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SUPPORT MAXIMUM PRINCIPLE FOR TIME-DELAYED SYSTEMS WITH FUNCTIONAL RESTRICTIONS–II*

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2. Formula for the deviation of the criterion.

On the trajectories of the system

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_1x(t) + A_1x(t-h) + bu(t),\ t \in [0, t^* + h] = T, \\ &x(\tau) = x_0(\tau),\ \tau \in [-h, 0[\ , x(0) = x^0, \end{aligned} \tag{1}$$

the problem of maximization

$$J(u) = c^T x(t^* + h) \to \max \tag{2}$$

stated in [1] is considered. Every piece-wise continuous function $u(t)$, $t \in T$, will be called an admissible control if

$$|u(t)| \le 1,\ t \in T, \tag{3}$$

and if for the corresponding trajectory of (1) the following restriction is satisfied:

$$d^T x(t) = y,\ t \in T^* = [t^*, t^* + h]. \tag{4}$$

Let us consider the deviation of the criterion for two admissible controls

$$\Delta J(u) = c^T \Delta x(t^* + h) = \int_0^{t^*+h} c^T F(t^* + h, t) b \Delta u(t) dt. \tag{5}$$

*Continuation of Serdica **19** (1993), 243-257

Then for the deviation $\Delta x(t)$ of the trajectory corresponding to $\Delta u(t)$ it follows:

$$
(6)\quad \begin{array}{lll}
d^T\Delta x(t) \equiv 0, & t \in \operatorname{int} T^*, & \\
d^T\Delta x^{(p)}(\mu_i + 0) = 0, & p = \overline{0,k_i} & i \in I^{+0}, \\
d^T\Delta x^{(p)}(\mu_i - 0) = 0, & p = \overline{0,k_{i-1}} & i \in I^{-0}, \\
d^T\Delta x^{(p)}(\tau_{ij}) = 0, & p = \overline{0,k_i+1} & j = \overline{1,s_i}, \quad i = \overline{0,\rho},
\end{array}
$$

where

$$
I^{+0} = \{i \in I^+ : \mu_i \neq \tau_{i1}\}, \; I^{-0} = \{i \in I^- \cup (\rho+1) : \mu_i \neq \tau_{i-1,s_{i-1}}\},
$$

or according to (1.15)*, we have:

$$
(7)\quad \begin{array}{l}
\sum\limits_{s=0}^{p} d_s^T(p)\Delta x(\mu_i - sh) + \sum\limits_{l=0}^{p-1}\sum\limits_{s=0}^{p-l-1} d_s^T(p-l-1)b\Delta u^{(l)}(\mu_i - sh + 0) = 0, \\
\qquad p = \overline{0,k_i}, \quad i \in I^{+0}; \\
\sum\limits_{s=0}^{p} d_s^T(p)\Delta x(\mu_i - sh) + \sum\limits_{l=0}^{p-1}\sum\limits_{s=0}^{p-l-1} d_s^T(p-l-1)b\Delta u^{(l)}(\mu_i - sh - 0) = 0, \\
\qquad p = \overline{0,k_{i-1}}, \quad i \in I^{-0};
\end{array}
$$

$$
(8)\quad \begin{array}{l}
\sum\limits_{s=0}^{p} d_s^T(p)\Delta x(\tau_{ij} - sh) + \sum\limits_{l=0}^{p-1}\sum\limits_{s=0}^{p-l-1} d_s^T(p-l-1)b\Delta u^{(l)}(\tau_{ij} - sh) = 0, \\
\qquad p = \overline{0,k_i+1}, \quad j = \overline{1,s_i}, \quad i = \overline{0,\rho}.
\end{array}
$$

Let us multiply the equalities (6)–(8) by function $\xi(t)$, $t \in T^*$, $\xi(t) \equiv 0$, $t \notin T^*$, and numbers v_i^p, $p = \overline{0,k_i}$, $i \in I^{+0}$, $p = \overline{0,k_{i-1}}$, $i \in I^{-0}$, v_{ij}^p, $p = \overline{0,k_{i+1}}$, $j = \overline{1,s_i}$, respectively $i = \overline{0,\rho}$. Let us sum up the results and add them to the right part of (5).

*Here and further on by (1.15) we denote formula (15) from part I in [1], and by (2.5) – formula (5) from part II of this paper.

We obtain:

$$\Delta J(u) = \int_0^{t^*+h} c^T F(t^*+h,t) b\Delta u(t)dt + \int_{t^*}^{t^*+h} \xi(t) d^T \Delta x(t)dt + \sum_{i\in I^{+0}} \sum_{p=0}^{k_i} v_i^p$$

$$\cdot \left(\sum_{s=0}^{p} d_s^T(p)\Delta x(\mu_i - sh) + \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b\Delta u^{(l)}(\mu_i - sh + 0) \right)$$

$$+ \sum_{i\in I^{-0}} \sum_{p=0}^{k_i} v_i^p \left(\sum_{s=0}^{p} d_s^T(p)\Delta x(\mu_i - sh) + \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b\Delta u^{(l)}(\mu_i - sh - 0) \right)$$

$$+ \sum_{i=0}^{\rho} \sum_{j=1}^{s_i} \sum_{p=0}^{k_i+1} v_{ij}^p \left(\sum_{s=0}^{p} d_s^T(p)\Delta x(\tau_{ij} - sh) + \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b\Delta u^{(l)}(\tau_{ij} - sh) \right).$$

Let $m_i = \max\{k_i, k_{i-1}\}$, $I^0 = I^{+0} \cup I^{-0}$. Then by using the Cauchy formula we obtain

$$\Delta J(u) = \int_0^{t^*+h} c^T F(t^*+h,t) b\Delta u(t)dt + \int_0^{t^*+h} \left(\int_{t^*}^{t^*+h} \xi(\tau) d^T F(\tau,t) d\tau \right) b\Delta u(t)dt$$

$$+ \int_{t^*}^{t^*+h} \left(\int_{t}^{t^*+h} \xi(\tau) d^T F(\tau,t) d\tau \right) b\Delta u(t)dt + \sum_{i\in I^0} \sum_{p=0}^{m_i} v_i^p \sum_{s=0}^{p} \int_0^{\mu_i - sh} d_s^T(p) F(\mu_i - sh, t)$$

$$\cdot b\Delta u(t)dt + \sum_{i=0}^{\rho} \sum_{j=1}^{s_i} \sum_{p=0}^{k_i+1} v_{ij}^p \sum_{s=0}^{p} \int_0^{\tau_{ij}-sh} d_s^T(p) F(\tau_{ij} - sh, t) b\Delta u(t)dt + \sum_{i\in I^{+0}} \sum_{p=0}^{k_i} v_i^p$$

$$\cdot \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b\Delta u^{(l)}(\mu_i - sh + 0) + \sum_{i\in I^{-0}} \sum_{p=0}^{k_{i-1}} v_i^p \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b$$

$$\cdot \Delta u^{(l)}(\mu_i - sh - 0) \sum_{i=0}^{\rho} \sum_{j=1}^{s} \sum_{p=0}^{k_i+1} v_{ij}^p \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} d_s^T(p-l-1) b\Delta u^{(l)}(\tau_{ij} - sh).$$

Since $F(\tau,t) \equiv 0$, $t > \tau$, then if we introduce a function

$$
\begin{aligned}
\Psi^T(t) &= c^T F(t^* + h, t) + \int_{t^*}^{t^*+h} \xi(\tau) d^T F(\tau, t) d\tau \\
&+ \sum_{i \in I^0} \sum_{p=0}^{m_i} \sum_{s=0}^{p} v_i^p d_s^T(p) F(\mu_i - sh, t) \\
&+ \sum_{i=0}^{\rho} \sum_{j=1}^{s_i} \sum_{p=0}^{k_i+1} \sum_{s=0}^{p} v_{ij}^p d_s^T(p) F(\tau_{ij} - sh, t), \quad t \in T,
\end{aligned} \tag{9}
$$

we will get:

$$
\begin{aligned}
\Delta J(u) &= \int_0^{t^*+h} \Psi^T(t) b \Delta u(t) dt \\
&+ \sum_{i \in I^{+0}} \sum_{p=0}^{k_i} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_i^p d_s^T(p-l-1) b \Delta u^{(l)}(\mu_i - sh + 0) \\
&+ \sum_{i \in I^{-0}} \sum_{p=0}^{k_{i-1}} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_i^p d_s^T(p-l-1) b \Delta u^{(l)}(\mu_i - sh - 0) \\
&+ \sum_{i=0}^{\rho} \sum_{j=1}^{s_i} \sum_{p=0}^{k_{i+1}} \sum_{l=0}^{p-1} \sum_{s=0}^{p-l-1} v_{ij}^p d_s^T(p-l-1) b \Delta u^{(l)}(\tau_{ij} - sh).
\end{aligned} \tag{10}
$$

Formula (10) is the final form of the deviation of the criterion.

Starting from Definition (9) of the function $\Psi(t)$, $t \in T$, we obtain a differential equations system, whose solution is $\Psi(t)$. Let:

$$
\widetilde{v}_{\rho+1}^p = \begin{cases} v_{\rho+1}^p, & p = \overline{0, k_\rho} \\ 0, & p = k_\rho + 1, \ \rho + 1 \in I^{-0}; \\ v_{\rho, s_\rho}^p, & p = \overline{0, k_\rho + 1}, \ \rho + 1 \notin I^{-0}; \end{cases} \tag{11}
$$

$$
\widetilde{v}_0^p = \begin{cases} v_0^p, & p = \overline{0, k_0} \\ 0, & p = k_0 + 1, \ 0 \in I^{+0}; \\ v_{0,1}^p, & p = \overline{0, k_0 + 1}, \ 0 \notin I^{+0}. \end{cases} \tag{12}
$$

Then from (9) we obtain the system

$$(13)\qquad \begin{aligned} \dot{\Psi}(t) &= -A_0^T\Psi(t) - A_1^T\Psi(t+h) + \xi(t)d, \quad t \in T, \\ \xi(t) &\equiv 0, \quad t \notin T^*, \quad \Psi(t) \equiv 0, \quad t > t^* + h, \end{aligned}$$

with a final condition

$$(14)\qquad \Psi(t^* + h - 0) = c + \sum_{p=0}^{k_\rho+1} \widetilde{v}_{\rho+1}^p d_0(p),$$

and jumps

$$(15)\qquad \begin{aligned} &\Psi(t^* - sh - 0) = \Psi(t^* - sh + 0) + \sum_{p=s}^{k_\rho+1} \widetilde{v}_0^p d_s(p) \\ &+ \sum_{p=s+1}^{k_\rho+1} \widetilde{v}_{\rho+1}^p d_{s+1}(p), \quad s = \overline{0, \max\{k_0+1, k_\rho+1\}}, \end{aligned}$$

$$(16)\quad \Psi(\mu_i - sh - 0) = \Psi(\mu_i - sh + 0) + \sum_{p=s}^{m_i} v_i^p d_s(p), \quad i = I^0 \setminus \{0, \rho+1\}, \quad s = \overline{0, m_i},$$

$$(17)\qquad \begin{aligned} &\Psi(\tau_{ij} - sh - 0) = \Psi(\tau_{ij} - sh + 0) + \textstyle\sum_{p=s}^{k_i+1} v_{ij}^p d_s(p), \\ &i = \overline{0,\rho}, \quad j = \overline{1, s_i}, \quad s = \overline{0, k_i + 1}, \quad \tau_{0,1} \neq t^*, \quad \tau_{\rho, s_\rho} \neq t^* + h. \end{aligned}$$

Definition. *The system* (13) – (17) *is called a conjugate system of problem* (1) – (4), *and the function* $\Psi(t)$, $t \in T$ *– a support cotrajectory. The scalar product* $\Delta(t) = \Psi^T(t)b$, $t \in T$, *is called co-control.*

Lemma. *If there exists a support of problem* (1) – (4) [1], *then there exist an unique set of numbers* v_i^p, $i \in I^0$, $p = \overline{0, m_i}$, v_{ij}^p, $i = \overline{0,\rho}$, $j = \overline{1, s_i}$, $p = \overline{0, k_i+1}$, *and a function* $\xi(t)$, $t \in T^*$, $\xi(t) \equiv 0$, $t \notin T^*$, *such that the solution of system* (13)–(17) *satisfies the conditions:*

$$\Delta(t) = \Psi^T(t)b \equiv 0, \ t \in \operatorname{int} T_i^{k_i}, \ i = \overline{0,\rho}, \quad \Delta(t_k) = \Psi^T(t_k)b = 0, \ k \in K_0.$$

The proof can be made by deriving the formula for the deviation of the criterion in another way. We omit the details.

3. Support maximum principle.

Definition. *The support control* [1] $\{u, S_{sup}\}$ *(in* [1] $\{u, S_{on}\}$*) is called non-degenerate, if*

$$|\lim_{\tau\to t\pm 0} u(\tau)| \neq 1 \quad \text{when } t \in \widetilde{T}_i^{k_i} \setminus \cup_{j=1}^{s_i} \tau_{ij}^{k_i},\ i = \overline{0,\rho}; \tag{1}$$

$$|(u(t_k+0) + u(t_k-0))/2| \neq 1,\ k \in K_0; \tag{2}$$

$$\dot{u}(\mu_i^{k_i}+0) \neq 0 \quad \text{when } i \in I^{-+} \cup I^{--} \text{ and } |u(\mu_i^{k_i}+0)| = 1, \tag{3}$$

$$\dot{u}(\mu_{i+1}^{k_i}-0) \neq 0 \quad \text{when } i \in I^{+-} \cup I^{--} \text{ and } |u(\mu_{i+1}^{k_i}-0)| = 1. \tag{4}$$

Let $\{u, S_{sup}\}$ be support control, $\Delta(t)$, $t \in T$, v_i^p, $p = \overline{0, m_i}$, $i \in I^0$; v_{ij}^p, $p = \overline{0, k_i+1}$, $j = \overline{1, s_i}$, $i = \overline{0,\rho}$, – the co-control and jumps corresponding to it.

Theorem (Optimality criterion). *For optimality of the admissible control* $u(t)$, $t \in T$, *it is sufficient, and in case of a non-degenerate support control* $\{u, S_{sup}\}$ *it is also necessary that*

$$\begin{cases} \Delta(t) \geq 0 & \text{for} \quad u(t) = 1, \\ \Delta(t) \leq 0 & \text{for} \quad u(t) = -1, \\ \Delta(t) = 0 & \text{for} \quad |u(t)| < 1,\ t \in T_n\ (T_n \text{ in } [1]), \end{cases} \tag{5}$$

$$\begin{cases} v_i^1 d^T b \geq 0 & \text{for} \quad u(\mu_i) = 1, \\ v_i^1 d^T b \leq 0 & \text{for} \quad u(\mu_i) = -1, \\ v_i^1 = 0 & \text{for} \quad |u(\mu_i)| < 1,\ i \in I^0; \end{cases} \tag{6}$$

$$v_i^p = 0, p = \overline{2, m_i}, \quad i \in I^0; \tag{7}$$

$$\begin{cases} v_{ij}^p d^T b \geq 0 & \text{for} \quad u(\tau_{ij}) = 1, \\ v_i^p d^T b \leq 0 & \text{for} \quad u(\tau_{ij}) = -1, \\ v_i^p = 0 & \text{for} \quad |u(\tau_{ij})| < 1,; \end{cases} \tag{8}$$

$$v_{ij}^p = 0,\ p = \overline{2, k_i+1},\ j = \overline{1, s_i},\ i = \overline{0,\rho}.$$

Proof. The sufficiency follows straight from formula (2.10) for the deviation.

Necessity. For the sake of simplicity we will give the proof for the case when $k_* = 2$ ($k_* \le 2$) and

$$\{t_k, k \in K_0\} \cap \left\{\mu_i^{k_i}, \mu_{i+1}^{k_i}, i = \overline{0,\rho}\right\} = \emptyset.$$

Let $I_1 = \{i \in \{0,1,\ldots,p+1\} : m_i = 1\}$, $I_2 = \{i \in \{0,1,\ldots,p+1\} : m_i = 2\}$ if t_k is a point of discontinuity of the control; $\gamma_k = \min\{1 - u(t_k), 1 + u(t_k)\}/2$ otherwise, $k \in K_0$.

First let us consider the simple case, when

$$s_i = 0,\ i = \overline{0,\rho}. \tag{9}$$

When $k_* \le 2$ and conditions (9) hold, formula (2.10) for the deviation is presented in the form:

$$\begin{aligned}\Delta J(u) = &\int_T \Delta(t)\Delta u(t)dt + \sum_{i\in I_1} v_i^1 d_0^T(0)b\Delta u(\mu_i)\\ &+\sum_{i\in I_2}\Big(\Delta u(\mu_i)\sum_{p=1}^{2} d_0^T(p-1)v_i^p + \Delta u(\mu_i - h)d_1^T(1)bv_i^2 + \Delta u^{(1)}(\mu_i)d_0^T(0)bv_i^2\Big).\end{aligned} \tag{10}$$

A) We will prove equalities (7) by supposing that the contrary holds. Assume that there exists an index $i \in I_2$, such that $v_{i_0}^2 d_0^T(0)b = v_{i_0}^2 d^T b > 0$. Since $i \in I_2$, then $k_{i_0} = 2$, $k_{i_0-1} \le 1$ (or $k_{i_0} \le 1$, $k_{i_0-1} = 2$). Let $k_{i_0-1} = 1$. The following cases are possible: a) $u(\mu_{i_0}) < 1$; b) $u(\mu_{i_0} = 1$.

Let us consider case a). Denote $\Theta = (\Theta_k, k \in K_0)$ and define the function $\Delta u^{\Theta\varepsilon}(t)$, $t \in T \setminus T_{sup}$ (T_{sup} is T_{on} in [1]).

$$\Delta u^{\Theta\varepsilon}(t) = \begin{cases} (t - \mu_{i_0} + \varepsilon)^2,\ t \in [\mu_{i_0} - \varepsilon, \mu_{i_0}[,\\ -(t - \mu_{i_0} - \varepsilon)^2 + 2\varepsilon^2,\ t \in [\mu_{i_0}, \mu_{i_0} + 2\varepsilon[,\\ (t - \mu_{i_0} - 3\varepsilon)^2,\ t \in [\mu_{i_0} + 2\varepsilon, \mu_{i_0} + 3\varepsilon],\\ 0,\ t \in [\mu_{i_0-1}, \mu_{i_0+1}] \setminus [\mu_{i_0} - \varepsilon, \mu_{i_0} + 3\varepsilon]; \end{cases} \tag{11}$$

$$\Delta u^{\Theta\varepsilon}(t) = 0,\quad t \in T_{i_0}^1; \tag{12}$$

$$\Delta u^{\Theta\varepsilon}(t) = 0,\ t \in T_i^k,\ k = \overline{0, k_i - 1},\ i \ne i_0,\ i \ne i_0 - 1,\ i = \overline{0,\rho}; \tag{13}$$

$$\Delta u^{\theta\varepsilon}(t) = \begin{cases} \gamma_k, & t \in [t_k, t_k + \Theta_k] \quad \text{when } \Theta_k \ge 0\\ -\gamma_k, & t \in [t_k + \Theta_k, t_k] \quad \text{when } \Theta_k < 0,\ k \in K_0;\\ 0 & \text{otherwise} \qquad t \in T_{nn}\ (T_{nn} \text{ in } T_{HH} \text{ in [1]}). \end{cases} \tag{14}$$

From (11) – (13) and (18) we get:

$$\begin{aligned}
\overline{g}^{\Theta\varepsilon}_{k_i+1}(t) &\equiv 0, \quad i=\overline{0,\rho}, \quad i\neq i_0, \quad i\neq i_0-1;\\
\overline{g}^{\Theta\varepsilon}_{k_{i_0}+1}(t) &= \sum_{j=1}^{2}\sum_{s=0}^{2-j} d_s^T(2-j)\Delta u^{\Theta\varepsilon(j)}(t-sh)+\sum_{s=0}^{1} d_s^T(2)b\Delta u^{\Theta\varepsilon}(t-sh)\\
&= \sum_{j=1}^{2} d_0^T(2-j)b\Delta u^{\Theta\varepsilon(j)}(t)+d_0^T(2)b\Delta u^{\Theta\varepsilon}(t)\\
&= d_0^T(2)b\Delta u^{\Theta\varepsilon}(t)+d_0^T(1)b\Delta \dot{u}^{\Theta\varepsilon}(t)+d_0^T(0)b\Delta \ddot{u}^{\Theta\varepsilon}(t)\\
&= \begin{cases} d_0^T(2)b[-(t-\mu_{i_0}-\varepsilon)^2+2\varepsilon^2]+d_0^T(1)b[-2(t-\mu_{i_0}-\varepsilon)] \\ \qquad -2d_0^T(0)b, \ t\in[\mu_{i_0},\mu_{i_0}+2\varepsilon[, \\ d_0^T(2)b(t-\mu_{i_0}-3\varepsilon)^2+d_0^T(1)b2(t-\mu_{i_0}-3\varepsilon)+2d_0^T(0)b, \\ \qquad t\in[\mu_{i_0}+2\varepsilon,\mu_{i_0}+3\varepsilon[, \\ 0, \ t\in[\mu_{i_0}+3\varepsilon,\mu_{i_0+1}], \end{cases}\\
\overline{g}^{\Theta\varepsilon}_{k_{i_0-1}+1}(t) &= d_0^T(2)b\Delta\dot{u}^{\Theta\varepsilon}+d_0^T(1)b\Delta u^{\Theta\varepsilon}(t)\\
&= \begin{cases} 0, \quad t\in T_{i_0-1}\setminus[\mu_{i_0}-\varepsilon,\mu_{i_0}], \\ d_0^T(1)b(t-\mu_{i_0}+\varepsilon)^2+2d_0^T(0)b(t-\mu_{i_0}+\varepsilon), \ t\in[\mu_{i_0}-\varepsilon,\mu_{i_0}]. \end{cases}
\end{aligned}$$

Let us choose arbitrary n–vectors z_0, z_1 ($k_*-1=1$) and let us consider the equations (see (1.50))

$$\begin{aligned}
\varphi_p(\Theta,\varepsilon,z_0,z_1) &= G_{p+1}[z^0(t^*+h)+\sum_{k\in K_0}\int_{t_k}^{t_k+\Theta_k}\Omega(t^*+h,\tau)b\gamma_k d\tau\\
&\quad +\sum_{s=0}^{k_*-1}\Omega(t^*+h,t^*-sh)z_s]-z_p=0, \ p=\overline{0,k_*-1};\\
\varphi_{pi}(\Theta,\varepsilon,z_0,z_1) &= r_p^T[z^0(\mu_i)+\sum_{k\in K_0}\int_{t_k}^{t_k+\Theta_k}\Omega(t^*+h,\tau)b\gamma_k d\tau\\
&\quad +\sum_{s=0}^{k_*-1}\Omega(\mu_i,t^*-sh)z_s]=\eta_{ip}, \ i=\overline{0,\rho}, \ p\in S_i.
\end{aligned} \tag{15}$$

where $z^0(\mu_i)$, $i=\overline{0,\rho}$, $z^0(t^*+h)$ are obtained from (1.25), (1.46), (1.44), and η_{ip} – from (1.16), (1.41) by using the functions $\Delta u^{\Theta,\varepsilon}$, $t\in T\backslash T_{\sup}$, $\overline{g}^{\Theta\varepsilon(t)}_{k_i}$, $i=\overline{0,g}$, expressed above, and functions $g_p(\mu_i)=0$, $p\in S_i$, $i=\overline{0,\rho}$, $i\neq i_0$, $g_2(\mu_{i0})=d_0^T(1)b\varepsilon^2+2d_0^T(0)b\varepsilon$ – from (1.16) ($S_{i_0}=\{2\}$).

The function $\varphi_p(\Theta,\varepsilon,z_0,z_1)$, $p=\overline{0,k_*-1}$; $\varphi_{pi}(\Theta,\varepsilon,z_0,z_1)$, $p\in S_i$, $i=\overline{0,\rho}$, are continuous and

$$\varphi_p(0,0,0,0)=0,\ p=\overline{0,k_*-1};\ \varphi_{pi}(0,0,0,0)=0,\ p\in S_i,\ i=\overline{0,\rho};$$

$$\text{(16)}\qquad \det\begin{bmatrix} \dfrac{\partial\varphi_p}{\partial\Theta}, & \dfrac{\partial\varphi_p}{\partial z_0}, & \dfrac{\partial\varphi_p}{\partial z_1} \\ & & p=\overline{0,k_*-1} \\ \dfrac{\partial\varphi_{pi}}{\partial\Theta}, & \dfrac{\partial\varphi_{pi}}{\partial z_0}, & \dfrac{\partial\varphi_{pi}}{\partial z_1} \\ & p\in S_i, & i=\overline{0,\rho} \end{bmatrix}\Bigg|_{\substack{\Theta=0,\ \varepsilon=0\\ z_0=z_1=0}} = \prod_{k\in K_0}\gamma_k \det P_{\sup}\neq 0$$

($P_{\sup}$ is P_{on} in [1]).
According to the theorem for implicit functions there exists functions:

$$\text{(17)}\qquad \Theta_k=\Theta_k(\varepsilon),\quad k\in K_0;\quad z_0=z_0(\varepsilon),\quad z_1=z_1(\varepsilon),$$

such that the following identities hold

$$\text{(18)}\qquad \begin{aligned} &\varphi_p(\Theta(\varepsilon),\varepsilon,z_0(\varepsilon),z_1(\varepsilon))\equiv 0,\quad p=\overline{0,k_*-1};\\ &\varphi_{pi}(\Theta(\varepsilon),\varepsilon,z_0(\varepsilon),z_1(\varepsilon))\equiv 0,\quad p\in S_i,\quad i=\overline{0,\rho}; \end{aligned}$$

when $\varepsilon\geq 0$ is sufficiently small.

It can be shown that the functions (17) are of order ε. Using (18), we can conclude, that functions $f(t)=0$, $t\in T^*$; $\Delta u^{\Theta(\varepsilon)\varepsilon}(t)$, $t\in T\backslash T_{\text{sup}}$ and vectors $z_0(\varepsilon)$, $z_1(\varepsilon)$ satisfy conditions (1)–(4) from [1] and relations (1.50) when $\varepsilon\geq 0$ is sufficiently small. Let us denote by $z^\varepsilon(t)$, $t\in T^*$, the solution of system (1.34) corresponding to them and let us define the function (1.55).

$$\text{(19)}\qquad \begin{aligned} \Delta u^\varepsilon(t-k_ih) &= \overline{f}_i(t)-\frac{1}{\alpha_i}r^T_{k_i+1}z^\varepsilon(t),\ t\in T_i,\ i\in I\backslash I_{k_*};\\ \Delta u^\varepsilon(t-k_ih) &= \overline{f}_i(t)-\frac{1}{\alpha_i}\overline{r}_{k_*+1}z^\varepsilon(t)-\frac{1}{\alpha_i}\int_0^{t-(k_*+1)h} d^T_{k_*+1}(k_*+1)\\ &\quad\cdot F(t-(k_*+1)h,\tau)b\Delta u^\varepsilon(\tau)d\tau,\ t\in T_i,\ i\in I_{k_*}. \end{aligned}$$

The control $\Delta u^\varepsilon(t)$ and the corresponding trajectory $\Delta x^\varepsilon(t)$ of the system (1.11) satisfy the condition

$$\text{(20)}\qquad d^T\Delta x^\varepsilon(t)\equiv 0,\quad t\in T^*.$$

According to [1], the control $\Delta u^\varepsilon(t)$, $t\in T$ is continuous at the points $\mu_i^{k_i}$, when $i\in I^{-+}\cup I^{--}$ and at the points $\mu_{i+1}^{k_i}$, if $i\in I^{+-}\cup I^{--}$.

We shall prove that for every sufficiently small $\varepsilon > 0$ there exists a number σ, $0 < \sigma = \sigma(\varepsilon) \le 1$ such that the control $u^\varepsilon(t) = u(t) + \sigma\Delta u^\varepsilon(t)$, $t \in T$, is admissible in problem (2.1)–(2.4). Since (20) holds, then it is sufficient to show that when $\varepsilon > 0$ is sufficiently small, there exists $\sigma = \sigma(\varepsilon) > 0$ such that the following inequalities hold:

$$|u(t) + \sigma\Delta u^\varepsilon(t)| \le 1, \quad t \in T. \tag{21}$$

By definition (see (11)–(14), (19)) $\Delta u^\varepsilon(t) = 0(\varepsilon)$, $t \in T_{\text{sup}}$ $|\Delta u^\varepsilon(t)| \le 1$, $t \in T \backslash T\text{sup}$. That is why without loss of generality we can think that $|\Delta u^\varepsilon(t)| \le 1$, $t \in T$.

The support control $\{u, S_{\text{sup}}\}$ non-degenerate, the control $u(t)$, $t \in T$, is continuous for $t \in T^*$ and $u(\mu_{i_0}) < 1$. Therefore, for moments $t \in T \backslash T_{\text{sup}} = T_{\text{nn}} \cup T_0$ the inequalities (21) hold for every σ, $0 < \sigma \le 1$, and for a sufficiently small $\varepsilon > 0$. Consider the moments $t \in T_{\text{sup}} = \bigcup\limits_{i=0}^{\rho} T_i^{k_i}$ and calculate steps

$$\sigma_i = \sigma_i(\varepsilon) = \min \sigma^\varepsilon(t), \quad t \in T_i^{k_i}, \quad i = \overline{0, \rho} \tag{22}$$

where

$$\sigma^\varepsilon(t) = \begin{cases} (1 - u(t))/\Delta u^\varepsilon(t) & \text{when} \quad \Delta u^\varepsilon(t) > 0, \\ (-1 - u(t))/\Delta u^\varepsilon(t) & \text{when} \quad \Delta u^\varepsilon(t) < 0, \\ \infty & \text{when} \quad \Delta u^\varepsilon(t) = 0, \quad t \in T_i^{k_i}. \end{cases}$$

Obviously, the inequalities (21) are true for every σ, $0 \le \sigma \le \sigma_i$, when $t \in T_i^{k_i}$ and $\varepsilon > 0$ is sufficiently small. We shall show that $\sigma_i > 0$, $i = \overline{0, \rho}$.

Let $i \in I^{++}$. Then [1] $\widetilde{T}_i^{k_i} = T_i^{k_i}$ and since the support control $\{u, S_{\text{sup}}\}$ is non-degenerate, we have

$$|u(t)| < 1, \quad t \in T_i^{k_i}. \tag{23}$$

From (22), (23) as $|\Delta u^\varepsilon(t)| \le 1$, $t \in T$, we obtain $\sigma_i \ge \sigma_{*i}$, where

$$\sigma_{*i} = \min_{t \in T_i^{k_i}} \min \{1 - u(t), 1 + u(t)\} > 0. \tag{24}$$

Suppose $i \in I^{-+}$. From the non-degeneration of $\{u, S_{\text{sup}}\}$ it follows that $|u(t)| < 1$, $t \in]\mu_i^{k_i}, \mu_{i+1}^{k_i}]$, $\dot{u}(\mu_i^{k_i}) \ne 0$ if $|u(\mu_i^{k_i})| = 1$.

If $|u(\mu_i^{k_i})| < 1$ then $\sigma_i > \sigma_{*i}$. If for sufficiently small $\partial = \partial(\varepsilon) > 0$ we have $u(\mu_i^{k_i}) = 1$, $\Delta u^\varepsilon(t) \le 0$, $t \in [\mu_i^{k_i}, \mu_i^{k_i} + \partial]$; or $u(\mu_i^{k_i}) = -1$, $\Delta u^\varepsilon(t) \ge 0$, $t \in [\mu_i^{k_i}, \mu_i^{k_i} + \partial]$, then $\sigma_i > \min\{1, \overline{\sigma}_{*i}\}$, where

$$\overline{\sigma}_{*i} = \min_{t \in \overline{T}_i^{k_i}} \min \{1 - u(t), 1 + u(t)\} > 0, \ \overline{T}_i^{k_i} = \overline{T}_i^{k_i} \backslash [\mu_i^{k_i}, \mu_i^{k_i} + \partial[.$$

Consider the case, when $u(\mu_i^{k_i}) = 1$, $\Delta u^\varepsilon(t) \geq 0$, $t \in [\mu_i^{k_i}, \mu_i^{k_i} + \partial]$ (or $u(\mu_i^{k_i}) = -1$, $\Delta u^\varepsilon(t) \leq 0$, $t \in [\mu_i^{k_i}, \mu_i^{k_i} + \partial]$).

According to (11)–(14) we have: $\Delta u^\varepsilon(\mu_i^{k_i} - 0) = 0$. As it was shown above, $\Delta u^\varepsilon(t)$, $t \in T$, is continuous at moment $\mu_i^{k_i}$, because $i \in I^{-+}$. Therefore $\Delta u^\varepsilon(\mu_i^{k_i} + 0) = 0$, $\Delta \dot{u}^\varepsilon(\mu_i^{k_i} + 0) \geq 0$ and for the step σ_i, the inequality holds:

$$\sigma_i > \min \left\{ -\dot{u}(\mu_i^{k_i} + 0) \,/ \Delta \dot{u}^\varepsilon(\mu_i^{k_i} + 0), \overline{\sigma}_{*i} \right\} > 0.$$

Reasoning by analogy, it can be shown that $\sigma_i > 0$, $i \in I^{--} \cup I^{+-}$.

Assign $\sigma_0 = \min\limits_{i=\overline{0,\rho}} \sigma_i$, $\sigma_* = \min\{1, \sigma_0\} > 0$.

Obviously, for a sufficiently small $\varepsilon > 0$ the inequalities (21) hold when $\sigma = \sigma_*$. It is proved that the deviation $\sigma_* \Delta u^\varepsilon(t)$, $t \in T$, is admissible.

Let us calculate the deviation (10) of criterion (2) for $\sigma_* \Delta u^\varepsilon(t)$, $t \in T$

$$\begin{aligned} \Delta J(u) &= \sigma_* \left(\sum_{k \in K_0} \int_{t_k}^{t_k + \Theta_k(\varepsilon)} \Delta(t) \gamma_k dt + \int_{\mu_{i_0} - \varepsilon}^{\mu_{i_0} + 3\varepsilon} \Delta(t) \Delta u^\varepsilon(t) dt \right. \\ &\left. + \sum_{p=1}^{2} d_0^T(p-1) v_{i_0}^p \Delta u(\mu_{i_0}) + d_0^T(0) b v_{i_0}^2 \Delta \dot{u}(\mu_{i_0}) \right). \end{aligned}$$

By definition

$$\Delta(t_k) = 0, \quad k \in K_0. \tag{25}$$

Therefore for a sufficiently small $\varepsilon > 0$ using (11), we have:

$$\Delta J(u) = \sigma_* \left(2\varepsilon d_0^T(0) b v_{i_0}^2 + o(\varepsilon) \right) > 0, \tag{26}$$

since by assumption $v_{i_0}^2 d_0^T(0) b > 0$. The inequality (26) contradicts the optimality of the control $u(t)$, $t \in T$. It yields that the assumption $v_{i_0}^2 d^T(0) b > 0$ is wrong.

In case b) we do the same as above of (11) we use

$$\Delta u^{\Theta\varepsilon}(t) = \begin{cases} -(t - \mu_{i_0} + 3\varepsilon)^2, \ t \in [\mu_{i_0} - 3\varepsilon, \mu_{i_0} - 2\varepsilon[; \\ (t - \mu_{i_0} + \varepsilon)^2 - 2\varepsilon^2, \ t \in [\mu_{i_0} - 2\varepsilon, \mu_{i_0}[; \\ -(t - \mu_{i_0} - 3\varepsilon)^2, \ t \in [\mu_{i_0}, \mu_{i_0} + \varepsilon]; \\ 0, \ t \in [\mu_{i_0 - 1}, \mu_{i_0 + 1} \setminus [\mu_{i_0} - 3\varepsilon, \mu_{i_0} + 3\varepsilon]. \end{cases}$$

By analogy it can be proved that the inequality $v_{i_0}^2 d^T b < 0$ is impossible.

Thus it is proved that the relations (7) are true under the assumption [1] $D^T b \neq 0$.

Let us assume that the relations (6) are not true. Suppose that there exists an index $i_0 \in I_1 \cup I_2$ such that $v^1_{i_0} d^T b > 0$, $u(\mu_{i_0}) < 1$. Let $i_0 \in I_2$. For example, let $k_{i_0} = 2$, $k_{i_0-1} = 1$. Define function $\Delta u^{\Theta\varepsilon}(t)$, $t \in T \setminus T_{\sup}$ through the following formulas:

$$\Delta u^{\Theta\varepsilon}(t) = \begin{cases} (t - \mu_{i_0} + 2\varepsilon)^2, \ t \in [\mu_{i_0} - 2\varepsilon, \mu_{i_0} - \varepsilon[; \\ -(t - \mu_{i_0})^2 + 2\varepsilon^2, \ t \in [\mu_{i_0} - \varepsilon, \mu_{i_0} + \varepsilon[; \\ (t - \mu_{i_0} - 2\varepsilon)^2, \ t \in [\mu_{i_0} + \varepsilon, \mu_{i_0} + 2\varepsilon]; \\ 0, \ t \in [\mu_{i_0-1}, \mu_{i_0+1}] \setminus [\mu_{i_0} - 2\varepsilon, \mu_{i_0} + 2\varepsilon]; \end{cases}$$

and (12)–(14).

Let us choose n–vectors z_2, z_3 and consider the equations (15), where $z^0(\mu_i)$ are expressed by (1.25), (1.46), (1.44), η_{ip} – by (1.16), (1.41) using the function $\Delta u^{\Theta\varepsilon}(t)$, $t \in T \setminus T_{\sup}$ and functions:

$$\begin{aligned} \overline{g}^{\Theta\varepsilon}_{k_{i_0-1}+1}(t) &= d_0^T(0) b \Delta \dot{u}^{\Theta\varepsilon}(t) + d_0^T(1) b \Delta u^{\Theta\varepsilon}(t), \ t \in T_{i_0-1}, \\ \overline{g}^{\Theta\varepsilon}_{k_{i_0}+1}(t) &= d_0^T(2) b \Delta u^{\Theta\varepsilon}(t) + d_0^T(1) b \Delta \dot{u}^{\Theta\varepsilon}(t) + d_0^T(0) b \Delta \ddot{u}^{\Theta\varepsilon}(t), \quad t \in T_{i_0}, \end{aligned}$$

estimated by (1.18) and using

$$\begin{aligned} g_1(\mu_{i_0} + 0) &= d_0^T(0) b \Delta u^{\Theta\varepsilon}(\mu_{i_0} + 0) = d_0^T(0) b \varepsilon^2 = d^T b \varepsilon^2, \\ g_2(\mu_{i_0} + 0) &= d_0^T(1) b \Delta u^{\Theta\varepsilon}(\mu_{i_0} + 0) + d_0^T(0) b \Delta \dot{u}^{\Theta\varepsilon}(\mu_{i_0} + 0) \\ &= 2 d_0^T(1) b \varepsilon^2, \ g_p(\mu_i) = 0, \ p \in S_i, \ i = \overline{0,\rho}, \ i \neq i_0 \end{aligned}$$

defined by (1.16).

We can notice, that

$$\frac{\partial z^0(\mu_i)}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0, \quad i = \overline{0, \rho+1}; \quad \frac{\partial g_p(\mu_i)}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0, \ p \in S_i, \ i = \overline{0,\rho}. \tag{27}$$

The functions $\varphi_p(\Theta, \varepsilon, z_0, z_1)$, $p = \overline{0, k_* - 1}$, $\varphi_{pi}(\Theta, \varepsilon, z_0, z_1)$, $p \in S_i$, $i = \overline{0,\rho}$, are continuous and for them relations (16) hold. Therefore according to the theorem for implicit functions there exist functions (17) such that for every sufficiently small $\varepsilon > 0$ the identities (18) hold. From (27) we have

$$\Theta_k(\varepsilon) = o(\varepsilon), \ k \in K_0. \tag{28}$$

In this way, functions $f(t) \equiv 0$, $t \in T^*$, $\Delta u^\varepsilon(t) = \Delta u^{\Theta(\varepsilon)\varepsilon}(t)$, $t \in T \backslash T_{\sup}$, and vectors $z_0(\varepsilon)$, $z_1(\varepsilon)$ satisfy conditions (1-4) from [1] and relations (1.50) for sufficiently small $\varepsilon > 0$. Denote by $z^\varepsilon(t)$, $t \in T^*$, the solution of the system (1.34) corresponding to them and define function $\Delta u^\varepsilon(t)$, $t \in T_{\sup}$, by means of formula (19).

Following the reasoning in case A), we can show that when the support control $\{u, S_{sup}\}$ is not-degenerate, the deviation $\sigma_* \Delta u^\varepsilon(t)$, $t \in T$, $\sigma_* > 0$, is admissible if $\varepsilon > 0$ is sufficiently small.

Let us estimate the deviation (10) for $\sigma_* \Delta u^\varepsilon(t)$:

$$\Delta J(u) = \sigma_* \left(\sum_{k \in K_0} \int_{t_k}^{t_k + \Theta_k(\varepsilon)} \Delta(t) \gamma_k dt + \int_{\mu_{i_0} - 2\varepsilon}^{\mu_{i_0} + 2\varepsilon} \Delta(t) \Delta u^\varepsilon(t) dt + 2\varepsilon^2 v^1_{i_0} d^T b \right).$$

Taking into account (25), (28) for a sufficiently small $\varepsilon > 0$ we get the inequality

$$\Delta J(u) = \sigma_* \left(2\varepsilon^2 v^1_{i_0} d^T b + o(\varepsilon^2) \right) > 0,$$

which contradicts the optimality of control $u(t)$, $t \in T$.

Now let $i_0 \in I_1$. For example $k_{i_0} = 1$, $k_{i_0 - 1} = 0$. Define function $\Delta u^{\Theta\varepsilon}(t)$, $t \in T \backslash T_{sup}$, through the following formulas

$$\Delta u^{\Theta \varepsilon}(t) = \begin{cases} -t + \mu_{i_0} + \varepsilon, \ t \in [\mu_{i_0}, \ \mu_{i_0} + \varepsilon[; \\ 0, \ t \in [\mu_{i_0} + \varepsilon, \ \mu_{i_0 + 1}[; \end{cases}$$

and (13), (14).

Next the reasoning proceeds as described above. Thus we obtain deviation $\sigma_* \Delta u^\varepsilon(t)$, $t \in T$, which satisfies the inequality

$$\Delta J(u) = \sigma_* \left(\varepsilon v^1_{i_0} d^T b + o(\varepsilon) \right) > 0,$$

for sufficiently small $\varepsilon > 0$. This inequality contradicts the optimality of control $u(t)$, $t \in T$. Relations (6) are proved.

Other possible cases of violated optimality conditions can be investigated by using the above scheme.

The optimality criterion can be formulated as a support maximum principle.

Maximum principle. Let $\{u, S_{sup}\}$ be support control, and let $x(t)$, $\Psi(t)$, $t \in T$, be the solutions of the systems (2.1) and (2.13)–(2.17) respectively. For optimality of the admissible control $u(t)$, $t \in T$, it is sufficient, that for $u(t)$, $x(t)$, $\Psi(t)$, $t \in T$, the Hamiltonian

$$H(x, z, \Psi, u, t) = \Psi^T (A_0 x + A_1 z + bu)$$

gets its maximum value:

$$\max_{|u| \le 1} H(x(t), x(t-h), \Psi(t), u, t) = H(x(t), x(t-h), \Psi(t), u(t), t), \ t \in T, \tag{29}$$

and the conditions of coordination (6)–(8) hold. Let $\{u, S_{sup}\}$ be a non–degenerate support control. Then for the optimality of the admissible control $u(t)$, $t \in T$, the conditions of maximum (29) and those of coordination (6)–(8) are also necessary.

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