Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
20 (1994) 68-73

A COMPLETE SET OF UNIMODAL DISTRIBUTIONS

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ABSTRACT. The limiting behavior of scale mixtures of renewal distributions is investigated. It is shown that every member of the class J of these mixtures and every (0) unimodal distribution can be obtained as the limit of a sequence contained in J.

1. Introduction. A distribution function $F(x)$ is called unimodal at the point $x = 0$ – in short : (0) unimodal – if $F(x)$ is convex in the interval $(-\infty, 0)$ and concave in the interval $(0, \infty)$ [11].

Let $F(x)$ be a distribution function in $(0, \infty)$ with finite mean μ and characteristic function $\varphi(u)$. The distribution function having probability density function defined by

$$\frac{1 - F(x)}{\mu}$$

and characteristic function defined by

$$\frac{\varphi(u) - 1}{i\mu u}$$

is (0) unimodal and it is called the renewal distribution function corresponding to the distribution function $F(x)$. The renewal distribution has many interesting applications in stochastic processes [8], [10] and in transformations of characteristic functions [3].

A distribution function $G(x)$ with characteristic function $\gamma(u)$ given by

$$\gamma(u) = \int_{-\infty}^{\infty} \frac{\varphi(ux) - 1}{i\mu ux} dH(x),$$

where $H(x)$ is a distribution function, is called a scale mixture of the renewal distribution function or a member of class J [2], [5]. Distribution functions of class J are (0) unimodal [cf. Medgyessy [7], Theorem 13].

The paper is devoted to the study of the limiting behavior of certain sequences of class J.

2. The Results. It may be noted that the results given in this section take for granted that the reader is aware of the details concerning the "Dominated Convergence Theorem" [cf. Ash [4], Theorem 1.6.9]. We also need the following lemma [cf. Lukacs [6], Lemma 5.5.3].

Lemma 1. *Let $\psi(u)$ be an infinitely divisible characteristic function. Then*

$$\lim_{n\to\infty} n\{[\psi(u)]^{1/n} - 1\} = \log\psi(u), \ u \in R.$$

We illustrate that J is complete, by showing that every member of J is an accumulation point, that it can be obtained as the limit of a sequence of characteristic functions contained in J.

Theorem 1. *Let $\gamma(u)$ be a characteristic function of class J. Then there exists a sequence $\{\gamma_n(u) : n = 1, 2, \ldots\}$ of characteristic functions of class J with $\gamma_n(u) \neq \gamma(u), 1$ such that*

$$\lim_{n\to\infty} \gamma_n(u) = \gamma(u), \ u \in R.$$

Proof. Consider the sequence of Poisson-type, infinitely divisible characteristic functions

$$\varphi_n(u) = \exp\left[\frac{1}{n}[\varphi(u) - 1]\right], \ n = 1, 2, \ldots$$

where $\varphi(u)$ is the characteristic function of a distribution function $F(x)$ in $(0, \infty)$ with finite mean μ. Let

$$\psi_n(u) = \frac{\left\{\exp\left[\frac{1}{n}(\varphi(ux) - 1)\right] - 1\right\}}{i\mu ux/n}, n = 1, 2, \ldots$$

the renewal characteristic function which corresponds to $\varphi_n(u)$ and set

$$\gamma_{n(u)} = \int_{-\infty}^{\infty} \frac{\exp\left[\frac{1}{n}(\varphi(ux) - 1)\right] - 1}{i\mu ux/n} dH(x), \tag{2.1}$$

where $H(x)$ is a distribution function. Then $\{\gamma_n(u) : n = 1, 2, \ldots\}$ is a sequence of characteristic functions of class J. Letting

$$\psi(u) = \exp\{(\varphi(u) - 1)\}$$

in Lemma 1 we get

$$\lim_{n\to\infty} \frac{\left\{\exp\left[\frac{1}{n}(\varphi(ux)-1)\right]-1\right\}}{i\mu ux/n} = \frac{\varphi(u)-1}{i\mu u} \tag{2.2}$$

which is the renewal characteristic function corresponding to $\varphi(u)$. From (2.2) and the "Dominated Convergence Theorem" it follows that

$$\lim_{n\to\infty} \gamma_n(u) = \int_{-\infty}^{\infty} \frac{\varphi(ux)-1}{i\mu ux} dH(x) = \gamma(u), \quad u \in R.$$

Corollary 1. *Let $\beta(u)$ be the characteristic function of a distribution function $B(x)$ having a convex density $b(x)$ in the half lines $x > 0$ and $x < 0$. Then there exists a sequence $\{\gamma_n(u) : n = 1,2,\ldots\}$ of characteristic functions of class J with $\gamma_n(u) \neq \beta(u), 1$ such that*

$$\lim_{n\to\infty} \gamma_n(u) = \beta(u), \quad u \in R.$$

P r o o f. Consider the characteristic function

$$\varphi(u) = \frac{e^{iu}-1}{iu} \tag{2.3}$$

of the uniform distribution in $(0,1)$. Substituting from (2.3) into (2.1) we get the sequence

$$\gamma_n(u) = \int_{-\infty}^{\infty} \frac{\left\{\exp\left[\frac{1}{n}\left(\frac{e^{iux}-1}{iux}-1\right)\right]-1\right\}}{iux/2n} dH(x).$$

of characteristic functions of class J. From (2.2) with $\varphi(u)$ defined by (2.3) and the "Dominated Convergence Theorem" it follows that

$$\lim_{n\to\infty} \gamma_n(u) = \int_{-\infty}^{\infty} \frac{2(1+iux-e^{iux})}{u^2x^2} dH(x).$$

The last formula agrees with Sakovic integral representation for characteristic functions of distribution functions having convex densities in the half lines $x > 0$ and $x < 0$ [9].

Corollary 2. *Let $\gamma(u)$ be a characteristic function of class J. Then there exists a sequence $\{\gamma_n(u) : n = 1,2,\ldots\}$ of characteristic functions of class J with $\gamma_n(u) \neq \gamma(u), 1$ such that*

$$\lim_{n\to\infty} \gamma_n(u) = \int_0^1 \gamma(uy)dy, \quad u \in R.$$

Proof. Let $\varphi(u)$ be the characteristic function of a distribution function in $(0,\infty)$ with finite mean μ. Then

$$\varphi_n(u) = \exp\left\{\frac{1}{n}\int_0^\infty \frac{e^{iux}-1}{x}(1-F(x))dx\right\}$$
$$= \exp\left\{\frac{1}{n}\int_0^1 \frac{\varphi(uy)-1}{y}dy\right\}, \quad n = 1,2,\ldots$$

is the characteristic function of an infinitely divisible distribution function in $(0,\infty)$ [cf. Lukacs [6], Theorem 11.2.2]. Let

$$\psi_n(u) = \frac{\left\{\exp\left(\frac{1}{n}\int_0^1 \frac{\varphi(uy)-1}{y}dy\right)-1\right\}}{i\mu u/n}$$

the renewal characteristic function which corresponds to $\varphi_n(u)$ and set

$$\gamma_n(u) = \int_{-\infty}^{\infty} \frac{\left\{\exp\left(\frac{1}{n}\int_0^1 \frac{\varphi(uyx)-1}{y}dy\right)-1\right\}}{i\mu ux/n}dH(x),$$

where $H(x)$ is a distribution function. Then $\{\gamma_n(u) : n = 1,2,\ldots\}$ is a sequence of characteristic functions of class J. From (2.2) with $\psi(u)$ defined by

$$\psi(u) = \exp\left[\int_0^1 \frac{\varphi(uy)-1}{y}dy\right],$$

and the "Dominated Convergence Theorem" it follows that

$$\lim_{n\to\infty}\gamma_n(u) = \int_{-\infty}^{\infty}\left(\int_0^1 \frac{\varphi(uyx)-1}{i\mu uyx}dy\right)dH(x)$$
$$= \int_0^1\left(\int_{-\infty}^{\infty} \frac{\varphi(uyx)-1}{i\mu uyx}dH(x)\right)dy = \int_0^1 \gamma(uy)dy, \quad u \in R.$$

Theorem 2 extends this investigation of convergent sequences in J by showing that every unimodal distribution can be obtained as the limit of a sequence of distributions contained in J. Theorem 2 is an extension of Theorem 1 of [1] since the renewal distribution is (0) unimodal.

Theorem 2. *Let $\beta(u)$ be the characteristic function of a (0) unimodal distribution function $B(x)$. Then there exists a sequence $\{\gamma_n(u) : n = 1,2,\ldots\}$ of characteristic functions of class J with $\gamma_n(u) \neq \beta(u)$, 1 such that*

$$\lim_{n\to\infty}\gamma_n(u/n\mu) = \beta(u), \quad u \in R,$$

where μ is a positive constant.

Proof. The (0) unimodality of the distribution function $B(x)$ and the integral representation for characteristic functions of (0) unimodal distribution functions [cf. Lukacs [6], Theorem 4.5.1] imply that

$$\beta(u) = \int_0^1 \eta(uy)dy,$$

where $\eta(u)$ is the characteristic function of a distribution function $H(x)$. Let $\varphi(u)$ be the characteristic function of a distribution function $F(x)$ in $(0, \infty)$ with finite mean μ. Then

$$\psi_n(u)\frac{\varphi^n(u) - 1}{in\mu u}, \quad n = 1, 2, \ldots \tag{2.4}$$

is the renewal characteristic function which corresponds to the characteristic function $\varphi^n(u)$. Set

$$\gamma_n(u) = \int_{-\infty}^{\infty} \frac{\varphi^n(ux) - 1}{in\mu ux} dH(x), \quad n = 1, 2, \ldots$$

Then $\{\gamma_n(u) : n = 1, 2, \ldots\}$ is a sequence of characteristic functions of class J. From Lukacs [6], Theorem 2.3.3 it follows that

$$\varphi(u) = 1 + i\mu u + o(u), \text{ as } u \to 0. \tag{2.5}$$

Let u be fixed but arbitrary real number ($u \neq 0$), then from (2.4) and (2.5) it follows that

$$\lim_{n\to\infty} \psi_n(u/n\mu) = \frac{e^{iu} - 1}{iu}, \quad u \in R. \tag{2.6}$$

From (2.6) and the "Dominated Convergence Theorem" it follows that

$$\lim_{n\to\infty} \gamma_n(u/n\mu) = \int_{-\infty}^{\infty} \frac{e^{iux} - 1}{iux} dH(x) = \int_0^1 \eta(uy)dy = \beta(u), \quad u \in R.$$

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Receive 17.05.1993
Revised 24.09.1993