénard equation and to apply a uniqueness theorem by Zhang Zhifen. After putting $R^2 = 1/v$, $dt/d\tau = v/2$, so as to simplify the system, we get:

$$\frac{dv}{d\tau} = -v(\lambda_1 v + \lambda_2 - \cos\varphi),$$

(8)

$$\frac{d\varphi}{d\tau} = 2(-v + \lambda_3 + \sin\varphi).$$

In a $(v, \varphi)$ phase plane the situation for the isoclines $dv/d\tau = 0$, $d\varphi/d\tau = 0$ is as indicated in Figure 1.

In case 1a of Figure 1 two singularities occur for $\varphi = \varphi_1$, $\varphi = \varphi_2$ ($\varphi_1 < \varphi_2$). It is easy to check that the singularity for $\varphi = \varphi_1$ ($\varphi = \varphi_2$) is a saddle (antisaddle). The limit cycle (s) should surround the singularity at $\varphi = \varphi_2$ and therefore next we confine ourselves to the interval $[\varphi_1, \varphi_1 + 2\pi)$ for $\varphi$. In case 1b no singularities and no limit cycles occur. The conditions on $\lambda_1$, $\lambda_2$, $\lambda_3$ to distinguish between the cases 1a and 1b are not given, because we will not use them. It is just assumed in the following that we are dealing with case 1a of Figure 1.

Putting $2(-v + \lambda_3 + \sin\varphi) = Y$, system (8) becomes:

$$\frac{dY}{d\tau} = h_1(\varphi) + h_2(\varphi)Y + \frac{\lambda_1}{2}Y^2,$$

(9)

$$\frac{d\varphi}{d\tau} = Y,$$

where $h_1(\varphi) = 2(\lambda_3 + \sin\varphi)(\lambda_1\lambda_3 + \lambda_2 + \lambda_1 \sin\varphi - \cos\varphi)$,
$h_2(\varphi) = -2\lambda_1\lambda 3 - \lambda_2 - 2\lambda_1 \sin\varphi + 3\cos\varphi$, which is of Liénard-type, except for the $\lambda_1 Y^2/2$-term in $dY/d\tau$. This term can be eliminated by applying the additional transformation:

(10) $$Y = ye^{\frac{\lambda_1}{2}\varphi}, \quad \frac{d\tau}{dt} = e^{-\frac{\lambda_1}{2}\varphi},$$

resulting in:

$$\frac{dy}{dt} = h_1(\varphi)e^{-\lambda_1\varphi} + h_2(\varphi)e^{-\frac{\lambda_1}{2}\varphi}y,$$

(11)

$$\frac{d\varphi}{dt} = y,$$

where $\varphi_1 \leq \varphi < \varphi_1 + 2\pi$.

Of course we could proceed from (11) by trying to apply some uniqueness theorem for Liénard equations. However, since we know explicitly a solution of (11) we can simplify the system further by transforming it into a generalized Liénard system using the method described in [9]. The known solution is the invariant line $v = 0$ in (8), which is transformed into:

$$y = 2e^{-\frac{\lambda_1}{2}\varphi}(\lambda_3 + \sin\varphi) = \psi(\varphi). \tag{12}$$

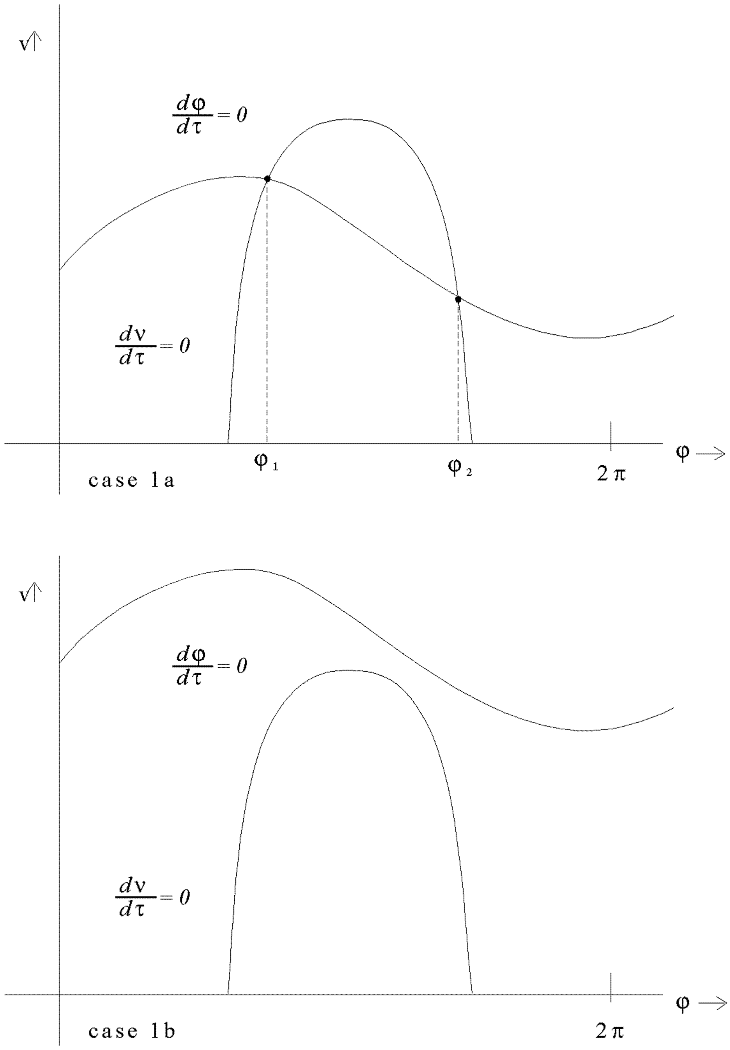


Figure 1. Isoclines $\frac{dv}{d\tau} = 0$, $\frac{d\varphi}{d\tau} = 0$ of system (8).

Since $y = \psi(\varphi)$ is a solution of (11), according to [9] we can put:

$$e^{y_1} = \frac{\psi(\varphi) - y}{\psi(0)}, \quad \frac{dt}{d\tau} = \frac{-1}{\psi(0)}, \tag{13}$$

which transforms (11) into (writing $x$ for $\varphi$, $y$ for $y_1$, $t$ for $\tau$):

$$(14)\qquad \begin{aligned} \frac{dx}{d\tau} &= k(y) - \int_0^x f(\tau)d\tau \equiv P_1, \\ \frac{dy}{d\tau} &= -g(x) \equiv Q_1, \end{aligned}$$

where $g(x) = (-\lambda_1\lambda_3 - \lambda_2 - \lambda_1 \sin x + \cos x)\dfrac{e^{-\frac{\lambda_1}{2}x}}{2\lambda_2}$,

$k(y) = e^y - 1$, $f(x) = (-\lambda_1\lambda_3 - \lambda_1 \sin x + 2\cos x)\dfrac{e^{-\frac{\lambda_1}{2}x}}{2\lambda_2}$, with $x \in [\varphi_1, \varphi_1 + 2\pi)$.

Notice that the zeros of $g(x)$ correspond to the singularities of (14). So we assume that $g(x) = 0$ for $x = \varphi_1$ (saddle), $x = \varphi_2$ (antisaddle).

**3. Application of Zhang Zhifen's theorem.** To prove the uniqueness of the limit cycle in (14) we use a theorem due to Zhang Zhifen [10], [11], [4], [5]:

**Lemma 1.** *Consider the generalized Liénard system* (15)*:*

$$(15)\qquad \begin{aligned} \frac{dx}{dt} &= k(y) - \int_0^x f(\tau)d\tau, \\ \frac{dy}{dt} &= -g(x). \end{aligned}$$

*Let $f(x), g(x)$ be continuously differentiable functions on the open interval $(r_1, r_2)$ where $r_1 < 0 < r_2$, and let $k(y)$ be a continuously differentiable function on $\mathbb{R}$, such that*

(i) $\dfrac{dk}{dy} > 0,$

(ii) $xg(x) > 0, \quad$ *for* $x \neq 0,$

(iii) $f(0)\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) < 0, \quad$ *for* $x \neq 0.$

*Then system* (14) *has at most one limit cycle and if it exists it is hyperbolic.*

In Lemma 1 the antisaddle inside the limit cycle is situated at $x = 0$, whereas in our case it is at $x = \varphi_2$. After a translation $x_1 = x - \varphi_2$, we arrive at the situation of Lemma 1 with $r_1 = \varphi_1 - \varphi_2 < 0$, $r_2 = \varphi_1 - \varphi_2 + 2\pi > 0$. Condition (i) is satisfied for system (14) because $dk/dy = e^y > 0$. Condition (ii) is also satisfied, because $g(x)$ becomes only zero at $x = \varphi_2$ on the interval $(\varphi_1, \varphi_1 + 2\pi)$ in system (14):

$(x-\varphi_2)g(x-\varphi_2) > 0$. The only relatively difficult condition to check is condition (iii). First we prove that $\frac{d}{dx}(\frac{f(x)}{g(x)})$ is of fixed sign. In our case that means to prove that:

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{d}{dx}\Big(\frac{-\lambda_1\lambda_3 - \lambda_1 \sin x + 2\cos x}{-\lambda_1\lambda_3 - \lambda_2 - \lambda_1 \sin x + \cos x}\Big) \tag{16}$$

is of fixed sign for $x \in [\varphi_1, \varphi_1 + 2\pi)$. In order to do this, we use a geometrical argument without explicitly calculating the derivative (16). Of course an analytical approach will also work, but the details are rather messy.

A necessary condition for limit cycles to exist is that $f(x)$ changes sign on $[\varphi_1, \varphi_1 + 2\pi)$, because $-f(x)$ is the divergence of (14) and according to Bendixson's criterion a fixed sign of $f(x)$ implies the nonexistence of limit cycles. So $f(x)$ will have on $[\varphi_1, \varphi_1 + 2\pi)$ two zeros $\varphi^*, \varphi^{**}$ (due to the structure of $f(x) \sim -\lambda_1\lambda_3 - \lambda_1 \sin x + 2\cos x$). We distinguish three situations for these zeros with respect to $\varphi_2$ (Figure 2).

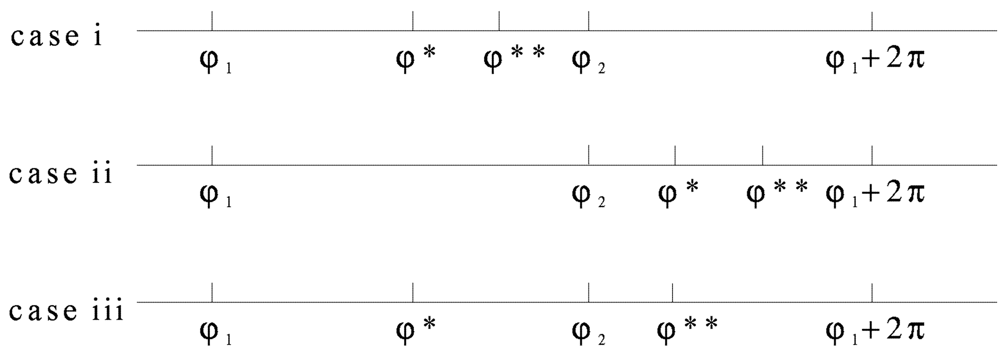


Figure 2. Three cases of relative position of the zeros $\varphi^*, \varphi^{**}$ of $f(x)$ with respect to the zeros $\varphi_1, \varphi_2$ of $g(x)$.

Consider the zeros of $f(x) - cg(x), c \in \mathbb{R}$. They are determined by:

$$-\lambda_1\lambda_3 + c\lambda_1\lambda_3 + c\lambda_2 + \lambda_1(-1+c)\sin x + (2-c)\cos x = 0. \tag{17}$$

It is clear from (17) that on the interval $[\varphi_1, \varphi_1 + 2\pi)$, $f(x) - cg(x)$ has at most two zeros. It implies that $f(x)/g(x)$ intersects every horizontal line $y = c$ at most at two points. With this property we can draw (Figure 3) the graph of $y = f(x)/g(x)$ for the three cases of Figure 2.

In the cases i, ii there exists a horizontal line $y = c^*$, having no intersections with $y = f(x)/g(x)$ (otherwise it would violate the maximum of two intersections mentioned above). We apply Dulac's criterion [6] with $B(x) = \exp(-c^*y)$ to system (14):

$$\mathrm{div}(BP_1, BQ_1) = e^{-c^*y}(-f(x) + c^*g(x)), \tag{18}$$

which is of fixed sign for $x \in [\varphi_1, \varphi_1 + 2\pi)$. It follows that in the cases i, ii no limit cycles occur.

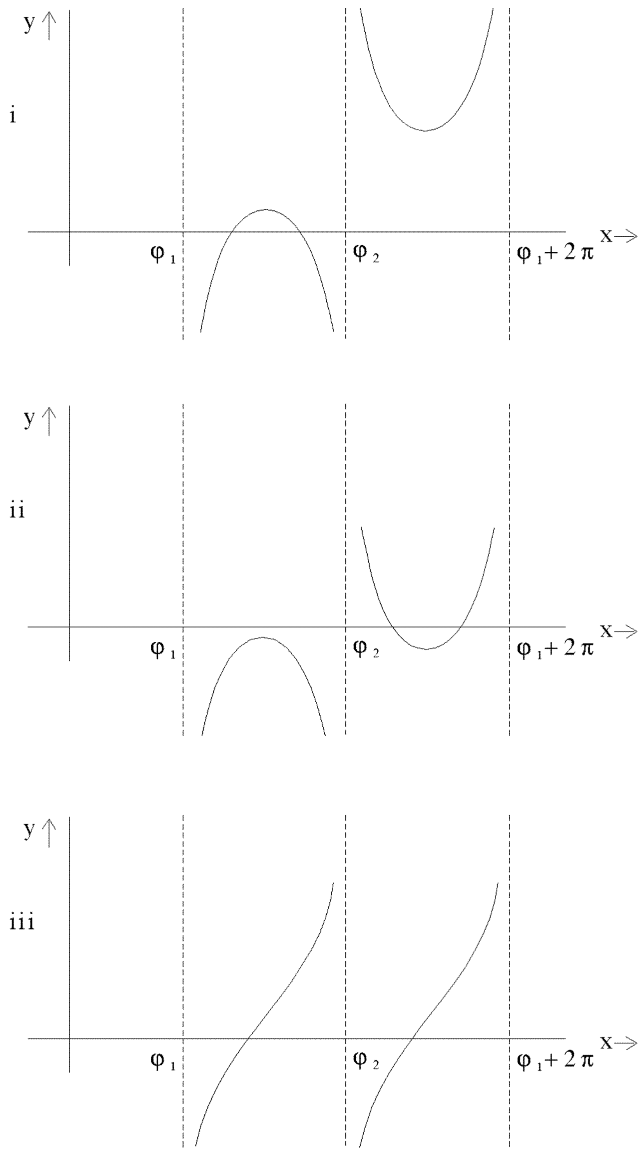


Figure 3. Graphs of $y = \frac{f(x)}{g(x)}$ for the three cases of Figure 2.

In case iii the graph of $y = f(x)/g(x)$ is monotonic. Again we can exclude relative extrema, because it would contradict that $f(x) - cg(x)$ has at most two zeros. It is easy to check by using Figure 3 that condition (iii) in Lemma 1 is satisfied, because

$f(x = \varphi_2)$ (which is $f(0)$ in Lemma 1 after the translation $x_1 = x - \varphi_2$) is negative. Therefore in case iii all the conditions of Lemma 1 are satisfied. At most one limit cycle exists in system (14) and if it exists it is hyperbolic. Summarizing our results we have:

**Theorem 1.** *System 1 has at most four limit cycles not surrounding the origin and if they appear, they are hyperbolic.*

**Corollary.** System 1 also happens to be the general cubic system with $2\pi/4$ rad rotational invariance. Therefore the statement of the theorem also holds with "System 1"replaced by "A cubic system with rotational invariance over $2\pi/4$ rad".

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