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UNIVERSAL FUNCTIONS AND CONFORMAL MAPPINGS

WOLFGANG LUH

Dedicated to Academician Ljubomir Iliev
on the Occasion of his Eightieth Birthday

ABSTRACT. Let $\mathcal{O} \subset \mathbf{C}$ be an open set with simply connected components. We prove the existence of a sequence (f_n) of "conformal mappings" of $\mathcal{O}$ onto $\mathcal{O}$ and a universal holomorphic function Φ on $\mathcal{O}$ with the following property: For all compact sets $B \subset \mathcal{O}$ with connected complement and all functions f continuous on B and holomorphic in the interior of B there exists a sequence (n_k) of natural numbers such that $(\Phi \circ f_{n_k})$ converges to f uniformly on B.

1. Introduction and statement of the result. Let $\mathcal{O}$ be an open set in the complex plane with simply connected components. Throughout the sequel we suppose, that $\mathcal{O}$ is represented in the form $\mathcal{O} = \bigcup_{\nu \in I} G_\nu$, where $I = \{1, 2, \ldots\}$ is a finite set or $I = \mathbf{N}$ and the G_ν are pairwise disjoint simply connected domains. By $H(\mathcal{O})$ we denote the set of all functions which are holomorphic on $\mathcal{O}$. A function $f \in H(\mathcal{O})$ is called a "conformal mapping" of $\mathcal{O}$ onto itself if for each $\nu \in I$ the restriction $f|G_\nu$ of f to G_ν is a conformal mapping of G_ν onto G_ν.

For a compact set $B \subset \mathbf{C}$ we abbreviate by $A(B)$ the space of all functions which are continuous on B and holomorphic in the interior of B. Introducing the uniform norm $A(B)$ becomes a Banach space.

In this note we deal with the existence of a universal sequence (f_n) of conformal mappings of $\mathcal{O}$ onto $\mathcal{O}$ and a universal function $\Phi \in H(\mathcal{O})$ such that the sequence $(\Phi \circ f_n)$ is dense in certain spaces of functions. We shall give a proof of the following result.

Theorem. *Let $\mathcal{O} \subset \mathbf{C}$ be an open set with simply connected components. Then there exists a sequence (f_n) of conformal mappings f_n of $\mathcal{O}$ onto $\mathcal{O}$ and a function $\Phi \in H(\mathcal{O})$ with the following property: For all compact sets $B \subset \mathcal{O}$ having connected complement the sequence $(\Phi \circ f_n)$ is dense in the space $A(B)$.*

There is an extensive literature on the field of functions which are "universal" in different respects. The development of a theory of universal holomorphic functions started with Birkhoff's result [1], where the existence of an entire function is proved which is universal under suitable translations. Seidel and Walsh [24] studied the non-euclidean analogue of the entire problem. Recent results ([7], [8], [10], [14], [18], [19], [20], [28]) deal with approximation properties in connection with the boundary behaviour of universal functions. In some papers ([4], [5], [21], [26]) universal approximation properties related to problems in summability theory have been investigated. There exist also universal functions with power series expansions possessing certain overconvergence phenomena ([2], [3], [15], [16]). For excellent treatices on the theory of overconvergence we refer to L. Iliev ([12, [13]).

2. An auxiliary result. Let $\mathcal{O} \subset \mathbf{C}$ be an open set; (S_n) is called an exhausting sequence for $\mathcal{O}$ if the S_n are bounded open sets with the properties

$$\overline{S_n} \subset S_{n+1} \subset \mathcal{O} \quad \text{for all} \quad n \in \mathbf{N},$$

for each compact set $B \subset \mathcal{O}$ there exists an $N \in \mathbf{N}$ such that $B \subset S_N$.

We start the following useful Lemma.

Lemma. *Let $G \subset \mathbf{C}$ be a simply connected domain. Then there exists a sequence (g_n) of conformal mappings of G onto itself and two sequences (K_n) and (L_n) of Jordan domains with the following properties:*

(1) $$\overline{K}_n \cap \overline{L}_n = \emptyset \quad \textit{for all} \quad n \in \mathbf{N},$$

(2) $$\overline{K}_n \subset L_m \quad \textit{for all} \quad m > n,$$

(3) $$\overline{K}_n \cap \overline{K}_m = \emptyset \quad \textit{for} \quad n \neq m,$$

(4) (L_n) *and* $(g_n(K_n))$ *are exhausting sequences for* G.

Proof. 1. Suppose first $G = \mathbf{C}$. Then we may choose $g_n(z) := z - (n+1)^3$ and

$$K_n := \{z : |z - (n+1)^3| < n+1\},$$

$$L_n := \{z : |z| < n^3 + n^2 + n + 1\}.$$

2. Suppose now that $G \neq \mathbf{C}$ and let φ be a conformal mapping of the unit disk $\mathbf{D} := \{z : |z| < 1\}$ onto G.

We first construct three sequences of real numbers. Let

$$r_1 := \frac{1}{2}, \quad t_1 := \frac{3}{4}, \quad s_1 := \frac{21}{22}$$

and suppose that for an $n \geq 2$ the numbers $r_1, \ldots, r_{n-1}; s_1, \ldots, s_{n-1}; t_1, \ldots, t_{n-1}$ have already been determined. We choose $1 - \frac{1}{2n} =: r_n < t_n < s_n < 1$ such that

$$\frac{s_{n-1} + r_{n-1}}{1 + r_{n-1}s_{n-1}} < t_n < \frac{s_n - r_n}{1 - s_n r_n}.$$

By induction we get the sequences $(r_n), (s_n), (t_n)$. The circles

$$k_n := \{z : |z - s_n \frac{1-r_n^2}{1-r_n^2 s_n^2}| < r_n \frac{1-s_n^2}{1-r_n^2 s_n^2}\}, \quad l_n := \{z : |z| < t_n\}$$

are subsets of the unit disk and have the following properties

$$\overline{k}_n \cap \overline{l}_n = \emptyset, \quad \text{for all} \quad n \in \mathbf{N},$$

$$\overline{k}_n \subset l_m \quad \text{for all} \quad m > n,$$

$$\overline{k}_n \cap \overline{k}_m = \emptyset \quad \text{for} \quad n \neq m.$$

The Möbius transform $\tau_n(z) := \dfrac{z - s_n}{1 - zs_n}$ maps the unit disk onto itself and satisfies

$$\tau_n(k_n) = \{\omega : |\omega| < 1 - \frac{1}{2n}\}.$$

The function $g_n := \varphi \circ \tau_n \circ \varphi^{-1}$ is a conformal mapping of G onto itself and it is easy to see that the Jordan domains

$$K_n := \varphi(k_n), \quad L_n := \varphi(l_n)$$

have the desired properties.

3. Proof of the Theorem. For each of the domains G_ν we choose a sequence $(g_{\nu n})_{n\in\mathbf{N}}$ of conformal mappings of G_ν onto G_ν and sequences $(K_{\nu n})_{n\in N}$ and $(L_{\nu n})_{n\in N}$ such that the properties (1)–(4) of the Lemma hold respectively. Let the conformal mappings g_n and f_n of $\mathcal{O}$ onto $\mathcal{O}$ be defined by

$$g_n(z) := g_{\nu n}(z), \quad f_n(z) := g_{\nu n}^{-1}(z) \quad \text{if} \quad z \in G_\nu \quad \text{for some} \quad \nu \in I.$$

The open sets

$$\mathcal{O}_n := \bigcup_{\nu\in I, \nu\leq n} L_{\nu n}, \quad U_n := \bigcup_{\nu\in I, \nu\leq n} K_{\nu n}.$$

consist of finitely many Jordan domains with pairwise disjoint closures.

Let (Q_n) be an enumeration of all polynomials with coefficients whose real-and imaginary-parts are rational; we construct a sequence (P_n) of polynomials as follows. Suppose that $P_0(z) \equiv 0$ and that for an $n \in \mathbf{N}$ the polynomials $P_0, \ldots, P_{n-1}$ have already been determined. By Runge's approximation theorem [6; p.92] we find a polynom P_n with the properties

$$\max_{\overline{\mathcal{O}}_n} |P_n(z) - P_{n-1}(z)| < \frac{1}{n^2}, \tag{5}$$

$$\max_{\overline{U}_n} |P_n(z) - Q_n \circ g_n(z)| < \frac{1}{n^2}, \tag{6}$$

Since $(\mathcal{O}_n)$ is an exhausting sequence for $\mathcal{O}$ is follows by (5) that the function

$$\Phi(z) := \sum_{m=1}^{\infty} \{P_m(z) - P_{m-1}(z)\} \quad (z \in \mathcal{O})$$

is holomorphic on $\mathcal{O}$, and we will show, that the sequence (f_n) and the function Φ satisfy the required properties.

For all $m > n$ we have $\overline{U}_n \subset \mathcal{O}_m$, and hence we obtain by (5)

$$\max_{\overline{U}_n} |\Phi(z) - P_n(z)| \leq \sum_{m=n+1}^{\infty} \max_{\overline{\mathcal{O}}_m} |P_m(z) - P_{m-1}(z)| < \frac{1}{n}.$$

Together with (6) this inequality implies

$$\max_{\overline{g_n(U_n)}} |\Phi \circ f_n(z) - Q_n(z)| \leq$$

$$\leq \max_{\overline{g_n(U_n)}} |\Phi \circ f_n(z) - P_n \circ f_n(z)| + \max_{\overline{g_n(U_n)}} |P_n \circ f_n(z) - Q_n(z)| =$$

$$= \max_{\overline{U}_n} |\Phi(z) - P_n(z)| + \max_{\overline{U}_n} |P_n(z) - Q_n \circ g_n(z)| < \frac{2}{n}.$$

Let now be given a compact set $B \subset \mathcal{O}$ with connected complement and a function $f \in A(B)$. It follows easily from Mergelyan's theorem [6; p.92] that we can choose a sequence (n_k) with $n_k \geq k$ and

$$\max_{B} |Q_{n_k}(z) - f(z)| < \frac{1}{k}.$$

Since $(g_n(U_n))$ is an exhausting sequence for $\mathcal{O}$ we have $B \subset g_{n_k}(U_{n_k})$ for all sufficiently great k and it follows for those k

$$\max_{B} |\Phi \circ f_{n_k}(z) - f(z)| \leq \max_{\overline{g_{n_k}(U_{n_k})}} |\Phi \circ f_{n_k}(z) - Q_{n_k}(z)| + \max_{B} |Q_{n_k}(z) - f(z)| < \frac{3}{k},$$

which proves the Theorem. □

Remark. By simple arguments (for details we refer to [20]) it follows that the sequence $(\Phi \circ f_n)$ is also dense in the space

• $H(U)$ for all open sets $U \subset \mathcal{O}$ with simply connected components (established with the topology of locally uniform convergence on U);

• $L(E)$ (consisting of the Lebesgue-measurable functions on E) for all Lebesgue-measurable sets $E \subset \mathcal{O}$ (established with the almost everywhere convergence on E).

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Universität Trier
Fachbereich IV/Mathematik
Postfach 38 25
D-5500 Trier
GERMANY