Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
19 (1993) 228-233

ON A CLASS OF MULTIPLIER SEQUENCES

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ABSTRACT. In the paper are given some results about sequences $\{\mu_k\}_{k=0}^{\infty}$ having the property that for every polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with zeros in the unit disc $|z| \le 1$ the zeros of the polynomial $\mu_0 a_0 + \mu_1 a_1 z + \cdots + \mu_n a_n z^n$ will be in the disc $|z| \le 1$ too.

Introduction. The theory of multipliers is primarily related to polynomial transformations preserving some regions of distribution of their zeros. The main problem in this theory is as follows: What should the sequence of numbers $\{\gamma_k\}_{k=0}^{\infty}$ be so that for every polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with zeros in the region D, the zeros of polynomial $\gamma_0 a_0 + \gamma_1 a_1 z + \cdots + \gamma_n a_n z^n$ will be in this region too.

The theory of multiplier sequences starts its development from Laguerre [3], with the case $D = \mathbf{R}$.

Definition 1 ([3], p. 5). *The infinite sequence $\{\alpha_k\}_{k=0}^{\infty}$ is called an α - sequence if for every polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with only real zeros, the polynomial $\alpha_0 a_0 + \alpha_1 a_1 z + \cdots + \alpha_n a_n z^n$ has real zeros only.*

Definition 2 ([3], p. 8). *Denote by L_1 the class of entire functions which are polynomials having real, nonpositive zeros only, or limits of such polynomials in every finite domain.*

Polya and Schur ([3], p. 16) established the following transcendental criterion for α – sequences:

$$\{\alpha_k\}_{k=0}^{\infty} \in \alpha \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} z^k \in L_1.$$

The case $D = \{|z| \le 1\}$ has been discussed by us in [2], where according to Polya's and Schur's idea, algebraic criteria have been found for the respective multiplier sequences and some elementary properties have been proved.

Definition 3. *Denote by K the class of the polynomials whose zeros lie in the unit disc $|z| \leq 1$.*

Definition 4 ([2]). *The infinite sequence $\{\mu_k\}_{k=0}^{\infty}$ is called a μ – sequence if it transforms every polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ from K into a polynomial $\mu p(z) = \mu_0 a_0 + \mu_1 a_1 z + \cdots + \mu_n a_n z^n$ in the same class.*

Definition 5. *The finite sequence $\{\mu_k\}_{k=0}^{n}$ is in M_n if for every polynomial $p(z)$ from the class K of degree n the polynomial $\mu p(z)$ belongs to the same class.*

2. Main results. In order to extend the scope of the applications of the μ – sequences, we shall prove some new properties.

Theorem 1. *Let $q(z) = \sum_{k=0}^{n} b_k z^k$, $b_n \neq 0$ be a polynomial such that all the zeros of the polynomial $q_1(z) = \sum_{k=0}^{n} \frac{b_k}{(n-k)!} z^k$ lie in the unit disc $|z| \leq 1$. Then for every polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ the following inequality is valid*

$$Z_k\left(\sum_{k=0}^{n} k! a_k b_k z^k\right) \geq Z_k\left(\sum_{k=0}^{n} a_k z^k\right),$$

where $Z_k(p(z))$ is the number of the zeros of $p(z)$ which lie in the unit disc.

Proof. We apply a corollary of Grace theorem established by Szegő([1], p. 143) to the polynomials $q_1(z)$ and $p(z)$. The zeros of the composed polynomial

$$r(z) = \sum_{k=0}^{n} k! a_k b_k z^k$$

are of the form $\xi = -k\beta v$, where $\beta_1, \beta_2, \ldots, \beta_n$ are the zeros of $p(z)$ and z belongs to the disc $|z| \leq 1$. From this it follows that the polynomial $r(z)$ has at least as many zeros in the disc $|z| \leq 1$ as $p(z)$.

This completes the proof of the Theorem. □

Corollary 1. *If $p(z) \in K$, then $r(z) \in K$.*

Corollary 2. *If $p(z) = \sum_{k=0}^{n} a_k z^k \in K$, then $p_1(z) = \sum_{k=0}^{n} \frac{c_k}{\binom{n}{k}} z^k \in K$.*

Let $\{\mu_k\}_{k=0}^{\infty}$ be arbitrary real sequence and let us denote $\Delta\mu_k = \mu_{k+1} - \mu_k$, $\Delta^2\mu_k = \mu_{k+1} - 2\mu_k + \mu_{k-1}$, $\mu_{-1} = 0$. It is known [2] that if $\Delta\mu_k \geq 0$, $k = 0, 1, 2, \ldots$,

$\mu_0 > 0$, then:

$$\text{(a)}\quad \{k!\mu_k\}_{k=0}^{\infty} \in \mu; \qquad \text{(b)}\quad \left\{\frac{\mu_k}{\binom{n}{k}}\right\}_{k=0}^{n} \in M_n.$$

Now we shall note the following

Corollary 3. *If* $\Delta^2\mu_k \ge 0$, $k = 0, 1, 2, \ldots$, $\mu_0 > 0$, *then:*

$$\text{(a)}\, \left\{(k!)^2\mu_k\right\}_{k=0}^{\infty} \in \mu;\ \text{(b)}\, \left\{\frac{\mu_k}{\binom{n}{k}}\right\}_{k=0}^{n} \in M_n;\ \text{(c)}\, \left\{k!\Delta\mu_k\right\}_{k=0}^{\infty} \in \mu;\ \text{(d)}\, \left\{\frac{\Delta\mu_k}{\binom{n}{k}}\right\}_{k=0}^{n} \in M_n.$$

The above corollary is a consequence of Theorem 1 and the following theorem due to Berman [5]: Let $p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$ be a polynomial with real coefficients and $a_0 > 0$. If $\Delta^2 a_k = a_{k+1} - 2a_k + a_{k-1} \ge 0$, $k = 0, 1, 2, \ldots, n-1$, $a_{-1} = 0$, then the zeros of $p(z)$ lie outside the unit disc.

Theorem 2. *If* $\left\{\mu_k\right\}_{k=0}^{\infty} \in \mu$, $\mu_k \ne 0$, $k = 0, 1, 2, \ldots$, *then* $|\mu_k| \le |\mu_{k+1}|$, $k = 0, 1, 2, \ldots$.

Proof. The following criterion for μ – sequences is known [2]:

$$\{\mu_k\}_{k=0}^{\infty} \in \mu \iff y_k(z) = \sum_{j=0}^{k} \binom{k}{j} \mu_j z^j \in K, \quad k = 1, 2, \ldots$$

Then also $y_{k+1}^{(k)}(z) = (k+1)!(\mu_k + z\mu_{k+1}) \in K$. Consequently $|\mu_k| \le |\mu_{k+}|$.

Theorem 3. *If* $\left\{\alpha_k\right\}_{k=0}^{\infty} \in \alpha$ *and* $|\alpha_k| \le |\alpha_{k+1}|$, $k = 0, 1, 2, \ldots$, *then* $\left\{\alpha_k\right\}_{k=0}^{\infty} \in M$.

Proof. It is known that if $\{\alpha_k\}_{k=0}^{\infty} \in \alpha$, then either all α_k have the same sign or they have alternating signs ([3], p. 5). So, we have

Case (a). Suppose that $\alpha_k \ge 0$, $k = 0, 1, 2, \ldots$. Then taking into account assumption $0 \le \alpha_k \le \alpha_{k+1}$, $k = 0, 1, 2, \ldots$ and using the Theorem of Graven and Csordas [4], we conclude that all the zeros of the polynomials

$$y_n(z) = \sum_{k=0}^{n} \binom{n}{k} \alpha_k z^k, \quad n = 1, 2, \ldots$$

lie in the interval $[-1, 0]$. It follows from the criterion for μ – sequences that $\left\{\alpha_k\right\}_{k=0}^{\infty} \in \mu$.

Case (b). Suppose that $\alpha_k \le 0$, $k = 0, 1, 2, \ldots$. Then we have $0 \ge \alpha_k \ge \alpha_{k+1}$, $k = 0, 1, 2, \ldots$. It follows from the same Theorem of Graven and Csordas that for any integer $n \ge 1$, $y_n(z) \in K$, i.e. $\{\alpha_k\}_{k=0}^{\infty} \in \mu$.

Case (c). Suppose that $(-1)^k\alpha_k \geq 0$, $k = 0,1,2,\ldots$. As in the case (a) we obtain that for any integer $n \geq 1$, the zeros of the polynomial

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha_k z^k,$$

lie in the interval $[-1,0]$. Then the zeros of $y_n(z)$, $n = 1,2,\ldots$ lie in $[0,1]$. Consequently $y_n(z) \in K$ and $\{\alpha_k\}_{k=0}^{\infty} \in \mu$.

Now we will show that μ – sequences can be obtained from every α – sequence.

Corollary 4. *Let* $\{\alpha_k\}_{k=0}^{\infty} \in \alpha$ *and let* $g_k(t) = \sum_{j=0}^{k}\binom{k}{j}\alpha_j t^j$.

(a) *If sign* $\alpha_k = sign\alpha_{k+1}$, $k = 0,1,2,\ldots$, *then* $\{g_k(t_0)\}_{k=0}^{\infty} \in \mu$ *for each fixed* $t_0 > 0$;

(b) *If sign* $\alpha_k = -sign\alpha_{k+1}$, $k = 0,1,2,\ldots$, *then* $\{g_k(t_0)\}_{k=0}^{\infty} \in \mu$ *for each fixed* $t_0 < 0$;

Proof. We will consider only case (a). The proof of case (b) is analogous. Suppose $\alpha_k \geq 0$, $k = 0,1,2,\ldots$. Let $f(z) = \sum_{k=0}^{\infty}\frac{\alpha_k}{k!}z^k$.

We note that $f(z) \in L_1$ and $e^z f(zt) = \sum_{k=0}^{\infty}\frac{g_k(t)}{k!}z^k$.

But $e^z f(zt_0) \in L_1$ for a fixed $t_0 > 0$. Therefore, by the transcendental criterion for α – sequences, for each fixed $t_0 > 0$, $\{g_k(t_0)\}_{k=0}^{\infty} \in \alpha$. It is evident that $0 \leq g_k(t_0) \leq g_{k+1}(t_0)$. By Theorem 3 we have $\{g_k(t_0)\}_{k=0}^{\infty} \in \mu$.

In particular, when $\alpha_k = 1$, $k = 0,1,2,\ldots$ and $t_0 = 1$, we obtain

$$\left\{\sum_{j=0}^{k}\binom{k}{j}\right\}_{k=0}^{\infty} \in \mu.$$

Theorem 4. *The sequence* $\{\mu_k\}_{k=0}^{\infty} \in \mu$, $\mu_k \neq 0$, $k = 0,1,2,\ldots$ *if and only if for every polynomial* $p(z)$ *the following inequality is valid*

$$Z_k(\mu p(z)) \geq Z_k(p(z)).$$

Proof. Suppose that the above inequality holds. Let $p(z)$, $\deg p(z) = n$ be an arbitrary polynomial from class K. Then $Z_k(\mu p(z)) \geq n$ and since $\deg\mu p(z)) = n$ it follows that $\mu p(z) \in K$., i.e. $\{\mu_k\}_{k=0}^{\infty} \in \mu$.

Conversely, let $\{\mu_k\}_{k=0}^{\infty} \in \mu$, $\mu_k \neq 0$, $k = 0, 1, 2, \dots$ and let

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_n \neq 0$$

be an arbitrary polynomial. Then the theorem of Szegő, when applied to polynomials $y_n(z)$ and $p(z)$ and the fact that $y_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mu_k z^k \in K$ show that the polynomial $\mu p(z)$ has at least as many zeros in the unit disc as $p(z)$.

Thus the proof of the Theorem is completed.

Theorem 5. *For any polynomial* $p(z)$

$$Z_k \left(\sum_{j=0}^{n} \frac{z^j p^{(j)}(z)}{j!} \right) \geq Z_k(p(z)),$$

where $n = \deg p(z)$.

Proof. Let $p(z) = \sum_{k=0}^{n} a_k z^k$. Since $\left\{ \sum_{k=0}^{k} \binom{k}{j} \right\}_{k=0}^{\infty} \in \mu$, by Theorem 4 we have

$$Z_k \left(\sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \binom{k}{j} \right) z^k \right) \geq Z_k(p(z))$$

But calculation shows that

$$\sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \binom{k}{j} \right) z^k = \sum_{j=0}^{n} \frac{z^j p^{(j)}(z)}{j!}$$

Thus the conclusion holds.

It is evident that if $p(z) \in K$, then $\sum_{j=0}^{n} \frac{z^j p^{(j)}(z)}{j!} \in K$.

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Received 12.12.1990
Revised 18.02.1993