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CONVOLUTION TYPE SYSTEMS IN SOME SPACES OF ANALYTIC FUNCTIONS

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Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

1. Introduction. Let D be a bounded convex domain in $\mathbf{C}$, and $H(D)$ denote the space of functions analytic on D with the topology of uniform convergence on compact subsets. Let $\varphi = \{\varphi_m\}$, $\varphi_m : D \to (0,\infty)$, $m \in \mathbf{N}$ be a decreasing sequence of functions bounded on compact subsets and some technical conditions are satisfied.

Let $H(\mathbf{C})$ be the space of entire functions with the topology of uniform convergence on compact subsets. Let $\varphi^* = \{\varphi^*_m\}$, $\varphi^*_m : \mathbf{C} \to (0,\infty)$ be a sequence defined by

$$\varphi^*_m(\lambda) = \sup_{z\in D}(\mathrm{Re}\lambda z - \varphi_m(z)), \quad \lambda \in \mathbf{C}, \quad m \in \mathbf{N}.$$

Consider two families of Banach spaces

$$H_{\varphi_m}(D) = \{f \in H(D) : \|f\|_m = \sup_D(|f(z)|/\exp\varphi_m(z)) < \infty\}, \quad m \in \mathbf{N},$$

$$P_m = P_{\varphi^*_m} = \{f \in H(\mathbf{C}) : \|f\|_m = \sup_{\mathbf{C}}(|f(\lambda)|/\exp\varphi^*_m(\lambda)) < \infty\}, \quad m \in \mathbf{N},$$

and define

$$H_\varphi = H_\varphi(D) = \lim\mathrm{pr}H_{\varphi_m}(D), \quad P = P_{\varphi^*} = \lim\mathrm{ind}P_{\varphi^*_m}.$$

In this paper we study the homogeneous convolution type system

$$(1) \qquad \langle S_z, f^{(n)}(z)\rangle = 0, \quad n \in \mathbf{N}_0, \quad f \in H_\varphi(D)$$

with $S \in H^*_\varphi(D)$. The sign $*$ denotes the strong duality. This system is a generalization of the homogeneous convolution equation

$$\langle S_z, f(z+t)\rangle = \langle S_z, \sum f^{(n)}(z)t^n/n!\rangle = \sum\langle S_z, f^{(n)}(z)\rangle t^n/n! = 0.$$

For any $F \in H_\varphi^*(D)$ define the Fourier-Laplace transform

$$\hat{F}(\lambda) = \langle F_z, \exp \lambda z \rangle, \quad \lambda \in \mathbf{C}.$$

Denote W all $f \in H_\varphi(D)$ satisfying (1). Denote $W^\perp$ all $F \in H_\varphi^*(D)$ such that

$$\langle F, f \rangle = 0 \quad \text{for all} \quad f \in W.$$

Let $I = (W^\perp)$. Fourier-Laplace transform also realize isomorphism $H_\varphi^* \to P$. Then $(H_\varphi^*/W^\perp) = P/I$. General reasons from functional analysis gives isomorphism $W^* = H_\varphi^*/W^\perp$. Isomorphisms above in a similar situation one can find in [2] and [3].

Let characteristic function $L(\lambda) = \hat{S}(\lambda)$ of convolution type system (1) have zeros $\Lambda = \{\lambda_j\}$, $j \in \mathbf{N}$, $0 \notin \Lambda$, and two estimates holds

$$|L(\lambda)| \le C_1 \exp(\varphi_n^*(\lambda) + \alpha \ln |\lambda|), \quad |\lambda| > R > 0, \tag{2}$$

$$|L'(\lambda_j)| \le C_2 \exp(\varphi_n^*(\lambda_j) - \alpha \ln |\lambda_j|), \qquad j \in \mathbf{N} \tag{3}$$

for some $n \in \mathbf{N}$, $\alpha > 0$.

Consider two families of Banach sequence spaces

$$P_{m,\Lambda} = \{a = (a_j) \in \mathbf{C}^{\mathbb{N}} : \|a\|_m = \sup_j(|a_j|/k_m(\lambda_j)) < \infty\},$$

$$\tilde{P}_{m,\Lambda}^* = \{b = (b_j) \in \mathbf{C}^{\mathbb{N}} : \|b\|_m^* = \sup_j(|b_j| k_m(\lambda_j)) < \infty\},$$

$m \in \mathbf{N}$ and define

$$P_\Lambda = \lim \operatorname{ind} P_{m,\Lambda}, \quad P_\Lambda^* = \lim \operatorname{pr} \tilde{P}_{m,\Lambda}^*.$$

As we will see in section 2 $(P_\Lambda)^* = P_\Lambda^*$, $(P_\Lambda^*)^* = P_\Lambda$.

Let $|_\Lambda$ denotes a restriction on Λ. As we will see in section 3 for any $(a_j) \in P_\Lambda$ there is $\omega \in P$ such that $\omega|_\Lambda = (a_j)$. This implies isomorphism $(P/I)|_\Lambda = P$ realized by restriction map $|_\Lambda$.

Note that Fourier-Laplace transform of $(b_j) \in P_\Lambda^*$ is $\sum b_j \exp \lambda_j z$.

So we have diagram constructed by isomorphisms above

$$\begin{array}{ccc}
W = \{\Sigma b_j exp \lambda_j z\} & \xleftarrow{\hat{}} & P_\Lambda^* = \{(b_j)\} \\
\downarrow * & & \uparrow * \\
H_\varphi^*/W^\perp \xrightarrow{\hat{}} P/I & \xrightarrow{|_\Lambda} & P_\Lambda = \{(a_j)\}
\end{array}$$

and we have

$$W = \{f(z) = \sum b_j \exp \lambda_j z : (b_j) \in P_\Lambda^*\}$$

with convergence in the topology of $H_\varphi(D)$.

2. Sequence spaces P_Λ and P^*_Λ. Let $\varphi^* = \{\varphi^*_m\}$, $\varphi^*_m : \mathbf{C} \to (0,\infty)$, $m \in \mathbf{N}$ be increasing sequence and for any $A > 0$, $m \in \mathbf{N}$ there is $p = p(m, A)$ such that

$$\varphi^*_{m+p}(\lambda) - \varphi^*_m(\lambda) > A \ln|\lambda|, \quad |\lambda| > r_{m,p} > 0. \tag{4}$$

Let $\Lambda, \{\lambda_j\}, j \in \mathbf{N}, 0 \notin \Lambda$ be a sequence such that

$$\sum |\lambda_j|^{-A} < \infty \tag{5}$$

for some $A > 0$.

Set $k_m(\lambda) = \exp \varphi^*_m(\lambda)$, $m \in \mathbf{N}$.

Lemma. 1°. *P^*_Λ is strong dual for P_Λ.*
2°. *P_Λ is strong dual for P^*_Λ.*

P r o o f. 1°. Let l be a linear continuous functional on P_Λ, i.e. l acts on $P_{m,\Lambda}$ for any $m \in \mathbf{N}$ and let $\|l\|'_m$ be its norm in $P_{m,\Lambda}, m \in \mathbf{N}$. Define

$$e_j = (\underbrace{0,\ldots,0,1}_{j},0,\ldots),\ b_j = l(e_j),\ j \in \mathbf{N}. \tag{6}$$

Let $a \in P_\Lambda$, i.e. $a \in P_{m,\Lambda}$ for some $m \in \mathbf{N}$. We formally write

$$l(a) = l(\sum a_j e_j) = \sum a_j b_j. \tag{7}$$

We have

$$|b_j k_m(\lambda_j)| = |k_m(\lambda_j) l(e_j)| = |l(k_m(\lambda_j)e_j)| \le \|l\|'_m \|k_m(\lambda_j)e_j\|_m = \|l\|'_m < \infty$$

Hence

$$\|b\|^*_m = \sup_j(|b_j|k_m(\lambda_j)) \le \|l\|'_m < \infty \tag{8}$$

for any $m \in \mathbf{N}$ and $b \in P_\Lambda$.

Fix $m \in \mathbf{N}, a = (a_j) \in P_{m,\Lambda}$. Let $A > 0$ be from (5) and $p = p(m, A)$ be from (4). Then

$$|a_j b_j| \le \|a\|_m k_m(\lambda_j)\|b\|^*_{m+p}/k_{m+p}(\lambda_j) \le \|a\|_m \|b\|^*_{m+p}|\lambda_j|^{-A}, \quad |\lambda_j| > r_{m,p} > 0$$

and $\sum a_j b_j$ converges absolutely. Also we have

$$\|a_j b_j\|_{m+p} = |a_j|\,\|e_j\|_{m+p} = |a_j|/k_{m+p}(\lambda_j) \le$$

$$\|a\|_m k_m(\lambda_j)/k_{m+p}(\lambda_j) \le \|a\|_m |\lambda_j|^{-A}, \quad |\lambda_j| > r_{m,p} > 0,$$

and $\Sigma a_j e_j$ converges normally in $P_{m+p,\Lambda}$. Hence it converges in P_Λ. So (7) is correct.

It has been shown, in fact, that every functional $l \in (P_\Lambda)^*$ corresponds to some $b = (b_j) \in P^*_\Lambda$ defined by (6). Also every $b = (b_j) \in P_\Lambda$ corresponds to some $l \in (P_\Lambda)^*$ if we use (4) as a definition for l. Indeed if $a \in P_{m,\Lambda}$, A be from (5), $p = p(m, A)$ be from (4) then

$$|l(a)| = |\sum a_j b_j| \le \|a\|_m \|b\|^*_{m+p} \sum |\lambda_j|^{-A},$$

and

(9) $$\|l\|'_m \le \|b\|^*_{m+p} \sum |\lambda_j|^{-A} < \infty.$$

Hence l is continuous. Linearity of l is evident. Isomorphism of $(P_\Lambda)^*$ and P^*_Λ is topological in virtue of (8) and (9).

2°. Let l be a linear continuous functional on P^*_Λ, i.e. l acts in $P^*_{m,\Lambda}$ for some $m \in \mathbf{N}$, and let $\|l\|''_m$ be a correspondent norm. Define

(10) $$a_j = l(e_j), \quad j \in \mathbf{N}.$$

Let $b \in P^*_\Lambda$, hence $b \in \tilde{P}^*_{m+p,\Lambda}$ for any P. We formally write

(11) $$l(b) = l(\sum b_j e_j) = \sum a_j b_j.$$

We have

$$|a_j/k_m(\lambda_j)| = |l(e_j/k_m(\lambda_j))| \le \|l\|''_m \|e_j/k_m(\lambda_j)\|^*_m = \|l\|''_m < \infty.$$

Hence

(12) $$\|a\|_m = \sup(|a_j|/k_m(\lambda_j)) \le \|l\|''_m < \infty$$

and $a \in P_\Lambda$. Let $A > 0$ be from (5) and $p = p(m, A)$ be from (4). Then

$$|a_j b_j| \le \|a\|_m k_m(\lambda_j) \|b\|^*_{m+p}/k_{m+p}(\lambda_j) \le \|a\|_m \|b\|^*_{m+p} |\lambda_j|^{-A}, \quad |\lambda_j| \ge r_{m,p} > 0$$

and $\sum a_j b_j$ converges absolutely. Also we have

$$\|b_j e_j\|^*_m = |b_j| \|e_j\|^*_m \le (\|b\|^*_{m+p}/k_{m+p}(\lambda_j)) k_m(\lambda_j) \le$$

$$\le \|b\|^*_{m+p} |\lambda_j|^{-A}, \quad |\lambda_j| \ge r_{m,p} > 0,$$

and $\sum b_j e_j$ converges normally in $\tilde{P}^*_{m,\Lambda}$ hence it converges in P^*_Λ.

So (11) is correct.

It has been shown, that every functional $l \in (P^*_\lambda)^*$ corresponds to some $a = (a_j) \in P_\Lambda$ defined by (10). Also every $a = (a_j) \in P_\Lambda$ corresponds to some $l \in (P^*_\Lambda)^*$ if we use (10) as a definition for l. Indeed let $a \in P_\Lambda$, i.e. $a \in P_{m,\Lambda}$ for some m. Let

$b \in P^*_\Lambda$, hence $b \in P^*_{m+p,\Lambda}$ for any p. If we take $A > 0$ from (5) and $p = p(m, A)$ from (4) then

$$|l(b)| = |\sum a_j b_j| \leq \|a\|_m \|b\|^*_{m+p} \sum |\lambda_j|^{-A} < \infty,$$

and

$$\|l\|''_{m+p} \leq \|a\|_m \sum |\lambda_j|^{-A} < \infty. \tag{13}$$

Hence l is continuous. Linearity of l is evident. Isomorphism of $(P^*_\Lambda)^*$ and P_Λ is topological in virtue of (12) and (13). Lemma is proved. □

Remark 1. $A = (k_m(\lambda_j))$, $j, m \in \mathbf{N}$ is a Kothe matrix, P^*_Λ is a Schwartz space, $P^*_\Lambda = \lambda^\infty(A)$ [1]. Due to (4) and (5) we have prove that $\lambda^\infty(A) = \lambda^1(A)$ in this case. We have isomorphism $(\lambda^1(A))^* = k^\infty(A)$ where $k^\infty(A)$ denotes P_Λ. Then $(P^*_\Lambda)^* = P_\Lambda$.

Remark 2. P_Λ is (LN^*)-space, P^*_Λ is (M^*)-space, hence they are reflexive [4]. Then 1° and 2° are equivalent.

3. Interpolation. Let $\varphi^* = \{\varphi^*_m\}$, $\varphi^*_m : \mathbf{C} \to (0, \infty)$, $m \in \mathbf{N}$ be an increasing sequence of functions such that

$$\varphi^*_{m+p}(\lambda) - \varphi^*_m(\lambda) \geq \ln|\lambda|, \quad p = p(m), \quad |\lambda| \geq r_m > 0, \tag{14}$$

$$|\varphi^*_m(\lambda) - \varphi^*_m(\mu)| \leq C, \quad |\lambda - \mu| \leq 1 \tag{15}$$

for all $m \in \mathbf{N}$. Let for every φ^*_m, φ^*_n, $m > n$ exist some subharmonic function $\Psi_{m,n}$ with the Riss measure $\mu_{m,n}$ such that

$$\mu_{m,n}\{\lambda \in \mathbf{C} : |\lambda - z| \leq t\} \leq C|z|^s t, \quad |z| \geq R > 0, \quad 0 \leq t \leq |z| \tag{16}$$

for some $s \geq 0$, and let

$$\varphi^*_m(\lambda) - \varphi^*_n(\lambda) \leq \Psi_{m,n}(\lambda), \tag{17}$$

$$\varphi^*_n(\lambda) + \Psi_{m,n}(\lambda) \leq \varphi^*_p(\lambda), \quad p = p(m, n), \tag{18}$$

for $|\lambda| \geq r_{m,n} > 0$.

Remark 1. Condition (16) is taken from [6]. If $\Psi = \Psi_{m,n} \in C^2$ and

$$|\partial\Psi/\partial z|, \ |\partial\Psi/\partial\bar{z}| \leq C|z|^s, \quad |z| \geq R > 0, \tag{19}$$

then Green formula implies (16).

Remark 2. Condition (14) implies that if $f(\lambda) \in P$ then $\lambda f(\lambda) \in P$.

Theorem 1. *Let $L(\lambda)$ be entire function satisfying (2) and (3). Then for any $(a_j) \in P$ exists $\omega \in P$ such that $\omega_{|\Lambda} = (a_j)$.*

Proof. We have $(a_j) \in P_\Lambda$, i.e. $(a_j) \in P_{m,\Lambda}$ for some m. Let n be taken from (2) and (3) and without lost of generality $m > n$. Let $\Psi = \Psi_{m,n}$ be a subharmonic function on $\mathbf{C}$ with (16), (17), (18). There is entire function $N(\lambda)$ such that

$$|\Psi(\lambda) - \ln|N(\lambda)|| \le \beta \ln|\lambda|, \quad |\lambda| \ge R > 0, \quad \lambda \notin E = \cup B_j, \quad \beta > 0,$$

$$B_j = \{\lambda \in \mathbf{C} : |\lambda - \lambda'_j| < C|\lambda'_j|^{-\gamma}\}, \quad B_i \cap B_j = \emptyset \quad \text{for} \quad i \ne j, C > 0, \gamma > 0$$

where $\Lambda = \{\lambda'_j\}, j \in \mathbf{N}$-zeros of $N(\lambda)$ [6]. We may choose $N(\lambda)$ and E such that $E \cap \Lambda = \emptyset$ [5].

Consider Lagrange series

$$\omega(\lambda) = \sum (a_j L(\lambda) N(\lambda) \lambda^s / (L'(\lambda_j) N(\lambda_j)(\lambda - \lambda_j)\lambda_j^s), \quad \lambda \in \mathbf{C}. \tag{20}$$

We have

$$|a_j| \le \|a\|_m \exp \varphi_m^*(\lambda_j), \quad j \in \mathbf{N},$$

$$|L'(\lambda_j)| \ge C_2 \exp(\varphi_m^*(\lambda_j) - \alpha \ln|\lambda_j|), \quad j \in \mathbf{N},$$

$$|N(\lambda_j)| \ge \exp(\Psi(\lambda_j) - \beta \ln|\lambda_j|), \quad |\lambda_j| \ge R > 0.$$

Then (17) implies

$$|a_j/(L'(\lambda_j)N(\lambda_j))| \le C|\lambda_j|^{\alpha+\beta}, \quad |\lambda_j| \ge \max(R, r_{m,n}) > 0.$$

We have

$$|L(\lambda)/(\lambda - \lambda_j)| \le |L(\lambda)| \le C_1 \exp(\varphi_n^*(\lambda) + \alpha \ln|\lambda|), \quad |\lambda - \lambda_j| \ge 1, \quad \lambda \ge R > 0, \quad j \in \mathbf{N}.$$

Now let $|\lambda - \lambda_j| < 1$ and λ' gives maximum to entire function $L(\lambda)/(\lambda - \lambda_j)$ on the circle $|\lambda - \lambda_j| = 1$. Then (15) implies

$$|L(\lambda)/(\lambda - \lambda_j)| \le |L(\lambda')| \le C_1 \exp(\varphi_n^*(\lambda') + \alpha \ln|\lambda'|) =$$

$$= C_1 \exp(\varphi_n^*(\lambda) + \alpha \ln|\lambda|) \exp(\varphi_n^*(\lambda') - \varphi_n^*(\lambda') + \alpha(\ln|\lambda'| - \ln|\lambda|)) \le$$

$$\le C \exp(\varphi_n^*(\lambda) + \alpha \ln|\lambda|), \quad |\lambda|, |\lambda_j| \ge R' > 0.$$

Hence (14) and (18) imply

$$|\lambda^s L(\lambda) N(\lambda)/(\lambda - \lambda_j)| \le C' \exp(\varphi_n^*(\lambda) + \Psi(\lambda) + (\alpha + \beta + s) \ln|\lambda|) \le$$

$$\le C' \exp(\varphi_p^*(\lambda) + (\alpha + \beta + \gamma) \ln|\lambda|) \le C' \exp \varphi_q^*(\lambda), \quad |\lambda| \ge R'' > 0,$$

for some q.

$L(\lambda)$ is a function of exponential type because of (2). Hence $\sum |\lambda_j|^{-1-\varepsilon} < \infty$ for any $\varepsilon > 0$. Choose $s > \alpha + \beta + 1, \quad s \in \mathbf{N}$. Then

$$\sum a_j / (\lambda_j^s L'(\lambda_j) N(\lambda_j))$$

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