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SUBDISTRIBUTIVE DE MORGAN TRIPLETS

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ABSTRACT. We study and characterize De Morgan triplets that verify the subdistributive property.

1. Introduction. Since the introduction of Fuzzy Sets Theory by Zadeh [6] in 1965, many authors have studied possible logical connectives to substitute Min, Max and $1-j$. In this context, t-norms and t-conorms have shown to be useful tools and the structure of De Morgan triplet is now frequently used in Fuzzy Set Theory [4].

On the other hand, it is well known that the lattice structure is lost when we use t-norms and t-conorms. Also the distributivity condition is only satisfied by De Morgan triplets of type $(\mathrm{Min}, \mathrm{Max}, N)$, where N is any strong negation. So, in order to provide other possibilities, we want to study conditions weaker than distributivity. In this way, note that in lattice theory, distributivity is given only by one inequality since the other $[x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)]$ is always satisfied.

Thus, in a De Morgan triplet we can also consider both inequalities separately. In this paper we study and characterize De Morgan triplets that verify one of the distributivity inequalities.

2. Preliminary notions. We begin with the following basic definitions and results

Definition 2.1. *A t-norm is a two place function T from $[0,1] \times [0,1]$ into $[0,1]$ which is associative, commutative, 1 is a unit and T is non-decreasing in each place.*

Definition 2.2. *A t-norm T is Archimedean if it is continuous and $T(x,x) < x$ for all x in $(0,1)$.*

Definition 2.3. *A t-norm T is strict if it is continuous and strictly increasing on $(0,1]^2$.*

Definition 2.4. *A t-conorm is a two place function S from $[0,1] \times [0,1]$ into $[0,1]$ which is associative, commutative, 0 is a unit and S is non-decreasing in each place. Moreover S is said to be Archimedean if it is continuous and $S(x,x) > x$ for all x in $(0,1)$, and strict if it is continuous and strictly increasing on $[0,1)^2$.*

By the representation theorem for the solutions of the associativity equation (see [1]) we have that any Archimedean t-norm can be represented in the form

$$T(x,y) = t^{[-1]}(t(x) + t(y)) \quad \text{for all } x, y \in [0,1], \tag{1}$$

where t is a continuous strictly decreasing function from $[0,1]$ into $[0,+\infty]$ such that $t(1) = 0$ and $t^{[-1]}$ is the pseudo-inverse of t defined by

$t^{[-1]}(x) = t^{-1}(x)$ whenever $0 \le x \le t(0)$ and $t^{[-1]}(x) = 0$ if $t(0) \le x$.

If $t(0) = +\infty$, then T is strict and $t^{[-1]} = t^{-1}$.

This function t is called an additive generator of T and it is unique up to a positive multiplicative constant. If T is a non-strict Archimedean t-norm, we can suppose $t(0) = 1$ and then, the zero-set of T, $Z(T) = \{(x,y) \in [0,1] \mid T(x,y) = 0\}$, is

$$Z(T) = \{(x,y) \in [0,1] \times [0,1] \mid y \le N_T(x)\},$$

where N_T is the strong negation given by

$$N_T(x) = t^{-1}(1 - t(x)) \quad \text{for all } x \in [0,1].$$

Similarly to the case of t-norms, any Archimedean t-conorm can be represented in the form

$$S(x,y) = s^{[-1]}(s(x) + s(y)) \quad \text{for all } x, y \in [0,1] \tag{1'}$$

with similar notations as above but s being a continuous strictly increasing function from $[0,1]$ into $[0,+\infty]$ such that $s(0) = 0$. If $s(1) = +\infty$, then S is strict and $s^{[-1]} = s^{-1}$. As before, this function s is called an additive generator of S.

Definition 2.5. *Let J be a finite or countable set. Consider a collection $\{T_i \mid i \in J\}$ of t-norms and a collection $\{(a_i, b_i) \mid i \in J\}$ of disjoint intervals to the intervals $\{(a_i, b_i) \mid i \in J\}$ the following t-norm*

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i}\right) & \textit{whenever } (x,y) \in (a_i, b_i)^2, \\[2ex] \operatorname{Min}(x,y) & \textit{otherwise.} \end{cases}$$

The following theorem gives us a general classification of continuous t-norms.

Theorem 2.1. *Let T be a continuos t-norm. Then T is Archimedean, $T = \text{Min}$ or T is an ordinal sum of Archimedean t-norms.*

Definition 2.6. *A strong negation is an involutive and decreasing function from $[0,1]$ into $[0,1]$.*

Definition 2.7. *(T, S, N) is a De Morgan triplet if T is a t-norm, N is a strong negation and S is the t-conorm N-dual of T (i.e. $S(x,y) = N(T(N(x), N(y)))$ for all x, y in $[0,1]$).*

3. Subdistributive De Morgan triplets.

First, if we consider in a De Morgan triplet the inequality corresponding to $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ that we have in lattice theory we obtain the following result:

Proposition 3.1. *Let (T, S, N) be a De Morgan tripet verifying*

$$T(x, S(y,z)) \geq S(T(x,y), T(x,z)) \quad \textit{for all} \quad x, y, z \quad \textit{in} \quad [0,1] \tag{2}$$

then $T = \text{Min}$ and $S = \text{Max}$.

Proof. Taking $y = z = 1$ in (2), we obtain

$$x = T(x, S(1,1)) \geq S(T(x,1), T(x,y)) = S(x,x) \geq \text{Max}(x,x) = x,$$

i.e. $S(x,x) = x$ for all x in $[0,1]$. Consequently $S = \text{Max}$ and by duality $T = \text{Min}$.

From this result, it seems quite natural to give the following

Definition 3.1. *A De Morgan triplet (T, S, N) is said to be subdistributive if it verifies,*

$$T(x, S(y,z)) \leq S(T(x,y), T(x,z)) \quad \textit{for all} \quad x, y, z \in [0,1]. \tag{3}$$

There are a lot of subdistributive De Morgan triplets as it is shown in the following examples:

Examples 3.1. It is easy to see that the following De Morgan triplets are subdistributive,

1. If we represent the t-form product by $\prod$, and its $(1-j)$-dual by $\prod^*$, then the De Morgan triplet $(\prod, \prod^*, 1-j)$ is subdistributive.

2. Let N be a strong negation with fixed point s and let T' be any t-norm. Consider the t-norm T given by the ordinal sum of only T' with respect to the interval $[0, s]$, represented in Figure 1. Let S' be the N-dual of T' and S the N-dual of T, represented in Figure 2. Then, the De Morgan triplet (T, S, N) is subdistributive.

Note that in the previous examples, there are no De Morgan triplets (T, S, N) with T being a non-strict Archimedean t-norm. The following theorem shows that it is not by chance.

Theorem 3.1. *Let (T, S, N) be a subdistributive De Morgan triplet. If T is an Archimedean t-norm, then T is strict.*

Proof. Suppose that T is a non-strict Archimedean t-norm. Then there is a strong negation N_T such that $T(x, y) = 0$ if and only if $y \leq N_T(x)$. Now, let s be the fixed point of N_T and put $x = y = z = s$ in (3). We obtain

$$T(s, S(s, s)) \leq S(T(s, s), T(s, s)) = 0.$$

i.e., $S(s, s) \leq s$ and consequently $S(s, s) = s$, and therefore $T(s, s) = s$ which is a contradiction.

Thus, if we want to study Archimedean subdistributive De Morgan triplets, we can reduce our research to the strict case. In this direction, we have the following.

Theorem 3.2. *Let (T, S, N) be a De Morgan triplet where T is a strict t-norm and let t be an additive generator of T. Consider the involution on $[0, +\infty]$ defined by $f = tNt^{-1}$ and let $F : [0, +\infty]^2 \to [0, +\infty]$ be the function given by $F(x, y) = f(f(x) + f(y))$. Then, (T, S, N) is subdistributive if and only if*

$$F(x + y, x + z) \leq x + F(y, z) \quad \text{for all } x, y, z \quad \text{in } [0, +\infty]. \tag{4}$$

Proof. We have by (1) $T(x, S(y, z)) \leq S(T(x, y), T(x, z))$ is equivalent to the inequality

$$t^{-1}(t(x) + tNt^{-1}(tN(y) + tN(z))) \leq Nt^{-1}(tNt^{-1}(t(x) + t(y)) + tNt^{-1}(t(x) + t(z))).$$

By changing $t(x) = u$, $t(y) = v$, $t(z) = w$, the above inequality holds if and only if

$$u + f(f(v) + f(w)) \geq f(f(u + v) + f(u + w)) \quad \text{for all} \quad u, v, w \quad \text{in} \quad [0, +\infty],$$

where f is precisely defined by $f = tNt^{-1}$. Finally, if we consider $F(x, y) = f(f(x) + f(y))$ then the result follows directly.

Remark 3.1. Note that, in the hypothesis of the previous theorem, inequality (4) is equivalent to

$$F(x, y + z) \leq F(x, y) + F(x, z),$$

i.e., the function F is subadditive in each variable. To see that, note that (4) is equivalent to

$$f(f(x + y) + f(x + z)) \leq x + f(f(y) + f(z)) = f(f(x)) + f(f(y) + f(z)) \iff$$

$$f(x+y)+f(x+z) \geq f(f(f(x)+f(f(y)+f(z))) = F(f(x), f(y)+f(z)) \iff$$

$$(\text{by taking } x=f(u), y=f(v), z=f(w))$$

$$f(f(u)+f(v))+f(f(u)+f(w)) \geq F(u, v+w), \text{ i.e. } F(u,v)+F(u,w) \geq F(u, v+w).$$

Remark 3.2. A geometrical interpretation of inequality (4) is the following: If we consider all the sections of the surface $z = F(x,y)$ by parallel planes to $x = y$, then all the secants of these sections have a slope less than or equal to $1/\sqrt{2}$.

Remark 3.3. Let us assume that f is a continious involution on $[0, +\infty]$ such that the function $F(x,y) = f(f(x)+f(y))$ satisfies condition (4) and also, let T be any strict t-norm and t an additive generator of T. Then taking $N = t^{-1}ft$ we obtain a strong negation on $[0,1]$, in such a way that (T, S, N) is a subdistributive De Morgan triplet, where S denotes the N-dual of T. In order to illustrate this fact, first let us give some examples.

Examples 3.2. a) Consider for any nonzero real number k, a function f_k given by $f_k(x) = k/x$. Then for all $k \neq 0$ we have,

$$F_k(x,y) = f_k(f_k(x)+f_k(y)) = \frac{k}{k/x + k/y} = \frac{xy}{x+y}.$$

Now a simple computation shows that F_k verifies condition (4).

b) Also we have already seen that $(\prod, \prod^*, 1-j)$ is a subdistributive De Morgan triplet. Hence, we have

$$f(x) = tNt^{-1}(x) = -\ln(1-e^{-x})$$

and we obtain a function

$$F(x,y) = f(f(x)+f(y)) = -\ln(e^{-x}+e^{-y}-e^{-(x+y)})$$

which clearly satisfies (4).

From these examples we can present many other subdistributive triplets by using Remark 3.3.

Theorem 3.3. *Let T be any strict t-norm and t an additive generator of T. Then*

a) *There exists a family of strong negations (N_k) with $k \neq 0$, such that (T, S_k, N_k) are subdistributive De Morgan triplets.*

b) *Taking $N(x) = t^{-1}(-\ln(1-e^{-t(x)}))$ and $S(x,y) = N(T(N(x), N(y)))$ we obtain that (T, S, N) is a subdistributive De Morgan triplet.*

Proof. Note that from Remark 3.3, b) is clear and a) follows by considering the negations:

$$N_k(x) = t^{-1}(k/t(x)) \text{ for all } x \text{ in } [0,1].$$

From these results, note that there exists a strong relation between De Morgan triplets (T, S, N) with T strict and their associate functions F defined ftom $f = t^{-1}Nt$. Thus, our interest now is to deal with functions

$$F(x,y) = f(f(x) + f(y)) \text{ for all } x, y \text{ in } [0,+\infty] \tag{5}$$

where f is a strictly decreasing continuous involution on $[0,+\infty]$. We can present first results in the differentiable case.

Theorem 3.4. *Let $f : [0,+\infty] \to [0,+\infty]$ be a strictly decreasing function such that $f(0) = +\infty$ and $f^2 = \mathbf{Id}$. Suppose also that f is differentiable on $(0,+\infty)$. If F is defined by (5), we see that F verifies (4) if and only if $1/f'$ is subadditive.*

Proof. First note that since f is an involution, f' does not vanish and consequently $1/f'$ take no infinite values on the interval $(0,+\infty)$. Now, suppose that F verifies (4). Let x, y be two fixed elements of $(0,+\infty)$ and consider the function $h : [0,+\infty] \to [0,+\infty]$ defined by $h(z) = F(x+z, y+z)$. We have $h(z) - h(0) \leq z$ and hence $h'(0) \leq 1$, i.e.

$$h'(0) = \frac{f'(x) + f'(y)}{f'(F(x,y))} \leq 1.$$

If we repeat this reasoning for all $x, y > 0$ we obtain (since $f' < 0$)

$$f'(x) + f'(y) \geq f'(F(x,y)) \text{ and by changing } u = f(x) \text{ and } v = f(y)$$

$$f'(f(u)) + f'(f(v)) \geq f'(f(u+v)) \text{ which is equivalent to } \frac{1}{f'(u)} + \frac{1}{f'(v)} \geq \frac{1}{f'(u+v)}.$$

Conversely, if $1/f'$ is subadditive it verifies

$$\frac{f'(x+z) + f'(y+z)}{f'(F(x+z, y+z))} \leq 1 \quad \text{for all } x, y \geq 0 \text{ and all } z > 0,$$

which is equivalent to $h'(z) \leq 1$ for all $z > 0$. Then, there exists c_z in $(0, z)$ such that

$$F(x+z, y+z) - F(x,y) = h(z) - h(0) = h'(c_z)z \leq z$$

and so F verifies (4).

Corollary 3.1. *Under the same assumptions of the above theorem, the following conditions are equivalent:*

a) $F(x,y) = f(f(x) + f(y))$ *verifies* (4).

b) $f' \circ f$ *is subadditive.*

c) $D_{1,1}F(x,y) \le 1$ *for all* $x, y > 0$ *(where* $D_{1,1}$ *stands for directional derivative in the direction of the vector* $(1,1)$*).*

On the other hand, if f is any strictly decreasing involution on $[0,+\infty]$, by taking $s(x) = f(-\ln(x))$ we can associate to F, the strict t-conorm S additively generated by s. Then we have the following theorem whose proof is immediate.

Theorem 3.5. *Let f be a strictly decreasing involution on $[0,+\infty]$. Consider the function $F(x,y) = f(f(x)+f(y))$. Then F verifies (4) if and only if S satisfies,*

$$S(xz, yz) \ge z\, S(x,y) \quad \text{for all } x, y, z \text{ in } [0,1]. \tag{6}$$

In order to present some results on S which will give us new information about f, we give one more definition (see [2] and [3]).

Definition 3.2. *A function $T : [0,1]^2 \to [0,1]$ is said to be totally negative of order 2 if for all $x_1 < x_2$ and $y_1 < y_2$ we have*

$$\begin{vmatrix} T(x_1,y_1) & T(x_1,y_2) \\ T(x_2,y_1) & T(x_2,y_2) \end{vmatrix} \le 0.$$

Theorem 3.6. *Let S be a strict t-conorm with additive generator s, satisfying (6). Then:*

a) *If s is a continuous differentiable function with derivative different from zero and such that*

$$\lim_{h\to 0} \frac{s'(h)}{s'(S(h,x))} \le 1, \quad \text{then } s \text{ is subadditive}$$

(i.e. $f(-\ln(x))$ is subadditive).

b) *S is totally negative of order 2 (in particular, $s(e^{-x}) = f(x)$ is convex).*

c) *If s is a differentiable function with derivative different from zero and such that*

$$\lim_{h\to 0} \frac{s'(xh)}{s'(h)} \le 1, \quad \text{then } s \text{ is convex} \quad (\text{i.e. } f + f' \text{ is increasing}).$$

Proof. **a)** From condition (6) we have

$$S(x,y) \le \lim_{h\to 0} \frac{S(xh,yh)}{h} = \lim_{h\to 0} \frac{xs'(xh)+ys'(yh)}{s'(S(xh,yh))} \le x + y,$$

which proves that $s(x) = f(-\ln(x))$ is subadditive.

b) Again from (6) we obtain $S(x, yz) \geq S(xz, yz) \geq z\ S(x, y)$ and so, S is totally negative of order 2. Consequently $s(e^{-x}) = f(x)$ is convex. The proof of these results is essentially the same as that for the case of totally positive strict t-norms given in [2] (Theorem 3.1.).

c) By putting $g(x) = s(z\ s^{-1}(x))$ we have that condition (6) is equivalent to the subadditivity of g and then,

$$\frac{g(x+y)-g(x)}{y} \leq \frac{g(y)}{y}.$$

Now, by taking limit when $y \to 0$,

$$z\ \frac{s'(zs^{-1}(x))}{s'(s^{-1}(x))} = g'(x) \leq \lim \frac{g(y)}{y} = \lim\ g'(y) = \lim\ z\ \frac{s'(zs^{-1}(y))}{s'(s^{-1}(y))} \leq z.$$

Thus, $s'(zx)/s'(x) \leq 1$, i.e. s' is increasing and s is convex (which is equivalent to $f' + f$ increasing).

In the last theorem we have proved that if the function $F(x, y) = f(f(x)+f(y))$ verifies (4), then f must be convex. In other words, if (T, S, N) is a subdistributive De Morgan triplet with T strict, then the function given by $f = t^{-1}Nt$ must be convex. To finish this work, we present the last example proving that the converse of this fact is not true.

Example 3.3. Let $a < 1$ be the unique solution of equation $x = -\ln(x)$ and consider the function f defined by,

$$f(x) = \begin{cases} -\ln(x) & \textit{whenever} \quad x \in [0, a] \\ e^{-x} & \textit{whenever} \quad x > a. \end{cases}$$

we trivially have that f is a strictly decreasing, continuous involution on $[0, +\infty]$ and convex in $[0, a) \cup (a, +\infty]$. To check that f is also convex at $x = a$, note that the left derivative of f at point a does not surpass the right one.

Thus f is convex in $[0, +\infty)$ but $F(x, y) = f(f(x) + f(y))$ does not verify (4) since by taking the values $y = z = 9a/10$ and $x = a/10$, we have

$$F(x+y, x+z) - F(y, z) = F(a, a) - F(9a/10, 9a/10) =$$

$$= f(2f(a)) - f(2f(9a/10)) = f(2a) - f(-2\ \ln(9a/10)) =$$

$$= e^{-2a} - e^{2ln(9a/10)} = a^2 - (9a/10)^2 = 19a^2/100 > a/10 = x$$

where the inequality holds because $a > 10/19$.

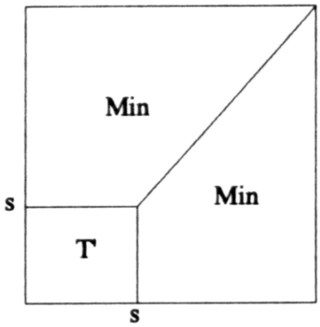

Figure 1.

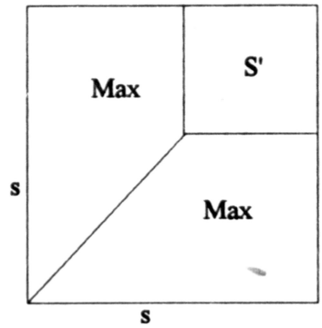

Figure 2.

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