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ON SOME METHODS OF SOLVING OF MIXED BOUNDARY VALUE PROBLEMS

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Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

ABSTRACT. In this paper we shall deal with such mixed boundery value problems when the kind of the boundary conditions are changing in the limits of the same surface of the border of the domain. Such problems represent a wide class of the boundary value problems of Mathematical Physics, Theory of Elasticity, Hydrodynamics etc. We may distinguish the next basic methods of solving of the mixed boundary value problems: the methods of theory of functions of complex variables [1], [2], the method of integral transforms [3], the method of orthogonal polynomials [4], asymptotic methods [4], variational methods [4], numerical methods and some others [4], [5], [6].

The method of dual, triple, N-tuple integral (or series) equations was widely used in last years. This method proved to be the most effective one among the contemporary analytical methods of solving of mixed boundary value problems [5], [6].

Let us consider in detail the method of dual (triple) integral equations. The first problem on dual integral equations was formulated by H. Weber [7] and solved by E. Beltramy [8]. Solving of dual (triple, N-tuple) integral equations with different kernels is the subject of many articles (see, for example, bibliography in [5], [6], [9], [10]).

The present paper deals with some questions of the theory of dual integral equations with hypergeometric functions in the kernels.

1. The system of l of N-tuple integral equations is a set of the integral equations:

$$\int_{\mathcal{D}_i^j} \sum_{k=1}^{l} a_{ik}^j(\tau)\varphi_k(\tau)K_i^j(\tau,x)d\tau = f_i^j(x); \tag{1}$$

$$(x \in I_i^j, \quad i=\overline{1,N},\ j=\overline{1,l}),$$

where $\mathcal{D}_i^j$, I_i^j are segments of real axis, $\varphi_k(\tau)$ are unknown functions defined on the following set

$$\mathcal{D} = \bigcup_{i=1}^{N} \bigcup_{j=1}^{l} \mathcal{D}_i^j,$$

$K_i^j(\tau, x)$ is i-kernel of j-equation, $f_i^j(x)$ are known functions defined on I_i^j, $a_{ik}^j(\tau)$ are known weight functions,

$$\mathcal{D}_1^j \cap \mathcal{D}_2^j \cap \ldots \cap \mathcal{D}_N^j = A_j,$$

here A_j is a set of a non-zero measure. It should be noted that in (1) we may take the line integral.

It can be noted also that definition (1) may be formulated in a matrix form, if on every I_i^j we define the vector function f in the form of a column matrix with l components, the set of N matrix integral operators is defined on $\mathcal{D}$ and provided the unknown vector-column $\varphi(\tau)$ satisfies N-matrix integral relations. It is easy to generalize (1) to the n-dimensional case.

If $l = 1$, $N = 2$ in (1), we have the following dual integral equations:

$$\text{(2)} \qquad \begin{aligned} &\int_{\mathcal{D}_1} a_{11}(\tau)\varphi(\tau)K_1(\tau, x)d\tau = f_1(x), \ x \in I_1, \\ &\int_{\mathcal{D}_2} a_{21}(\tau)\varphi(\tau)K_2(\tau, x)d\tau = f_2(x), \ x \in I_2. \end{aligned}$$

When $l = 1$, $N = 3$, then we have the triple integral equations. The N-tuple integral equations have the following form:

$$\text{(3)} \qquad \begin{gathered} \int_0^\infty a_i(\tau)\varphi(\tau)K_i(\tau, x)d\tau = f_i(x) \\ (e_i < x < e_{i+1}, \ i = \overline{1, N}, \ e_1 = 0, \ e_{N+1} = \infty). \end{gathered}$$

Below we state some results from the theory of the dual integral equations.

Lemma 1. *The dual integral equations*

$$\text{(4)} \qquad \begin{aligned} &\int_0^\infty \psi(\tau)K_1(\tau, x)d\tau = g_1(x), \ (0 < x < a), \\ &\int_0^\infty G_1(\tau)\psi(\tau)K_2(\tau, x)d\tau = g_2(x), \ (a < x < \infty) \end{aligned}$$

are equivalent to the following dual integral equations:

$$\text{(5)} \qquad \begin{aligned} &\int_0^\infty G(\tau)\varphi(\tau)K_1(\tau, x)d\tau = f_1(x), \ (0 < x < a), \\ &\int_0^\infty \varphi(\tau)K_2(\tau, x)d\tau = f_2(x), \ (a < x < \infty), \end{aligned}$$

if the function $G_1(\tau)$ does not vanish on a set of a non-zero measure.

The result follows immediately by the substitution

$$\varphi(\tau) = G_1(\tau)\psi(\tau), \quad G(\tau) = [G_1(\tau)]^{-1},$$
$$g_i(x) = f_i(x), \quad (i = 1, 2).$$

Lemma 2. *The dual integral equations (5) can be reduced to the pair of dual integral equations:*

$$\tag{6} \begin{aligned} &\int_0^\infty G(\tau)\Phi(\tau)K_1(\tau, x)d\tau = F_1(x), \ (0 < x < a), \\ &\int_0^\infty \Phi(\tau)K_2(\tau, x)d\tau = 0, \ (a < x < \infty), \end{aligned}$$

if the appropriate integral transforms with kernels $\widetilde{K_1}(\tau, x)$, $\widetilde{K_2}(\tau, x)$ exist.

To prove this lemma we introduce a new unknown function $\varphi(\tau)$:

$$\tag{7} \varphi(\tau) = \Phi(\tau) + \int_a^\infty \widetilde{K_2}(\tau, x)f_2(x)dx + \int_0^a \widetilde{K_1}(\tau, x)g(x)dx,$$

where $g(x)$ is arbitrary function but such that integral $\int_0^a K_1(\tau, x)g(x)dx$ exists and has a finite value

$$F_1(x) = f_1(x) - \int_0^\infty K_1(\tau, x)G(\tau)\Big[\int_a^\infty \widetilde{K_2}(\tau, y)f_2(y)dy + \int_0^a \widetilde{K_1}(\tau, y)g(y)dy\Big]d\tau.$$

Lemma 3. *The dual integral equations* (4) *may be reduced to the following dual integral equations:*

$$\tag{8} \begin{aligned} &\int_0^\infty \Psi(\tau)K_1(\tau, x)d\tau = 0, \ (0 < x < a), \\ &\int_0^\infty \Phi(\tau)G_1(\tau)K_2(\tau, x)d\tau = R(x), \ (a < x < \infty), \end{aligned}$$

if the appropriate integral transforms with kernels $\widetilde{K_1}(\tau, x)$, $\widetilde{K_2}(\tau, x)$ exist.

It is evidently that if the function $G_1(\tau)$ does not vanish on a set of a non-zero measure, then dual integral equations (4) can be easily reduced to dual equations (5) by application of Lemma 2. If the function $G_1(\tau)$ vanishes on a set of a non-zero measure, then we introduce a new unknown function $\Psi(\tau)$:

$$\psi(\tau) = \Psi(\tau) + \int_a^\infty \widetilde{K_2}(\tau, x)g(x)dx + \int_0^a \widetilde{K_1}(\tau, x)g_1(x)dx,$$

where

$$R(x) = g_2(x) - \int_0^\infty K_2(\tau,x)G(\tau)\Big[\int_0^a \widetilde{K_1}(\tau,y)g_1(y)dy + \int_a^\infty \widetilde{K_2}(\tau,y)g(y)dy\Big]d\tau.$$

2. Dual integral equations involving Gauss hypergeometric function. Consider the pair of dual integral equations:

$$\int_1^\infty (xt)^{b-1}\ {}_2F_1(a,b;c;-xt)\varphi(t)dt = f_1(x), \quad (0 < x < 1) \tag{9}$$

$$\int_1^\infty (xt)^{b-1}\ {}_2F_1(a,b;c+\alpha;-xt)\varphi(t)dt = f_2(x), \quad (1 < x < \infty) \tag{10}$$

where $\varphi(t)$ is unknown function, $f_i(x)$, $(i = 1,2)$ are given functions on its intervals and the parameters a, b, c, α satisfy the conditions:

$$\alpha > 0,\ \ c > 0,\ \ 0 < a < 1,\ \ b - a < \frac{1}{p} < c - b,\ \ 0 < \frac{1}{p} \le 1. \tag{11}$$

Let us introduce and study the following analogue of the integral operator B from [11]:

$$\widetilde{B}^{a,b}_{c,\lambda} f(x) = g(x) \equiv x^\lambda \int_1^\infty (xt)^{b-1}_2 F_1(a,b;c;-xt)f(t)dt. \tag{12}$$

The new composition relations are expressed in the following theorems.

Theorem 1. *If $\alpha > 0$, $c > 0$, $0 < a < 1$, $\beta + \lambda + b > c + \alpha$, then the relation*

$$I^\alpha_{c-\lambda-b,\beta}\widetilde{B}^{a,b}_{c,\lambda} f(x) = \frac{\Gamma(c)}{\Gamma(c+\alpha)}\widetilde{B}^{a,b}_{c+\alpha,\beta+\lambda} f(x), \tag{13}$$

holds, where

$$I^\alpha_{\nu,\beta} f \equiv \frac{x^{\beta-\nu-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^\nu f(t)dt \tag{14}$$

is an Erdélyi-Kober fractional integral of order $\alpha > 0$ (see [9]).

Theorem 2. *If $0 < \frac{1}{p} \le 1$, $\frac{1}{p} - 1 < \nu < \frac{1}{p} - \alpha$, $0 < \alpha < 1$, $\alpha + \nu < 1$, $\frac{1}{p} - \beta \ge 0$, then the following relation*

$$\widetilde{I}^\lambda_{0,\lambda} K^\alpha_{\nu,\alpha+\nu} f = K^{\alpha+\lambda}_{\nu,\alpha+\lambda+\nu} f, \tag{15}$$

where

$$\widetilde{I}^{\lambda}_{\alpha,\beta}f(x) = \frac{x^a}{\Gamma(\lambda)}\int_x^\infty (t-x)^{\lambda-1}t^{\beta-\alpha-\lambda}f(t)dt \tag{16}$$

is the Weyl-type fractional integral of order $\lambda > 0$ [9] and

$$K^{\alpha}_{\nu,\beta}f(x) = \frac{x^{\beta-\nu-\alpha}}{\Gamma(\alpha)\sin\pi\alpha}\int_0^\infty (x+t)^{\alpha-1}t^{\nu}f(t)dt. \tag{17}$$

Theorem 3. (Inversion formula for $\widetilde{B}^{a,b}_{c,\lambda}$). *If $0 < \frac{1}{p} \le 1$, $0 \le \lambda + \frac{2}{p} - 1 < c - b < \lambda + \frac{1}{p} < 1$, $0 < a < 1$, $b - a < \frac{1}{p} < c - b$, then*

$$f(x) = x^{1-b}\Gamma^{-1}(a)D^{1-a}(1+x)\ \underset{n\to\infty}{\mathrm{l.i.m}}\ L_n h(x), \tag{18}$$

where

$$h(x) = (1-x)[r(x) - r(1)],$$

$$r(x) = Cx^{\frac{2}{p}-a}U\left\{t^{1-b}\frac{d}{dt}\left\{t^{\frac{2}{p}}I^{1-c+b}_{c-b-\lambda,1-\frac{2}{p}-\lambda}g(t)\right\}\right\}(x),$$

$$g = \widetilde{B}^{a,b}_{c,\lambda}f;$$

$$L_n f(x) = \frac{(-x)^{n-1}}{n!(n-2)!}\frac{d^{2n-1}}{dx^{2n-1}}\{x^n f(x)\},$$

$$L_1 f(x) = \frac{d}{dx}\{xf(x)\};\quad xr(x)\to 0,\quad x\to 0;$$

$$r(x)\to 0,\quad x\to\infty;$$

and

$$Uf(x) = x^{-\frac{2}{p}}f(\frac{1}{x}).$$

Now let us go back to dual integral equations (9), (10). Applying the operator $I^{\alpha}_{c-\lambda-b,\beta}$ to equation (9), using composition relation (13) (see Theorem 1) for equation (9), we receive that the dual integral equations are reduced to a single integral equation

$$\int_1^\infty (xt)^{b-1}\varphi(t)\,{}_2F_1(a,b;c+\alpha;-xt)dt = R(x), \tag{19}$$

where

$$R(x) = \begin{cases} \dfrac{\Gamma(\alpha+c)}{\Gamma(c)}I^{\alpha}_{c-\lambda-b,\beta}x^{\lambda}f_1(x), & (0 < x < 1) \\ x^{\beta+\lambda}f_2(x), & (1 < x < \infty). \end{cases} \tag{20}$$

By the help of formula (18) we write the solution of (9), (10) in the closed form:

$$\varphi(x) = \left(\widetilde{B}^{a,b}_{c+\alpha,\beta+\lambda}\right)^{-1} R(x). \tag{21}$$

3. The hybrid dual integral equations. Let us consider the following hybrid dual integral equations:

$$\int_1^\infty t^{\nu-1}\ {}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;-xt)\varphi(t)dt = p_1(x),\quad (0 < x < 1), \tag{22}$$

$$\int_1^\infty t^{\alpha_2-1}\ {}_2F_1(\alpha_1,\alpha_2,b_1;-xt)\varphi(t)dt = p_2(x),\quad (1 < x < \infty). \tag{23}$$

where ${}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;-xt)$ is a generalized hypergeometric function [12], $\varphi(t)$ is the unknown function, $p_1(x)$ and $p_2(x)$ are given functions, $0 < \mu < 1$, $\nu > 0$, $0 < \alpha_1 < 1$, $0 < \frac{1}{p} \le 1$, $0 \le \lambda + \frac{2}{p} - 1 < b_1 - \alpha_2 < \lambda + \frac{1}{p} < 1$, $\alpha_2 > 0$, $b_1 > 0$, $\alpha_2 - \alpha_1 < \frac{1}{p} < b_1 - \alpha_2$.

Using the integral relation between functions ${}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;z)$ and ${}_2F_1(a,b;c;z)$ [13], we receive

$$\begin{aligned} &Ax^{\mu-\nu-1}\ {}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;-xt) = \\ &\int_0^x z^{\nu-1}(x-z)^{\mu-1}\ {}_2F_1(\alpha_1,\alpha_2;b_1;-zt)dz, \end{aligned} \tag{24}$$

where

$$A = \Gamma(\mu)\Gamma(\nu)\Gamma^{-1}(\mu+\nu).$$

Considering (24) as Abel's integral equation with respect to function ${}_2F_1$, we find the following expression for ${}_2F_1$:

$$\begin{aligned} &z^{\nu-1}\ {}_2F_1(\alpha_1,\alpha_2;b_1;-zt) = \\ &A\pi^{-1}\sin\mu\pi\frac{d}{dz}\int_0^z x^{\mu+\nu-1}\ {}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;-xt)(z-x)^{-\mu}dx. \end{aligned} \tag{25}$$

After multiplying equation (22) by $Ax^{\mu+\nu-1}(z-x)^{-\mu\pi-1}\sin\mu\pi$, integrating with respect to x form 0 to z and then differentiating with respect to z, we obtain

$$\int_1^\infty t^{\nu-1}\varphi(t)A\pi^{-1}\sin\mu\pi\left(\frac{d}{dz}\int_0^z x^{\mu+\nu-1}{}_3F_2(\nu,\alpha_1,\alpha_2;\mu+\nu,b_1;-xt)(z-x)^{-\mu}dx\right)dt =$$

$$= \widetilde{p_1}(z),\quad (0 < z < 1),$$

where

$$\widetilde{p_1}(z) = A\pi^{-1}\sin\mu\pi\frac{d}{dz}\int_0^z x^{\mu+\nu-1}(z-x)^{-\mu}p_1(x)dx.$$

Now set $\nu = \alpha_2$ and rewrite equation (23) in the form:

$$\int_1^\infty (tz)^{\alpha_2-1}\varphi(t)\ {}_2F_1(\alpha_1,\alpha_2;b_1;-zt)dt = z^{\alpha_2-1}p_2(z).$$

It should be noted that now the pair of equations (22), (23) are reduced to a single integral equation

$$\int_1^\infty (tz)^{\alpha_2-1}\ {}_2F_1(\alpha_1,\alpha_2;b_1;-zt)\varphi(t)dt = P(z), \tag{26}$$

where

$$P(z) = \begin{cases} z^{\alpha_2-1+\lambda}\widetilde{p_1}(z), & (0<z<1), \\ z^{\alpha_2-1+\lambda}p_2(z), & (1<z<\infty). \end{cases} \tag{27}$$

Applying the inverse operator (18), we can write the saught solution in the closed form:

$$\varphi(x) = \left(\widetilde{B}^{\alpha_1,\alpha_2}_{b_1,\lambda}\right)^{-1}P(x). \tag{28}$$

It is worth mentioning that one can solve the following dual integral equations

$$\begin{aligned} &\int_0^\infty \tau^{\alpha_1-1}\Phi(\alpha_1,\beta_1;-x\tau)\varphi(\tau)d\tau = g_1(x), \quad (0<x<1), \\ &\int_0^\infty \tau^{\alpha_1-1}\Phi(\alpha_2,\beta_2;-x\tau)\varphi(\tau)d\tau = g_2(x), \quad (1<x<\infty). \end{aligned} \tag{29}$$

by analogy with these in section 2.

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