Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae mathematicae publicationes
20 (1994) 42-47

ON SOME COVER PROPERTIES

S. A. ANUFRIENKO

ABSTRACT. The present paper deals with some generalizations of paracompactness which can arise by using the concept of weakly discrete families of sets and related properties. These generalizations and their particularities are in contrast to those which come by varying the power of an open cover or local and point covering properties. In particular, some generalizations of paracompactness considered in the paper are not invariants of perfect mappings and do not imply metacompactness even if perfect normality is provided.

Throughout this paper a space will mean a T_1–space, i.e., every finite subset of the space is closed.

1. Some weaker forms of paracompactness. We will begin with the following definitions.

Definition 1.1. *A family $\gamma = \{U\}$ of sets in a space X is said to be weakly discrete if every set $\{x(U) \in U : U \in \gamma\}$ is a discrete set in X (see [1]). If the set $\cup\{F(U) \subseteq U : U \in \gamma\}$ is discrete in X for every finite (resp. discrete in X) $F(U)$, then we say that γ is f-family (resp. d-family).*

Definition 1.2. *A space X is said to be weakly–d–paracompact (resp. d–paracompact) if every open cover of X has a weakly discrete (resp. d–) refinement. By requiring that every open cover of X has an open weakly discrete refinement (resp. open f–refinement, open d–refinement) we introduce the class of weakly–f–paracompact (resp. f–paracompact, strong-f–paracompact) spaces. We will use abbreviations, e.g., wdp space for weakly–d–paracompact space.*

The d–families and the class of d–paracompact spaces were introduced by N. V. Veličko in [6].

Remark 1.3. Since a locally finite cover is a d–family, paracompactness implies sfp property implies dp property implies wdp property. And sfp implies fp implies

wfp implies wdp. In general, there are no other implications. We shall consider some special cases.

Definition 1.4. *A space X is called a q–space (see [4]) if every $x \in X$ has a sequence of neighbourhoods $\{U_i : i \in N\}$ such that if $x_i \in U_i$, for every $i \in N$, then $\{x_i : i \in N\}$ has a limit point in X.*

Remark 1.5. Let γ be an open weakly discrete cover of a q–space X. Let $A = \{x \in X :$ any neighbourhood of x meet infinitely many members of $\gamma\}$. Then every $x \in A$ is isolated in X and A is closed. Thus, $\gamma' = \{U \setminus A : U \in \gamma\} \cup \{\{x\} : x \in A\}$ is an open loccaly finite refinement.

Using Remark 1.3 and Remark 1.5 we have the following

Corollary 1.6. *A q–space is paracompact if and only if it is wfp.*

Thus, every weakly discrete family of sets in a Fréchet space X is closure–preserving. At the same time, there exists a regular space of countable tightness which is not paracompact (see Example 3.4.). In the realm of k–spaces we have the following

Proposition 1.7. *Every closed weakly discrete family in a k–space X is closure–preserving.*

Proof. Let $\mathcal{F} = \{F\}$ be a closed weakly discrete family which is not closure–preserving in a k–space X. Then, there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $\cup\mathcal{F}'$ is not closed in X, and thefore, is not closed in a compact $K \subseteq X$. Since every $F \in \mathcal{F}$ is closed, then for every finite $\mathcal{F}'' \subseteq \mathcal{F}'$, $(\cup(\mathcal{F}' \setminus \mathcal{F}'')) \cap K \neq \emptyset$. Thefore, we can choose infinite $\{x_i : i \in N\} \subseteq (\cup\mathcal{F}') \cap K$ such that distinct elements of the sequence belong to the distinct elements of $\mathcal{F}'$. $\{x_i : i \in N\}$ has a limit point, but $\mathcal{F}$ is weakly discrete. A contradiction.

We will show that the above generalizations and paracompactness are equivalent for every linearly ordered space. To prove it, some preliminary notations are needed. Let $(X, <)$ be an ordered set. A gap (A, B) in X is a left Q–gap if either $A = \emptyset$ or there exists an increasing transfinite sequence $x_0, x_1, \ldots, x_\xi, \ldots, \xi < \alpha$ of elements of A which is cofinal in A such that for every limit ordinal number $\lambda < \alpha$ the set $\{x_\xi : \xi < \lambda\}$ has no least upper bound in X; a right Q–gap in X is defined analogously. By a Q–gap in a linearly ordered set X we mean a gap in X which is both a left Q–gap and a right Q–gap. In [5] Gillman and Henriksen proved that a linearly ordered space X is paracompact if and only if every gap in X is a Q–gap.

Theorem 1.8. *Let X be a linearly ordered space. Then the following conditions are equivalent:*

1. *X is paracompact.*
2. *X is sfp.*

3. *X is fp.*
4. *X is wfp.*
5. *X is dp.*
6. *X is wdp.*

Proof. According to Remark 1.3. it is sufficient to prove that 6) $\Rightarrow$ 1). On the contrary, suppose that X is not paracompact. According to the result of Gillman and Henriksen there exists a gap (A, B) which is not left Q–gap (if the gap is not right Q–gap, the proof is the same). Since A is a closed subspace in X, A is *wdp* (it can be shown by standart argument). Let $\{x_\xi : \xi < \alpha\}$ be an increasing transfinite sequence which is a cofinal set in A. Let us remark that α is a limit ordinal, otherwise $\alpha = \beta + 1$ and $\{x_\beta\}$ is a cofinal set in A. Define $U_\xi = \{x \in A : x < x_\xi\}$. The open covering $\mathcal{U} = \{U_\xi\}$ has a weakly discrete refinement $\mathcal{V}$. Then $\{V_\xi : \xi < \alpha\}$, where $V_\xi = \cup\{V : V \in \mathcal{V}, \xi = \min\{\mu < \alpha : V \subseteq U_\mu\}\}$ is weakly discrete too and refines $\mathcal{U}$.

Prove that $W = \{\xi : \xi < \alpha, V_\xi \setminus \cup\{U_\mu : \mu < \xi\} \neq \emptyset\}\}$ is cofinal in α. Otherwise there is an ordinal number $\beta < \alpha$, such that for every $\xi \in W$ it follows that $\xi \leq \beta$ and $V_\mu \subseteq U_\beta$ for every $\mu < \alpha$. Therefore $x_\beta \notin \cup\{V_\xi : \xi < \alpha\}$. Thus, W is cofinal in α.

Choose an $y_\xi \in V_\xi \setminus \cup\{U_\mu : \mu < \xi\}$ for every $\xi \in W$. Obviously, $y_\xi < y_\mu$ for every $\xi < \mu$ and $\xi, \mu \in W$. Moreover, $\{y_\xi : \xi \in W\}$ is cofinal in A. To prove the latter, suppose, on the contrary, that there exists a x_μ, $\mu < \alpha$ such that $y_\xi < x_\mu$ for every $\xi \in W$. But then $\xi < \mu + 1$ for every $\xi \in W$. Since α is a limit ordinal and W is cofinal in α we obtain a contradiction.

Since (A, B) is not a left Q–gap, then $\{y_\xi : \xi \in W\}$ has a limit point. But $\{V : \xi < \alpha\}$ is a weakly discrete. Thus is a contradiction which completes the proof.

2. Invariance of the generalizations. In this section, we establish some results which distinguish the considered generalizations from the standart ones. To begin with let us mention one type of mappings. A mapping $f : X \to Y$ is said to be weakly closed if for every discrete A in X the image $f(A)$ is discrete in Y. These mappings were introduced by N. V. Veličko in [6]. He also proved that d–paracompactness is an invariant of weakly discrete mappings. Without essential changes, it can be proved that wd–paracompactness is an invariant of weakly discrete mappings too and that wf–paracompactness and sf–paracompactness are invariants of open weakly discrete mappings.

Let $TW(w_0 + 1)$ be the ordinal space $\{\alpha : \alpha \leq \omega_0\}$.

Lemma 2.1. *If X is sfp, but not metacompact, then $X \times TW(\omega_0 + 1)$ is not wfp.*

P r o o f. By hypothesis, X has an open cover $\gamma = \{U\}$ which possesses no point–finite refinement. Then $\gamma' = \{\pi_X^{-1}(U) : U \in \gamma\}$ has not any weakly discrete refinement. On the contrary, suppose γ' has a weakly discrete refinement $\gamma'' = \{V\}$. For every $x \in X$ the point (x, ω_0) belongs to finitely many members of γ''. But then $\{V \cap X \times \{\omega_0\} : V \in \gamma''\}$ is a point–finite refinement of γ. A contradiction.

Theorem 2.2. *An inverse image of a sfp (resp fp) space Y under any perfect mapping is a sfp (resp fp) space if and only if Y is metacompact.*

P r o o f. It follows from Lemma 2.1 that these properties are not inverse invariants of open perfect mappings, so that it is sufficient to prove that if $g : X Y$ is a perfect mapping onto a metacompact sfp space Y, then X is the same.

Consider an open cover $\{U_s : s \in S\}$ of the space X and for every $y \in Y$ choose a finite set $S(y) \subseteq S$ such that $g^{-1}(y) \subseteq \cup\{U_s : s \in S(y)\}$. Since g is closed, there exists a neighbourhood V_y of the point y such that $g^{-1}(y) \subseteq g^{-1}(V_y) \subseteq \cup\{U_s : s \in S(y)\}$. The open cover $\{V_y : y \in Y\}$ has an open point–finite d–refinement $\{W_t : t \in T\}$. The family $\{g^{-1}(W_t) : t \in T\}$ is an open cover of X and for every $t \in T$ there exists $y_t \in Y$ satisfying $g^{-1}(W_t) \subseteq g^{-1}(V_{y_t}) \subseteq \cup\{U_s : s \in S(y_t)\}$. We shall show that the family $\{g^{-1}(W_t) \cap U_s : t \in T, s \in S(y)\}$ is a d–family.

Choose an arbitrary discrete set $F_{t,s} \subseteq g^{-1}(W_t) \cap U_s$ for every $t \in T$ and $s \in S(y_t)$. The set $F_t = \cup\{F_{t,s} : s \in S(y_t)\}$ is a discrete set in X. Let us show that $\cup\{F_t : t \in T\}$ is a discrete set in X. On the contrary, suppose there exists $T' \subseteq T$ such that $A = \overline{\cup\{F_t : t \in T'\}} \setminus \cup\{F_t : t \in T\}$ is not empty. Choose an $x \in A$. Let us note that $g(F_t) \subseteq W_t$ and $g(F_t)$ is discrete for every $t \in T$. Since $\{W_t : t \in T\}$ is a d–family, there exists for every $t \in T'$ finite $F'_t \subseteq F_t$ such that $x \in \overline{\cup\{F'_t : t \in T'\}} \setminus \cup\{F'_t : t \in T'\}$ and $g(\cup\{F'_t : t \in T'\}) = g(x)$. But $g(x)$ belongs to finitely many members of $\{W_t : t \in T\}$. That assertion contradicts the condidtion that $g(x)$ is an image of infinitely many members of $\{F'_t : t \in T'\}$.

The proof is the same when Y is a metacompact fp space.

Corollary 2.3. *If X is a fp space which is an inverse image of sfp space Y under a perfect mapping, then X is sfp.*

P r o o f. Let us follow the proof of the above theorem. The open cover $\{g^{-1}(W_t) : t \in T\}$ has an open f–refinement $\{G_t : t \in T\}$. It is easy to check that $\{G_t \cap U_s : t \in T, s \in S(y_t)\}$ is a d–refinement of $\{U_s : s \in S\}$.

Corollary 2.4. *d–paracompactness is an inverse invariant of perfect mappings.*

P r o o f. Since a pairwise disjoint d–family $\{W_t : t \in T\}$ is a point–finite then the proof is the same as in the theorem above.

3. Examples.

Example 3.1. A perfectly normal metacompact fp space which is not collectionwise normal. Michael's subspace [3] of Bing's space [2] is metacompact and fp. To obtain a perfectly normal space satisgying these properties, one can use Bing's construction [2] to Michael's subspace.

Example 3.2. A perfectly normal fp space which is not metacompact. Replacing σ–product in Michael's space by Σ–product, we obtain the normal fp space which is not metacompact. Use Bing's construction again.

Example 3.3. A regular dp space which is not fp. N. V. Veličko [6] gives an example of a regular space which is an image of a paracompact space under a weakly discrete mapping. This regular space is dp, σ–sfp, but it is not wfp. The latter may be verified directly.

Example 3.4. A wfp regular space of countable tightness which is not fp. Let $D[\aleph_0]$ be a countable discrete space. Let $D[i,\alpha] = D[\aleph_0]$ for every $i \in N$ and $\alpha < \omega_1$. Let X be the set $\{q_i : i \in N\} \cup \{p_\alpha : \alpha < \omega_1\} \cup \oplus\{D[i,\alpha] : i \in N, \alpha < \omega_1\}$ equipped with the following topology. The sets $O_n(q_i) = \{q_i\} \cup (\cup\{D[i,\alpha] \setminus f[i,\alpha] : \alpha \in \omega_1 \setminus K\})$, where K is any finite set of ordinals and $|F[i,\alpha]| \leq n$ for every $\alpha \in \omega_1 \setminus K$, are neighbourhoods of q_i. The sets $O_n(p_\alpha) = \{p_\alpha\} = \{p_\alpha\} \cup (\cup\{D[i,\alpha] \setminus E[i,\alpha] : i \in N \setminus K\}$, where $K \subseteq N$ is any finite set and $|E[i,\alpha]| \leq n$ for every $i \in N \setminus K$, are neighbourhoods of p_i. Every open cover which contains a neighbourhood of q_i and p_α for every $i \in N$ and $\alpha < \omega_1$ and points not belonging to the union of the neighbourhoods above, is a weakly discrete family. Using the restrictions $|F[i,\alpha]| \leq n$ for every $\alpha \in \omega_1 \setminus K$ and $|E[i,\alpha]| \leq n$ for every $i \in N \setminus K$ one can easily check that X is a space of countable tightness. Finally, X is not normal. The proof of this fact is similar to the proof that the Tychonoff plane is not normal.

4. Questions. 1. Let X be a hereditarily collectionwise normal wdp space. Is X paracompact?

2. Is X paracompact if X is collectionwise normal and wdp?

3. Is X paracompact if X is sequential and wdp?

4. Are the wdp, fp and sfp properties invariants of perfect mappings?

REFERENCES

[1] A. V. Arhangel'skii, V. I. Ponomarev. Foundations of general topology through problems and exercises. Moscow, 1974.

[2] R. H. BING. Metrization of topological spaces. *Canad. J. Math.*, **3** (1951), 175-186.

[3] E. MICHAEL. Point–finite and locally finite coverings. *Canad. J. Math.*, **7** (1955), 255

[4] E. MICHAEL. A note on closed maps and compact sets. *Isr. J. Math.*, **2**(1964), 173-279.

[5] L. GILLMAN, M. HENRIKSEN. Concerning rings of continuous functions. *Trans. Amer. Math. Soc.*, **77** (1954), 340-362.

[6] N. V. VELIČKO. On continuous mappings of topological spaces. *Sib. Math. J.*, **13** (1973), 541-557.

S. A. Anufrienko
Dept of Math.
The Urals State Univ
620083 Ekaterinburg
pr. Lenina
RUSSIA
E-mail address:
postmaster@urgu.e-burg.su

Received 26.04.93