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CONTACTLY CONFORMAL TRANSFORMATIONS OF ALMOST CONTACT MANIFOLDS WITH B-METRIC

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ABSTRACT. In this paper, we study the main classes of almost contact manifolds with B−metric. The groups of contactly conformal transformations with respect to which the considered classes are closed or contactly conformally equivalent to the class of such manifolds with covariantly constant structure tensors are introduced. Examples of almost contact manifolds with B-metric belonging to these classes are constructed.

1. Almost contact manifolds with B-metric. A $(2n+1)$−dimensional real differentiable manifold M is said to have a (φ,ξ,η)-structure, or an almost contact structure, if M admits a tensor field φ of type $(1,1)$, a vector field ξ and 1-form η satisfying the conditions [2]:

$$\eta(\xi)=1; \quad \varphi^2=-id+\eta\otimes\xi. \tag{1.1}$$

Further, X,Y,Z will stand for arbitrary differertiable vector fields on M and x,y,z – arbitrary vectors in tangential space T_pM to M at an arbitrary point p in M.

Definition 1.1. *If a manifold M with a (φ,ξ,η)-structure admits a metric g such that*

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{1.2}$$

then M is called **an almost contact manifold with** ***B*-metric g** [4].

The equalities (1.1) and (1.2) imply immediately

$$\eta\circ\varphi=0,\ \varphi\xi=0,\ \eta(X)=g(X,\xi),\ g(\varphi X,Y)=g(X,\varphi Y).$$

The tensor $\tilde{g}$ given by

$$\tilde{g}(X,Y)=g(X,\varphi Y)+\eta(X)\eta(Y) \tag{1.3}$$

is a B-metric associated with the metric g. Both metrics g and $\tilde{g}$ are indefinite of signature (n+1,n) [4].

Let ∇ be the Levi-Civita connection of the metric g.

The tensor F of type $(0,3)$ on M is defined by $F(x,y,z) = g((\nabla_x\varphi)y,z)$.We have

$$(\nabla_x\eta)y = g(\nabla_x\xi,y) = F(x,\varphi y,\xi). \tag{1.4}$$

Because of (1.1) and (1.2), the tensor F has the following properties:

$$\begin{aligned} &F(x,y,z) = F(x,z,y),\\ &F(x,\varphi y,\varphi z) = F(x,y,z) - \eta(y)F(x,\xi,z) - \eta(z)F(x,y,\xi). \end{aligned} \tag{1.5}$$

The 1-forms θ,θ^* and ω associated with the tensor F is defined by

$$\theta(x) = g^{ij}F(e_i,e_j,x),\ \theta^*(x) = g^{ij}F(e_i,\varphi e_j,x),\ \omega(x) = F(\xi,\xi,x), \tag{1.6}$$

where $\{e_i,\xi\}$ $(i = 1,2,\ldots,2n)$ is a basis of T_pM, and g^{ij} are the components of the inverse matrix of g.

A classification of the almost contact manifolds with B-metric is given in [5]. The basic eleven classes $\mathcal{F}_i$ of these manifolds are characterized by the following conditions for the tensor F:

$$\begin{aligned}
&\mathcal{F}_1 : F(x,y,z) = 1/(2n)\{g(x,\varphi y)\theta(\varphi z) + g(x,\varphi z)\theta(\varphi y)\\
&\qquad +g(\varphi x,\varphi y)\theta(\varphi^2 z) + g(\varphi x,\varphi z)\theta(\varphi^2 y)\};\\
&\mathcal{F}_2 : F(\xi,y,z) = F(x,\xi,z) = 0,\ F(x,y,\varphi z) + F(y,z,\varphi x)+\\
&\qquad F(z,x,\varphi y) = 0,\ \theta = 0;\\
&\mathcal{F}_3 : F(\xi,y,z) = F(x,\xi,z) = 0,\ F(x,y,z) + F(y,z,x) + F(z,x,y) = 0;\\
&\mathcal{F}_4 : F(x,y,z) = -(\theta(\xi))/(2n)\{g(\varphi x,\varphi y)\eta(z) + g(\varphi x,\varphi z)\eta(y)\};\\
&\mathcal{F}_5 : F(x,y,z) = -(\theta^*(\xi))/(2n)\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\};\\
&\mathcal{F}_6 : F(x,y,z) = -F(\varphi x,\varphi y,z) - F(\varphi x,y,\varphi z) = -F(y,z,x) + F(z,x,y)-\\
&\qquad -2F(\varphi x,\varphi y,z),\ \theta(\xi) = \theta^*(\xi) = 0;\\
&\mathcal{F}_7 : F(x,y,z) = -F(\varphi x,\varphi y,z) - F(\varphi x,y,\varphi z) = -F(y,z,x) - F(z,x,y);\\
&\mathcal{F}_8 : F(x,y,z) = F(\varphi x,\varphi y,z) + F(\varphi x,y,\varphi z) =\\
&\qquad = -F(y,z,x) + F(z,x,y) + 2F(\varphi x,\varphi y,z);\\
&\mathcal{F}_9 : F(x,y,z) = F(\varphi x,\varphi y,z) + F(\varphi x,y,\varphi z) = -F(y,z,x) - F(z,x,y);\\
&\mathcal{F}_{10} : F(x,y,z) = \eta(x)F(\xi,\varphi y,\varphi z);\\
&\mathcal{F}_{11} : F(x,y,z) = \eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\}.
\end{aligned} \tag{1.7}$$

The class $\mathcal{F}_0$ of almost contact manifolds with B-metric is defined by the condition $F = 0$. This special class is contained in each of the basic classes.

Let us point out that $\mathcal{F}_0$ is analogous to the class of cosymplectic manifolds of the metric almost contact manifolds [2],[3].

The Nijenhuis tensor N for the structure (φ, ξ, η) is given by [1]:

$$N(X,Y) = [\varphi,\varphi](X,Y) + d\eta(X,Y)\xi,$$

where

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \tag{1.8}$$

$$d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X. \tag{1.9}$$

By means of the covariant derivatives of φ and η, the tensor N is expressed by the equality

$$\begin{aligned} N(X,Y) = \; & (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X - \varphi(\nabla_X\varphi)Y + \varphi(\nabla_Y\varphi)X + \\ & +(\nabla_X\eta)Y.\xi - (\nabla_Y\eta)X.\xi. \end{aligned} \tag{1.10}$$

The associated tensor $\tilde{N}$ with N is defined by

$$\begin{aligned} \tilde{N}(X,Y) = \; & (\nabla_{\varphi X}\varphi)Y + (\nabla_{\varphi Y}\varphi)X + \varphi(\nabla_X\varphi)Y + \varphi(\nabla_Y\varphi)X \\ & +(\nabla_X\eta)Y.\xi + (\nabla_Y\eta)X.\xi. \end{aligned} \tag{1.11}$$

In this paper we study the special class $\mathcal{F}_0$ and the classes $\mathcal{F}_1$, $\mathcal{F}_4$ and $\mathcal{F}_5$ arised from the main components of F.

Theorem 1.1. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-mertic. We have*

a) *If $M \in \mathcal{F}_0$, then $N = \tilde{N} = [\varphi,\varphi] = d\eta = F = \nabla\varphi = \nabla\eta = \nabla\xi = \theta = \theta^* = \omega = 0$;*

b) *If $M \in \mathcal{F}_1$, then $N = [\varphi,\varphi] = d\eta = \nabla\eta = \nabla\xi = \theta(\xi) = \theta^*(\xi) = \omega = 0$;*

c) *If $M \in \mathcal{F}_4$, then $N = [\varphi,\varphi] = d\eta = \theta^* = \omega = 0$;*

d) *If $M \in \mathcal{F}_5$, then $N = [\varphi,\varphi] = d\eta = \theta = \omega = 0$.*

P r o o f. Having in mind the characterization conditions (1.7) of $\mathcal{F}_i$ $(i = 0,1,4,5)$ and the equalities (1.6), (1.8), (1.9), (1.10), (1.11) we obtain the statement immediately.

Let us point out that Theorem 1.1 implies $N \equiv 0$ for the classes $\mathcal{F}_i$ $(i = 0,1,4,5)$, i.e. these classes are an analogue of the normal metric almost contact manifolds. Moreover, on account of $[\varphi,\varphi] = 0$ and $d\eta = 0$, these classes have the character of strong normal manifolds [2].

We define subclasses $\mathcal{F}_i^0$ of $\mathcal{F}_i$ ($i = 1,4,5$) by conditions for the 1-forms θ and θ^*.

$$\begin{aligned} &\text{a) } M \in \mathcal{F}_1^0 \text{ iff } \theta \text{ and } \theta^* \text{ are closed;}\\ &\text{b) } M \in \mathcal{F}_4^0 \text{ iff } \theta \text{ is closed;}\\ &\text{c) } M \in \mathcal{F}_5^0 \text{ iff } \theta^* \text{ is closed.}\end{aligned} \tag{1.12}$$

Lemma 1.1. a) *If $M \in \mathcal{F}_1$, then θ (respectively θ^*) is closed iff $(\nabla_X \theta)Y = (\nabla_Y \theta)X$ (respectively $(\nabla_X \theta)\varphi Y = (\nabla_Y \theta)\varphi X)x$);*

b) *If $M \in \mathcal{F}_4$, then θ is closed iff $X\theta(\xi) = \eta(X)\xi\theta(\xi)$;*

c) *If $M \in \mathcal{F}_5$, then θ^* is closed iff $X\theta^*(\xi) = \eta(X)\xi\theta^*(\xi)$.*

We shall say that M is an $\mathcal{F}_i^0$-manifold (respectively an $\mathcal{F}_0$-manifold), if $M \in \mathcal{F}_i^0$ ($i = 1,4,5$) (respectively $M \in \mathcal{F}_0$).

2. Contactly conformal transformations.

Definition 2.1. *Let M be with an almost contact structure with B-metric (φ,ξ,η,g). The transformation $c : c(\varphi,\xi,\eta,g) = (\bar\varphi,\bar\xi,\bar\eta,\bar g)$ defined by means of the equalities:*

$$\bar\varphi = \varphi,\ \bar\xi = \xi,\ \bar\eta = \eta,$$

$$\bar g = e^{2u}\cos 2v\ g + e^{2u}\sin 2v\ \tilde g + (1 - e^{2u}\cos 2v - e^{2u}\sin 2v)\eta\otimes\eta,\ u,v \in \mathcal{F}M, \tag{2.1}$$

will be called contactly conformal transformation of the structure (φ,ξ,η,g). We shall say (φ,ξ,η,g) and $(\bar\varphi,\bar\xi,\bar\eta,\bar g)$ are contactly conformally related structures, (M,φ,ξ,η,g) and $(M,\bar\varphi,\bar\xi,\bar\eta,\bar g)$ – contactly conformally related manifolds.

We have to point out that the metric $\bar g$ has a maximal rank for arbitrary u and v belonging to $\mathcal{F}M$. According to (1.3), the equality (2.1) takes the form

$$\bar g(X,Y) = e^{2u}\cos 2v\ g(X,Y) + e^{2u}\sin 2v\ g(X,\varphi Y) + (1 - e^{2u}\cos 2v)\eta(X)\eta(Y). \tag{2.1'}$$

Since $\bar g(\xi,\xi) = 1$, $\bar g(X,\xi) = \eta(X)$ and $\bar g(\varphi X,\varphi Y) = -\bar g(X,Y) + \eta(X)\eta(Y)$, then the structure $(\varphi,\xi,\eta,\bar g)$ is also an almost contact structure with B-metric $\bar g$ on M.

Lemma 2.1. *The contactly conformal transformations on an almost contact manifold with B-metric form a group.*

P r o o f. Let (φ,ξ,η,g) be an almost contact structure with B-metric on M. From (2.1′) we get

$$\begin{aligned} g(X,Y) = {} & e^{-2u}\cos 2v\bar g(X,Y) - e^{-2u}\sin 2v\bar g(X,\varphi Y)\\ & + (1 - e^{-2u}\cos 2v)\eta(X)\eta(Y),\end{aligned} \tag{2.2}$$

i.e. the inverse transformation c^{-1} of c is determined by the pair of functions $(-u,-v)$. Thus, c^{-1} is also a contactly conformal transformation. Let c' and c'' be contactly conformal transformations determined by the pair of functions (u',v') and (u'',v''), respectively. We consider the transformation $c = c'' \circ c'$. Applying consecutively (2.1′) for the transformations c' and c'', we obtain that the composition c is a contactly conformal transformation determined by the pair of functions $(u = u'+u'')$, $v = v'+v''$.

The group of contactly conformal transformations on an almost contact manifold with B-metric will be denoted by C.

Next, we take into consideration local contact conformal transformations.

Theorem 2.1. *Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric and 1-forms θ,θ^* and ω, associated with F and $(M,\varphi,\xi,\eta,\bar{g})$ – the contactly conformally related manifold to (M,φ,ξ,η,g) by a transformation $c \in C$. Then $(M,\varphi,\xi,\eta,\bar{g})$ is an almost contact manifold with B-metric and 1-forms $\bar{\theta},\bar{\theta}^*$ and $\bar{\omega}$, associated with $\bar{F}$, so that*

$$
\begin{aligned}
&2\bar{F}(X,Y,Z) = 2e^{2u}\cos 2v F(X,Y,Z)\\
&+e^{2u}\sin 2v\big[\eta(Z)F(X,\varphi Y,\xi) + \eta(Y)F(X,\varphi Z,\xi)\\
&+F(\varphi Y,Z,X) - F(Y,\varphi Z,X) + F(\varphi Z,X,Y) - F(Z,X,\varphi Y)\big]\\
&+(1-e^{2u}\cos 2v)\Big\{\eta(X)\big[F(Y,Z,\xi) + F(\varphi Z,\varphi Y,\xi)\\
&+F(Z,Y,\xi) + F(\varphi Y,\varphi Z,\xi)\big]\\
&+\eta(Y)\big[F(X,Z,\xi) + F(\varphi Z,\varphi X,\xi)\big]\\
&+\eta(Z)\big[F(X,Y,\xi) + F(\varphi Y,\varphi X,\xi)\big]\Big\}\\
&-\gamma(Z)g(\varphi X,\varphi Y) - \gamma(Y)g(\varphi X,\varphi Z) - \tilde{\gamma}(Z)g(X,\varphi Y) - \tilde{\gamma}(Y)g(X,\varphi Z),\\
&\big(\gamma = d(e^{2u}\cos 2v)\circ\varphi + d(e^{2u}\sin 2v),\\
&\tilde{\gamma} = d(e^{2u}\cos 2v) - d(e^{2u}\sin 2v)\circ\varphi\big);
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\bar{\theta} &= \theta + 2n[du\circ\varphi + dv]\\
\bar{\theta}^* &= \theta^* + 2n[du - dv\circ\varphi]\\
\bar{\omega} &= \omega
\end{aligned}
\tag{2.4}
$$

P r o o f. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of g and $\bar{g}$. Then, using the well known equality for a metric and its Levi-Civita connection and taking into account (2.1) and (1.4), we get

$$
\begin{aligned}
(2.5)\qquad & g(\bar{\nabla}_X Y - \nabla_X Y, Z)\\
&= -\frac{1}{4}\sin 4v\big[F(X,Y,\varphi^2 Z) + F(Y,\varphi^2 Z, X) - F(\varphi^2 Z, X, Y)\big]\\
&-\frac{1}{2}\sin^2 2v\big[F(X,Y,\varphi Z) + F(Y,\varphi Z, X) - F(\varphi Z, X, Y)\big]\\
&+\frac{1}{2}e^{2u}\sin 2v\big[F(X,Y,\xi) + F(Y,\xi,X) - F(\xi,X,Y)\big]\eta(Z)\\
&+(1 - e^{2u}\cos 2v)(\nabla_X\,\eta)Y.\eta(Z)\\
&+e^{2u}\cos 2v\,\eta(Z)\big[du(\xi)g(\varphi X,\varphi Y) - dv(\xi)g(X,\varphi Y)\big]\\
&-e^{2u}\sin 2v\,\eta(Z)\big[du(\xi)g(X,\varphi Y) + dv(\xi)g(\varphi X,\varphi Y)\big]\\
&-du(X)g(\varphi Y,\varphi Z) - du(Y)g(\varphi X,\varphi Z)\\
&+dv(X)g(Y,\varphi Z) + dv(Y)g(X,\varphi Z)\\
&+\Big\{-\cos^2 2v\,du(\varphi^2 Z) - \frac{1}{2}\sin 4v\,du(\varphi Z)\\
&+\frac{1}{2}\sin 4v\,dv(\varphi^2 Z) + \sin^2 2v\,dv(\varphi Z)\Big\}g(\varphi X,\varphi Y)\\
&+\Big\{\frac{1}{2}\sin 4v\,du(\varphi^2 Z) + \sin^2 2v\,du(\varphi Z)\\
&+\cos^2 2v\,dv(\varphi^2 Z) + \frac{1}{2}\sin 4v\,dv(\varphi Z)\Big\}g(X,\varphi Y)
\end{aligned}
$$

Let $\bar{F}(X,Y,Z) = \bar{g}((\bar{\nabla}_X\,\varphi)Y, Z)$. By means of (2.5) and (2.2) we obtain (2.3). The equalities (2.4) come directly from (2.3), taking into account (1.6) and the consequence of (2.1′)

$$\bar{g}^{ij} = e^{-2u}\cos 2v\,g^{ij} - e^{-2u}\sin 2v\,\varphi^j_k g^{ik} + (1 - e^{-2u}\cos 2v)\xi^i\xi^j.$$

We define some subsets of C using the following conditions:

$$(2.6)\qquad C_0 = \{c \in C \mid du\circ\varphi = dv\circ\varphi^2,\ du(\xi) = dv(\xi) = 0\};$$

$$(2.7)\qquad C_1 = \{c \in C \mid du(\xi) = dv(\xi) = 0\};$$

$$(2.8)\qquad C_4 = \{c \in C \mid du\circ\varphi = dv\circ\varphi^2,\ du(\xi) = 0\};$$

$$(2.9)\qquad C_5 = \{c \in C \mid du\circ\varphi = dv\circ\varphi^2,\ dv(\xi) = 0\};$$

$$C_{45} = \{c \in C \mid du \circ \varphi = dv \circ \varphi^2\}; \tag{2.10}$$

$$C_{15} = \{c \in C \mid du(\xi)=0\}; \tag{2.11}$$

$$C_{14} = \{c \in C \mid dv(\xi)=0\}. \tag{2.12}$$

The subsets C^*_{ij} of C_{ij} $(i,j = 1,4,5;\ i \neq j)$ are defined by

$$C^*_{ij} = \{c \in C_{ij} \mid c \notin C_i, c \notin C_j\}. \tag{2.13}$$

Definition 2.2. *The real function $u \in \mathcal{F}M$ is said to be φ-pluriharmonic, if $du(\xi) = 0$ and the 1-form $du \circ \varphi$ is closed, i.e. $d(du \circ \varphi) = 0$.*

Definition 2.3. *The real pair of functions (u,v), $u, v \in \mathcal{F}M$, is said to be a φ-holomorphic pair of functions, if $du = dv \circ \varphi$ and $dv = -du \circ \varphi$.*

It is clear that, if (u,v) is φ-holomorphic pair of functions, then each of them is φ-pluriharmonic function, but the inverse assertion is not always true.

Let $C^0_i \subset C_i$ $(i = 1,4,5)$ be sets of contactly conformal transformations defined by the conditions:

$$C^0_1 = \{c \in C_1 \mid u, v \text{ are } \varphi\text{−pluriharmonic}\}, \tag{2.14}$$

$$C^0_4 = \{c \in C_4 \mid u \text{ is } \varphi\text{−pluriharmonic}\}, \tag{2.15}$$

$$C^0_5 = \{c \in C_5 \mid v \text{ is } \varphi\text{−pluriharmonic}\}. \tag{2.16}$$

Let $C^{0'}_i$ $(i = 1,4,5)$ be sets of contactly conformal transformations so that

$$C^{0'}_1 = \{c \in C^0_1 \mid dv = 0\}, \tag{2.17}$$

$$C^{0'}_4 = \{c \in C^0_4 \mid du = 0\}, \tag{2.18}$$

$$C^{0'}_5 = \{c \in C^0_5 \mid dv = 0\}. \tag{2.19}$$

Lemma 2.2. *The sets of contactly conformal transformations C_0, C_i, C^0_i, $C^{0'}_i$, C_{ij}, C^*_{ij} $(i,j = 1,4,5;\ i \neq j)$ are subgroups of C, furthermore $C_0 \subset C^0_i \subset C_i \subset C_{ij} \subset C$.*

The proof of this lemma is an anologous to the proof of Lemma 2.1.

We shall give a geometrical interpretation of the defined groups of contactly conformal transformations in the following three theorems.

Theorem 2.2. *The class $\mathcal{F}_i$ is closed with respect to the transformations of the group C_i $(i = 0, 1, 4, 5)$. This group is the maximal such subgroup of the contactly conformal group C. The corresponding basic 1-forms satisfy the equalities:*

a) $\bar{\theta} = \theta = 0,$	$\bar{\theta}^* = \theta^* = 0$	for $i = 0;$
b) $\bar{\theta} = \theta + 2n[du \circ \varphi - dv \circ \varphi^2],$	$\bar{\theta}^* = \theta^* - 2n[du \circ \varphi^2 + dv \circ \varphi]$	for $i = 1;$
c) $\bar{\theta} = \theta + 2ndv(\xi)\eta,$	$\bar{\theta}^* = \theta^* = 0$	for $i = 4;$
d) $\bar{\theta} = \theta = 0,$	$\bar{\theta}^* = \theta^* + 2ndu(\xi)\eta$	for $i = 5.$

Proof. Taking into account the characterization conditions (1.7) of $\mathcal{F}_i$ and the defining conditions (2.6) – (2.9) of C_i $(i = 0, 1, 4, 5)$, the equality (2.3) implies that, if $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_i$ $(i = 0, 1, 4, 5)$, then the manifold $C_i(M, \varphi, \xi, \eta, g)$ belongs to the same class of manifolds.

Let c be an arbitrary transformation from C and let $(M, \varphi, \xi, \eta, g)$ and $c(M, \varphi, \xi, \eta, g)$ belong to $\mathcal{F}_i$ $(i = 0, 1, 4, 5)$. We effect the characterization conditions (1.7) for F and $\bar{F}$ in (2.3). Thereby we obtain that $c \in C_i$ $(i = 0, 1, 4, 5)$. Hence C_i $(i = 0, 1, 4, 5)$ is the maximal subgroup of C with such a property. The equalities a),b),c),d) follow from (2.4), (1.5), (1.6) and Theorem 1.1.

In an analogical way, as in Theorem 2.2, we ascertain the truthfulness of the next

Theorem 2.3. *The classes $\mathcal{F}_i \oplus \mathcal{F}_j$ and $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$ are closed with respect to the transformations of the groups C_{ij} $(i, j = 1, 4, 5;\ i \neq j)$ and C, respectively. These groups are the maximal such subgroups of the contactly conformal group C. The corresponding basic 1-forms satisfy the equalities:*

a) $\bar{\theta} = \theta + 2ndv(\xi)\eta,$	$\bar{\theta}^* = \theta^* + 2ndu(\xi)\eta$	for $i = 4, j = 5;$
b) $\bar{\theta} = \theta + 2n[du \circ \varphi - dv \circ \varphi^2],$	$\bar{\theta}^* = \theta^* + 2n[du - dv \circ \varphi]$	for $i = 1, j = 5;$
c) $\bar{\theta} = \theta + 2n[du \circ \varphi + dv],$	$\bar{\theta}^* = \theta^* - 2n[du \circ \varphi^2 + dv \circ \varphi]$	for $i = 1, j = 4;$
d) $\bar{\theta} = \theta + 2n[du \circ \varphi + dv],$	$\bar{\theta}^* = \theta^* + 2n[du - dv \circ \varphi]$	for $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5$ and$C$.

Theorem 2.4. *The classes $\mathcal{F}_i$ and $\mathcal{F}_j$ are contactly conformally related with respect to the transformations of the group C^*_{ij} $(i, j = 1, 4, 5;\ i \neq j)$.*

The method of proof of Theorem 2.4 is the same as that of Theorem 2.2. We omit the details.

From Theorem 2.2, taking into account the definition conditions (1.12) of the subclasses $\mathcal{F}_i^0$ and (2.14), (2.15), (2.16) of the subgroups C_i^0 ($i = 1,4,5$, we ascertain the next

Theorem 2.5. *The class $\mathcal{F}_i^0$ is closed with respect to the transformations of the group C_i^0 ($i = 1,4,5$). This group is the maximal such subgroup of the contactly conformal group C.*

Since the class $\mathcal{F}_0$ is contained in each of $\mathcal{F}_i^0$, then Theorem 2.5 implies $C_i^0(\mathcal{F}_0) \subset \mathcal{F}_i^0$ ($i = 1,4,5$). The inverse inclusions are also valid. Thus we obtain characterizations of the classes $\mathcal{F}_1^0$, $\mathcal{F}_4^0$ and $\mathcal{F}_5^0$ stated in the following three theorems.

Theorem 2.6. *The class $\mathcal{F}_1^0$ is the class of the manifolds, which are contactly conformally equivalent to the $\mathcal{F}_0$-manifolds by the transformations of the group C_1^0, i.e. $\mathcal{F}_1^0 = C_1^0(\mathcal{F}_0)$.*

P r o o f. It is sufficient to prove that $\mathcal{F}_1^0 \subset C_1^0(\mathcal{F}_0)$. Let $(M,\varphi,\xi,\eta,g) \subset \mathcal{F}_1^0$, i.e. θ and θ^* are closed 1-forms on M. We consider the equation

$$du'(X) = -\frac{\theta^*(X)}{2n} \tag{2.20}$$

for $u' \in \mathcal{F}M$ and an arbitrary vector field X on M. Since $\theta^*(\xi) = 0$ for an $\mathcal{F}_1^0$-manifold then

$$du'(\xi) = 0. \tag{2.21}$$

Substituting φX for X into (2.20) and taking into account (1.6) and (1.5) we get

$$du'(\varphi X) = -\frac{\theta(X)}{2n}. \tag{2.22}$$

Solving locally the equation (2.20), we find the function u' satisfying the conditions (2.21) and (2.22). Since θ is a closed 1-form, then from (2.22) it appears that u' is a φ-pluriharmonic function.

Let c' be a contactly conformal transformation determined by the pair of functions $(u', v' = \text{const})$. According to (2.21) and (2.22), the transformation c' belongs to $C_1^{0'}$, but $c' \notin C_0$. Since $C_1^{0'} \subset C_1^0$ and according to (2.20), (2.22) and Theorem 2.2, it follows that the manifold $(M,\varphi,\xi,\eta,g') = c'(M,\varphi,\xi,\eta,g)$ belongs to $\mathcal{F}_0$.

Let $c'' \in C_0$ be determined by a φ-holomorphic pair of functions (u'', v''). Theorem 2.2 implies that $c''(M,\varphi,\xi,\eta,g') = (M,\varphi,\xi,\eta,\bar{g})$ belongs to $\mathcal{F}_0$. Since $C_0 \subset C_1^0$ and $c' \notin C_0$, then the transformation $c = c'' \circ c'$ belongs to C_1^0, but $c \notin C_1^{0'}$, i.e. $C_1^0(\mathcal{F}_1^0) \subset \mathcal{F}_0$. C_1^0 is a group and because of that from the last including, it follows $\mathcal{F}_1^0 \subset C_1^0(\mathcal{F}_0)$.

Theorem 2.7. *The class $\mathcal{F}_4^0$ is the class of the manifolds, which are contactly conformally equivalent to the $\mathcal{F}_0$-manifolds by the transformations of the group C_4^0, i.e. $\mathcal{F}_4^0 = C_4^0(\mathcal{F}_0)$.*

Proof. It is sufficient to prove that $\mathcal{F}_4^0 \subset C_4^0(\mathcal{F}_0)$. Let $(M,\varphi,\xi,\eta,g) \in \mathcal{F}_4^0$ and the function $v' \in \mathcal{F}M$ satisfies the equations

$$dv'(\xi) = -\frac{\theta(\xi)}{2n} \tag{2.23}$$

and

$$dv' \circ \varphi = 0. \tag{2.24}$$

Since $M \in \mathcal{F}_4^0$, then $\theta(X) = \eta(X)\theta(\xi)$. Therefore the conditions (2.23) and (2.24) are equivalent to

$$dv'(X) = -\frac{\theta(X)}{2n}. \tag{2.25}$$

Solving locally the equation (2.25), we find the function v'. Let c' be a contactly conformal transformation determined by the pair of functions (u' =const, v'). According to (2.23) and (2.24) we have $c' \in C_4^{0'}$, $c' \notin C_0$. Since $C_4^{0'} \subset C_4^0$, from the conditions (2.23), (2.24) and Theorem 2.2, it follows that the manifold $(M,\varphi,\xi,\eta,g') = c'(M,\varphi,\xi,\eta,g)$ belongs to $\mathcal{F}_0$. Let $c'' \in C_0$ be determined by the φ-holomorphic pair of functions (u'',v''). Theorem 2.2 implies that $c''(M,\varphi,\xi,\eta,g') = (M,\varphi,\xi,\eta,\bar{g})$ belongs to $\mathcal{F}_0$. Since $C_0 \subset C_4^0$ and $c' \notin C_0$, then the transformation $c = c'' \circ c'$ belongs to C_4^0, but $c \notin C_4^{0'}$, i.e. $C_4^0(\mathcal{F}_4^0) \subset \mathcal{F}_0$. C_4^0 is a group and because of that, from the last including, it is clear, that $\mathcal{F}_4^0 \subset C_4^0(\mathcal{F}_0)$.

By analogy with the proof of Theorem 2.7 (changing the roles of the functions u' and v') we ascertain the truthfulness of the next

Theorem 2.8. *The class $\mathcal{F}_5^0$ is the class of the manifolds, which are contactly conformally equivalent to the $\mathcal{F}_0$-manifolds by the transformations of the group C_5^0, i.e. $\mathcal{F}_5^0 = C_5^0(\mathcal{F}_0)$.*

Taking into account, that C_0 is a subgroup of C_i and C_i^0 $(i = 1,4,5)$, from Theorem 2.2, Theorem 2.5 and Theorem 2.1, we get that the classes $\mathcal{F}_i$ and $\mathcal{F}_i^0$ $(i = 1,4,5)$ are closed with respect to the transformations of the group C_0.

Having in mind (2.4), we establish that the basic 1-forms θ, θ^* and ω are invariant with respect to the group C_0. Taking into account (2.6) and (2.3), it is easy to verify that every class $\mathcal{F}_i$ $(i = 6,7,8,9,10,11)$ is closed with respect to the group C_0.

Let $C_{ij}^{*0} = \{c = c_2 \circ c_1 \mid c_1 \in C_i^0, c_2 \in C_j^0 \; (i,j = 1,4,5;\; i \neq j)\}$. It is clear, that every C_{ij}^{*0} $(i,j = 1,4,5;\; i \neq j)$ is a group. Then from Theorems 2.4, 2.6, 2.7, 2.8 it follows the next

Theorem 2.9. *The classes $\mathcal{F}_i^0$ and $\mathcal{F}_j^0$ are contactly conformally equivalent with respect to the group C_{ij}^{*0}, i.e. $\mathcal{F}_j^0 = C_{ij}^{*0}(\mathcal{F}_i^0)$ for every $i,j = 1,4,5;\; i \neq j$.*

3. Some examples of almost contact manifolds with B-metric. An example of an almost contact manifold with B-metric in the class $\mathcal{F}_0$ is given in [5].

Example 1. Let $R^{2n+1} = \{(x^1, x^2, \dots, x^n;\ y^1, y^2, \dots, y^n;\ t) \mid x^i, y^i, t \in R\}$.

The structure (φ, ξ, η, g) is defined on R^{2n+1} in the following way:

$$\xi = \frac{\partial}{\partial t}, \quad \eta = dt;$$

$$\varphi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad \varphi\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0;$$

$$g(z,z) = -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j + \nu^2,$$

where $z = \lambda^i \frac{\partial}{\partial x^i} + \mu^i \frac{\partial}{\partial y^i} + \nu \frac{\partial}{\partial t}$ and δ_{ij} are Kronecker's symbols. From this definition it follows that

$$g(\xi, z) = \eta(z), \quad g(\varphi z, \varphi z) = -g(z,z) + \eta(z)\eta(z)$$

for an arbitrary vector z.

If ∇ is the Levi-Civita connection of the metric g, it is easy to check that $\nabla\varphi = 0$.

Hence, $(R^{2n+1}, \varphi, \xi, \eta, g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_0$.

Using the results from Theorem 2.6, Theorem 2.7, Theorem 2.8 and Example 1, further we construct examples of almost contact manifolds with B-metric, belonging to the classes $\mathcal{F}_1^0$, $\mathcal{F}_4^0$ and $\mathcal{F}_5^0$, respectively.

Let $M \in \mathcal{F}_0$ and $u \in \mathcal{F}M$ be a φ-pluriharmonic function on M, i.e. the 1-form $du \circ \varphi$ is closed. Taking into account that $\nabla\varphi = 0$ and the condition $d(du \circ \varphi) = 0$, we obtain

$$(\nabla_x \sigma)\varphi y = (\nabla_y \sigma)\varphi x, \tag{3.1}$$

where $\sigma = du$. The equality (3.1) regarding a local basis $\left\{\frac{\partial}{\partial x^i}\right\}$ $(i = 1, 2, \dots, 2n+1)$ is expressed in the following way

$$\varphi_i^k \nabla_j \sigma_k = \varphi_j^k \nabla_i \sigma_k, \tag{3.2}$$

where φ_i^k are the local coordinates of the tensor φ and $\sigma_k = \sigma\left(\frac{\partial}{\partial x^k}\right)$, i.e. (3.2) is a necessary and sufficient condition for the function u to be φ-pluriharmonic.

Example 2. Let $(M{=}R^5, \varphi, \xi, \eta, g) \in \mathcal{F}_0$ be a manifold of the kind given in Example 1. Then regarding a local basis $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right\}$ $(i = 1, 2)$ the conditions (3.2) are equivalent to the equalities

$$\begin{aligned} \frac{\partial^2 u}{\partial x^1 \partial y^2} &= \frac{\partial^2 u}{\partial x^2 \partial y^1}, & \frac{\partial^2 u}{\partial x^2 \partial x^2} &= -\frac{\partial^2 u}{\partial y^2 \partial y^2}, \\ \frac{\partial^2 u}{\partial x^1 \partial x^1} &= -\frac{\partial^2 u}{\partial y^1 \partial y^1}, & \frac{\partial^2 u}{\partial x^1 \partial x^2} &= -\frac{\partial^2 u}{\partial y^1 \partial y^2}. \end{aligned} \tag{3.3}$$

We consider the function

$$u = \ln\Big[(x^1 + y^2)^2 + (x^2 - y^1)^2\Big] \quad \text{on} \quad R^5, \tag{3.4}$$

where $x^1 + y^2 \neq 0$ or $x^2 - y^1 \neq 0$. We verify directly, that the function u satisfies (3.3) and $du(\xi) = 0$, i.e. the function u, defined by (3.4), is φ-pluriharmonic. Hence, the contactly conformal transformation c determined by the pair of functions $(u, v = \text{const})$ belongs to the group $C_1^{0'} \subset C_1^0$, but $c \notin C_0$. Then Theorem 2.6 implies, that the almost contact manifold with B-metric $(R^5, \varphi, \xi, \eta, \bar{g}) = c(R^5, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_1^0$, but it does not belong to $\mathcal{F}_0$, since (u, v) is not a φ-holomorphic pair of functions.

Example 3. Let R^{2n+1} be supplied with an almost contact structure with B-metric in the class $\mathcal{F}_0$ as in Example 1. Let us consider the functions

$$u = \ln \sqrt{\prod_{i=1}^{n} [(x^i)^2 + (y^i)^2]}, \qquad v' = \sum_{i=1}^{n} \arctan \frac{x^i}{y^i}, \quad u, v' \in \mathcal{F}R^{2n+1}. \tag{3.5}$$

Using Definition 2.3 and the conditions $\varphi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$, $\varphi\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$, we check immediately, that the functions (3.5) are a φ-holomorphic pair. Let $f(t)$ be an arbitrary differentiable function on R^{2n+1} and $f'(t) \neq 0$. Hence $df = f'(t)\eta$. Then the pair of functions $(u,\ v = v' + f(t))$ satisfies the conditions in (2.8), moreover u is φ-pluricharmonic. The contactly conformal transformation determined by this pair of functions (u, v) belongs to the group C_4^0, but $c \notin C_0$. Theorem 2.4 implies that the manifold $c(R^{2n+1}, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_4^0$, but it does not belong to the class $\mathcal{F}_0$, since $\theta = 2ndv(\xi)\eta$ and $dv(\xi) = f'(t) \neq 0$.

Example 4. Let $(R^{2n+1}, \varphi, \xi, \eta, g)$ be the manifold of Example 3 and let u, v' be the functions defined by (3.5). The contactly conformal transformation

$$c : \bar{g} = e^{2u'} \cos 2v' g + e^{2u'} \sin 2v' \tilde{g} + (1 - e^{2u'} \cos 2v' - e^{2u'} \sin 2v')\eta \otimes \eta,$$

determined by the pair of functions $(u' = u + f(t), v')$, where the function $f(t)$ is given in Example 3, belongs to the group C_5^0. According to Theorem 2.8, R^{2n+1} together with the transformed structure $(\varphi, \xi, \eta, \bar{g})$ is a manifold in the class $\mathcal{F}_5^0$, but it does not belong to the class $\mathcal{F}_0$, since $\theta^* = 2ndu(\xi)\eta$ and $du(\xi) = f'(t) \neq 0$.

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