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FURTHER RESULTS ON THE NON-COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

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Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

ABSTRACT. The non-commutative neutrix product of the distributions $x_+^s \ln x_+$ and $\delta^{(r)}(x)$ is evaluated for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Further neutrix products are then deduced.

In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \qquad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2] or [4].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [5] and generalizes Definition 1.

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$\mathrm{N}-\lim_{n\to\infty} \langle f(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

Note that if

$$\lim_{n\to\infty} \langle f(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle,$$

we simply say that the *product* $f.g$ exists and equals h, see [4].

It is obvious that if the product $f.g$ exists then the neutrix product $f \circ g$ exists and $f.g = f \circ g$. Further, it was proved in [4] that if the product fg exists by Definition 1 then the product $f.g$ exists by Definition 2 and $fg = f.g$. Note also that although the product defined in Definition 1 is always commutative, the product and neutrix product defined in Definition 2 is in general non-commutative.

The following two theorems hold, see [5].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the interval (a, b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists on the interval (a, b) and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a, b).

Theorem 2. *The neutrix products $x_+^s \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ x_+^s$ exist and*

$$x_+^s \circ \delta^{(r)}(x) = \delta^{(r)}(x) \circ x_+^s = \frac{(-1)^s r!}{2(r-s)!}\delta^{(r-s)}(x)$$

for $s = 0, 1, 2, \ldots$ and $r = s, s+1, \ldots$.

The following extension of Theorem 1 was proved in [7].

Theorem 3. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g^{(i)}$ (or $f^{(i)} \circ g$) exist on the interval (a, b) for $i = 0, 1, 2, \ldots, r$. Then the neutrix products $f^{(k)} \circ g$ (or $f \circ g^{(k)}$) exist on the interval (a, b) for $k = 1, 2, \ldots, r$ and*

$$f^{(k)} \circ g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \circ g^{(i)}]^{(k-i)} \tag{1}$$

or

$$f \circ g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f^{(i)} \circ g]^{(k-i)} \tag{2}$$

on the interval (a, b) for $k = 1, 2, \ldots, r$.

The next two theorems were proved in [5].

Theorem 4. *The neutrix products $\ln x_+ \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ \ln x_+$ exist and*

$$\ln x_+ \circ \delta^{(r)}(x) = [c(\rho) + \tfrac{1}{2}\psi(r)]\delta^{(r)}(x), \tag{3}$$

$$\delta^{(r)}(x) \circ \ln x_- = c(\rho)\delta^{(r)}(x) \tag{4}$$

for $r = 0, 1, 2, \ldots$, where

$$c(\rho) = \int_0^1 \ln t \rho(t)\, dt$$

and

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^{r} i^{-1}, & r \geq 1. \end{cases}$$

Theorem 5. *The neutrix products $(x_+^r \ln x_+) \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ (x_+^r \ln x_+)$ exist and*

$$\begin{aligned} x_+^r \ln x_+ \circ \delta^{(r)}(x) &= (-1)^r r! [c(\rho) + \tfrac{1}{2}\psi(r)]\delta(x), \\ &= \delta^{(r)}(x) \circ (x_+^r \ln x_+) \end{aligned}$$

for $r = 1, 2, \ldots$.

It was shown in [6] that by suitable choice of the function ρ, $c(\rho)$ can take any negative value.

In the next theorem, which was proved in [4], the distributions x_+^{-r} and x_-^{-r} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)(r),$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov [8].

Theorem 6. *The neutrix products $x_+^{-s} \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ x_+^{-s}$ exist and*

$$x_+^{-s} \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{2(r+s)!}\delta^{(r+s)}(x), \tag{5}$$

$$\delta^{(r)}(x) \circ x_+^{-s} = 0 \tag{6}$$

for $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$.

The following theorem was proved in [7] using Theorem 3.

Theorem 7. *The neutrix products $x_-^{-r} \circ x_+^s$ and $x_+^s \circ x_-^{-r}$ exist and*

$$\begin{aligned} x_-^{-r} \circ x_+^s &= x_-^{-r} x_+^s = 0, \\ x_+^s \circ x_-^{-r} &= x_+^s x_-^{-r} = 0 \end{aligned}$$

for $r = 1, 2, \ldots$ and $s = r, r+1, \ldots$ and

$$\begin{aligned} x_-^{-r} \circ x_+^s &= \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!}[c(\rho) + \tfrac{1}{2}\psi(i-s-1)]\delta^{(r-s-1)}(x), \\ x_+^s \circ x_-^{-r} &= \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!} c(\rho)\delta^{(r-s-1)}(x), \end{aligned}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

We now prove the following generalization of Theorems 4 and 5 also using Theorem 3.

Theorem 8. *The neutrix products $(x_+^s \ln x_+) \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ (x_+^s \ln x_+)$ exist and*

$$(x_+^s \ln x_+) \circ \delta^{(r)}(x) = (x_+^s \ln x_+).\delta^{(r)}(x) = 0, \tag{7}$$

$$\delta^{(r)}(x) \circ (x_+^s \ln x_+) = \delta^{(r)}(x).(x_+^s \ln x_+) = 0, \tag{8}$$

for $s = 1, 2, \ldots$ and $r = 0, 1, \ldots, s-1$ and

$$(x_+^s \ln x_+) \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{(r-s)!}[c(\rho) + \tfrac{1}{2}\psi(s)]\delta^{(r-s)}(x) +$$

$$(9) \qquad - \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^i}{2(i-s)} \delta^{(r-s)}(x),$$

$$(10) \qquad \delta^{(r)}(x) \circ (x_+^s \ln x_+) = \frac{(-1)^s r!}{(r-s)!}[c(\rho) + \tfrac{1}{2}\psi(s)]\delta^{(r-s)}(x),$$

for $s = 0, 1, 2, \dots$ *and* $r = s, s+1, \dots$.

Proof. We define the function $f(x_+, s)$ by

$$f(x_+, s) = \frac{x_+^s \ln x_+ - \psi(s) x_+^s}{s!}$$

and it follows easily by induction that

$$f^{(i)}(x_+, s) = f(x_+, s - i),$$

for $i = 0, 1, \dots, s$. In particular,

$$f^{(s)}(x_+, s) = \ln x_+,$$

so that

$$f^{(i)}(x_+, s) = (-1)^{i-s-1}(i - s - 1)! x_+^{-i+s},$$

for $i = s + 1, s + 2, \dots$. Now $f^{(i)}(x_+, s)$ is a continuous function which is zero at the origin for $i = 0, 1, \dots, s - 1$ and so

$$(11) \qquad f^{(i)}(x_+, s).\delta(x) = 0,$$

for $s = 0, 1, \dots, s - 1$. Using equation (3) we have

$$(12) \qquad f^{(s)}(x_+, s) \circ \delta(x) = c(\rho)\delta(x)$$

and using equation (5) we have

$$(13) \qquad f^{(i)}(x_+, s) \circ \delta(x) = -\frac{1}{2(i-s)} \delta^{(i-s)}(x)$$

for $i = s + 1, s + 2, \dots$.

Using Theorem 3 and equation (11) we now have

$$\begin{aligned} f(x_+, s).\delta^{(r)}(x) &= \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f^{(i)}(x_+, s).\delta(x)]^{(r-i)} \\ &= 0, \end{aligned}$$

for $s = 1, 2, \dots$ and $r = 0, 1, \dots, s - 1$. Equations (7) follows on noting that

$$x_+^i.\delta(x) = 0,$$

for $i = 1, 2, \ldots$.

When $r \geq s$ we have

$$\begin{aligned} f(x_+, s) \circ \delta^{(r)}(x) &= \sum_{i=s}^{r} \binom{r}{i} (-1)^i [f^{(i)}(x_+, s) \circ \delta(x)]^{(r-i)} \\ &= \binom{r}{s} (-1)^s c(\rho) \delta^{(r-s)}(x) - \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^i}{2(i-s)} \delta^{(r-s)}(x) \end{aligned}$$

on using Theorem 3 and equations (11), (12) and (13). It now follows that

$$(x_+^s \ln x_+) \circ \delta^{(r)}(x) = s! f(x_+, s) \circ \delta^{(r)}(x) + \psi(s) x_+^s \circ \delta^{(r)}(x)$$

and equation (9) follows on using Theorem 2.

We now consider the product $\delta^{(r)}(x) \circ (x_+^s \ln x_+)$. As above, we have

$$\delta(x).f^{(i)}(x_+, s) = 0, \tag{14}$$

for $s = 0, 1, \ldots, s-1$. Using equation (4) we have

$$\delta(x) \circ f^{(s)}(x_+, s) = c(\rho)\delta(x) \tag{15}$$

and using equation (6) we have

$$\delta(x) \circ f^{(i)}(x_+, s) = 0, \tag{16}$$

for $i = s+1, s+2, \ldots$.

Using equations (1) an (14) we now have

$$\begin{aligned} \delta^{(r)}(x).f(x_+, s) &= \sum_{i=0}^{r} \binom{r}{i} (-1)^i [\delta(x).f^{(i)}(x_+, s)]^{(r-i)} \\ &= 0, \end{aligned}$$

for $s = 1, 2, \ldots$ and $r = 0, 1, \ldots, s-1$. Equations (8) follow on noting that

$$\delta(x).x_+^i = 0,$$

for $i = 1, 2, \ldots$.

When $r \geq s$ we have

$$\begin{aligned} \delta^{(r)} \circ f(x_+, s) &= \sum_{i=s}^{r} \binom{r}{i} (-1)^i [\delta(x) \circ f^{(i)}(x_+, s)]^{(r-i)} \\ &= \binom{r}{s} (-1)^s c(\rho) \delta^{(r-s)}(x) \end{aligned}$$

on using equations (1), (14), (15) and (16). It now follows that

$$\delta^{(r)}(x)\circ(x_+^s\ln x_+) = s!\delta^{(r)}(x)\circ f(x_+,s)+\psi(s)\delta^{(r)}(x)\circ x_+^s$$

and equation (9) follows on using Theorem 2. This completes the proof of the theorem. □

Note that by putting $s=0$ in equation (9) and comparing with equation (3) we prove that

$$\psi(r)=\sum_{i=1}^{r}\binom{r}{i}\frac{(-1)^i}{i},$$

for $r=1,2,\dots$.

Corollary 1. *The neutrix products* $(x_-^s\ln x_-)\circ\delta^{(r)}(x)$ *and* $\delta^{(r)}(x)\circ(x_-^s\ln x_-)$ *exist and*

$$\begin{aligned}(x_-^s\ln x_-)\circ\delta^{(r)}(x) &= (x_-^s\ln x_-).\delta^{(r)}(x)=0,\\ \delta^{(r)}(x)\circ(x_-^s\ln x_-) &= \delta^{(r)}(x).(x_-^s\ln x_-)=0,\end{aligned}$$

for $s=1,2,\dots$ *and* $r=0,1,\dots,s-1$ *and*

$$\begin{aligned}(x_-^s\ln x_-)\circ\delta^{(r)}(x) &= \frac{r!}{(r-s)!}[c(\rho)+\tfrac{1}{2}\psi(s)]\delta^{(r-s)}(x)+\\ &\quad -\sum_{i=s+1}^{r}\binom{r}{i}\frac{(-1)^{i-s}}{2(i-s)}\delta^{(r-s)}(x),\\ \delta^{(r)}(x)\circ(x_-^s\ln x_-) &= \frac{r!}{(r-s)!}[c(\rho)+\tfrac{1}{2}\psi(s)]\delta^{(r-s)}(x),\end{aligned}$$

for $s=0,1,2,\dots,$ *and* $r=s,s+1,\dots$.

Proof. The results follow immediately on replacing x by $-x$ in equations (7), (8), (9) and (10). □

Corollary 2. *The neutrix products* $(x^s\ln|x|)\circ\delta^{(r)}(x)$ *and* $\delta^{(r)}(x)\circ(x^s\ln|x|)$ *exist and*

$$\begin{aligned}(x^s\ln|x|)\circ\delta^{(r)}(x) &= (x^s\ln|x|).\delta^{(r)}(x)=0,\\ \delta^{(r)}(x)\circ(x^s\ln|x|) &= \delta^{(r)}(x).(x^s\ln|x|)=0,\end{aligned}$$

for $s=1,2,\dots$ *and* $r=0,1,\dots,s-1$ *and*

$$\begin{aligned}(x^s\ln|x|)\circ\delta^{(r)}(x) &= \frac{(-1)^s r!}{(r-s)!}[2c(\rho)+\psi(s)]\delta^{(r-s)}(x)+\\ &\quad -\sum_{s+1}^{r}\binom{r}{i}\frac{(-1)^i}{i-s}\delta^{(r-s)}(x),\\ \delta^{(r)}(x)\circ(x^s\ln|x|) &= \frac{(-1)^s r!}{(r-s)!}[2c(\rho)+\psi(s)]\delta^{(r-s)}(x),\end{aligned}$$

for $s = 0, 1, 2, \ldots$ *and* $r = s, s+1, \ldots$.

Proof. The results follow on noting that

$$x^s \ln|x| = x_+^s \ln x_+ + (-1)^s x_-^s \ln x_-$$

and that the neutrix convolution product is distributative with respect to addition. □

REFERENCES

[1] J.G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959-60), 291-398.

[2] B. Fisher, The product of distributions, *Quart. J. Math. Oxford (2)* **22** (1971), 291-298.

[3] B. Fisher, On defining the product of distributions, *Math. Nachr.* **99** (1980), 239-249.

[4] B. Fisher, A non-commutative neutrix product of distributions, *Math. Nachr.* **108** (1982), 117-127.

[5] B. Fisher, Some results on the non-commutative neutrix product of distributions, *Trabajos de Matematica* **44**, Buenos Aires, 1983.

[6] B. Fisher, The non-commutative neutrix product of the distributions x_+^{-r} and $\delta^{(p)}(x)$, *Indian J. Pure Appl. Math.* **14** (1983), 1439-1449.

[7] B. Fisher, E. Savaş, S. Pehlivan and E. Özçağ, Results on the non-commutative neutrix product of distributions, *Math. Balkanica* (to appear).

[8] I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. I, Academic Press, 1964.

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