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ISOMORPHISM PROBLEMS FOR RINGS OF FUNCTIONS

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ABSTRACT. In the present paper we investigate the concept of a complete algebra of functions. The α Baire class of functions on a Tychonov space is a complete algebra of functions. Some criteria of the isomorphism of algebras of functions are obtained.

1. Introduction. In the present paper some problems of the theory of rings of functions on given sets are discussed.

The content of the paper is as follows:

1. Section 1 presents the basic definitions and notations.

2. In Section 2 we discuss the concept of complete algebras of functions. For every complete algebra E of functions on a set S, $\beta_E S$ denotes the analogue of the Stone-Čech compactification and $\nu_E S$ denotes the analogue of the Hewitt real compactification. It is proved that $\nu_E S$ is a dense subspace of the compact space $\beta_E S$ and $\nu_E S = \beta_E S \setminus \{H \subseteq \beta_E S \setminus S : H$ is a zero-set in $\beta_E S\}$.

3. Let E be a complete algebra of functions on a set S and $T_\alpha E$ be a sequential closure of E of index $\alpha \leq \Omega$, where Ω is the first uncountable ordinal. The set $t_\alpha E$ is a complete algebra of functions. In section 3 it is proved that for every function $f \in T_\alpha E$ there exists a unique extension $\nu_E f$ on $\nu_E S$ such that $\nu_E f$ is a Baire function of class α. Extensions of the Baire measurable mappings are studied in Section 10.

4. In Section 4 we study the zero-set spaces introduced by H. Gordon [12]. Theorem 4.3 and 4.5 affirm that the algebra E of functions on a set S is complete if and only if E coincides with the set of all mapping-set functions for a zero-set structure on S. It is proved that every Lindelöf space admits a unique zero-set structure.

5. In Sections 5 and 6 the fundamental theorems of Stone, Banach, Gelfand and Kolmogorov characterizing the maximal ideals are applied to the problems of isomorphism of b-complete and complete algebras of functions.

6. In Section 7 we establish some properties of topological spaces $\nu_E S$ and $\beta_E S$.

7. In Section 8 we discuss the notion of Z-perfect mappings.

8. In Section 9 the factorization theorem for Baire measurable mappings is proved (see [4]). In Section 11 this fact is playing the key role in determining the non-isomorphism of different algebras of Baire sets and of Baire functions in some spaces.

9. In all sections we discuss algebras of Baire sets and algebras of Baire functions.

The present paper is connected with the results of W. J. Bade [2], M. M. Choban [3, 4, 5, 6, 7], F. K. Dashiell [8], R. M. Dudley [9], L. Gilmar and M. Jerison [11], H. Gordon [12], J. E. Jayne [13, 14, 22], J. E. Jayne and F. Jellett [15], R. Levy and M. D. Rice [17], E. R. Lorch [18], P. R. Mayer [19] and Ju. E. Ochan [20, 21].

1. Preliminary results and definitions. We consider only Tychonov spaces. We shall use the notations and terminology from [10, 11, 16, 24]. In particular, βX is the Stone-Čech compactification of a space X, νX is the Hewitt real compactification of X, $w(X)$ is the weight of the space X, the cardinality of a set Y is denoted by $|Y|$, $cl\, H$ or $cl_X H$ denotes the closure of a set H in X, $N = \{1, 2, \ldots\}$, the symbol $\mathbf{R}$ will denote the field of real numbers, $C(X)$ is the space of all continuous real-valued functions on a space X, $C^*(X)$ stands for all bounded functions in $C(X)$.

A space is real compact if it is homeomorphic to a closed subspace of a product of real lines.

Let S be a set, $B(S)$ be the space of all real-valued functions on S and $B^*(S) = \{f \in B(S) : f \text{ is bounden on } S\}$. With respect to pointwise operations the sets $B(S)$ and $B^*(S)$ are lattice-ordered algebras. The space $B(S)$ is a Banach algebra with the supremum norm $\|f\| = \sup\{|f(x)| : x \in S\}$. If $\alpha \in \mathbf{R}$, then $\alpha_S(x) = \alpha$ for every $x \in S$. If $f \in B(S)$ and $\alpha \in \mathbf{R}$, then we put $f \vee \alpha = f \vee \alpha_S$ and $f \wedge \alpha = f \wedge \alpha_S$.

If $E \subseteq B(S)$, then T_E is the topology on S generated by E and it has the base consisting of all sets of the form $\cap\{f_i^{-1}U_i : i = 1, \ldots, n\}$, where $n \in \mathbf{N}$, $f_1, \ldots, f_n \in E$ and $U_1, \ldots, U_n$ are open subsets of $\mathbf{R}$. The space E separates the set S if for each pair of distinct points $x, y \in S$ there exists $f \in E$ such that $f(x) \neq f(y)$. The space (S, T_E) is Tychonov if and only if E separates the set S.

Let a subspace E of $B(S)$ separate the set S. Then the mapping $w_E : S \to R^E$, where $w_E(x) = \{f(x) : f \in E\}$, is an embedding of (S, T_E) in R^E. The closure $\nu_E S$ of the set $S = w_E(S)$ in R^E is a real compactification of the space (S, T_E). The space $\nu_E S$ is compact if and only if $E \subseteq B^*(S)$.

Let E and F be separating subspaces of $B(S)$. The symbol $\nu_E S > \nu_F S$ means that there exists a continuous mapping $\pi = \pi(E, F) : \nu_E S \to \nu_F S$ such that $\pi(x) = x$ for all $x \in S$.

Property 1.1. *Let $F \subseteq E \subseteq B(S)$ and F separate the set S. Then $\nu_E S > \nu_F S$*

Property 1.2. *Let $E \subseteq B^*(S)$ separate the set S. Then $\nu_E S$ is the smallest compactification of the space (S, T_E) such that all functions of E are continuously extendable over $\nu_E S$.*

Property 1.3. *$\beta X = \nu_{C^*(X)} X$ and $\nu X = \nu_{C(X)} X$ for every space X.*

On $B(S)$ let p denote the usual pointwise topology and let u be the topology of uniform convergence. We have $u - \lim f_n = f$ if and only if $\lim \|f - f_n\| = 0$.

Fix $E \subseteq B(S)$. By t denote the largest topology on E in which pointwise convergent sequences $\{f_n : n \in \mathbf{N}\}$ are topologically convergent. Hence, $t - \lim f_n = f$ if and only if $\lim f_n(x) = f(x)$ for every $x \in S$. Let $[A]_{t,E} = \{f \in E : f = t - \lim f_n$ for some $\{f_n \in A : n \in \mathbf{N}\}\}$ for every $A \subseteq S$. The set A is closed in the t-topology if and only if $A = [A]_{t,E}$. Denote by $[A]_{p,E}$ the p-closure of A in E and by $[A]_{u,E}$ the u-closure of A in E. It is clear that $[A]_{u,E} \subseteq [A]_{t,E} \subseteq [A]_{p,E}$.

Let $t_0 E = E$ and $t_\alpha E = [\cup\{t_\beta E : \beta < \alpha\}]_{t,B(S)}$, $t^*_\alpha E = t_\alpha E \cap B^*(S)$ for every $\alpha \leq \Omega$, where Ω is the first uncountable ordinal. By construction $t_\Omega E = [T_\Omega E]_{t,B(S)}$.

For every $f \in B(S)$ we denote $Z(f) = f^{-1}(0)$ and $S \setminus Z(f) = CZ(f)$. If $E \subseteq B(S)$, then $Z(E) = \{Z(f) : f \in E\}$ and $CZ(E) = \{CZ(f) : f \in E\}$.

Fix a space X. Let $B_0(X) = C(X)$, $B_\alpha(X) = t_\alpha C(X)$ and $B^*_\alpha(X) = t^*_\alpha C(X)$ for every $\alpha \leq \Omega$. The functions in $B_\alpha(X)$ are called Baire functions of class α. Let $Z_\alpha(X) = Z(B_\alpha(X))$, $CZ_\alpha(X) = CZ(B_\alpha(X))$, $A_\alpha(X) = Z_\alpha(X) \cup C_\alpha Z(X)$. The class $Z_\alpha(X)$ (class $CZ_\alpha(X)$) is a multiplicative (additive) class α of Baire sets of the space X. The sets $A_\alpha(X)$ are called the Baire sets of ambiguous class α.

Fix a space X. Let PX be the set X with the topology generated by G_δ-sets in X. The topology of the space PX is called the Baire topology of the space X. For every subspace E of $B_\Omega(X)$, where $B^*_1(X) \subseteq E$, we have $PX = (X, T_E)$. If $\alpha \geq 0$, then $Z_\alpha(X)$, $CZ_{1+\alpha}(X)$, $A_{1+\alpha}(X)$ are bases of the space PX.

A space $b_\alpha PX = \nu_{B^*_\alpha(X)}X$ is called the Baire compactification of class $\alpha \geq 1$ of the space PX and $r_\alpha PX = \nu_{B_\alpha(X)}X$ is the Baire real compactification of PX.

A space X is called a P-space if $X = PX$.

Property 1.4. *$Z_\alpha X = \{H \cap X : H \in Z_\alpha(\beta X)\}$ and $CZ_\alpha(X) = \{H \cap X : H \in CZ_\alpha(\beta X)\}$ for every $\alpha \leq \Omega$.*

2. Complete algebras of functions. Fix a set S.

A subspace E of $B(S)$ is called a b-complete algebra of functions on the set S if it is a Banach subalgebra of $B^*(S)$ with the following properties:
1. E contains all constant functions.
2. E separates the set S.

A subspace E of $B(S)$ is called a complete algebra of functions on the set S if it has the following properties:
3. $E \cap B^*(S)$ is a b-complete algebra of functions on S.
4. If $(f \wedge (-n)) \vee n \in E$ for every $n \in \mathbf{N}$, then $f \in E$.
5. If $f \in E$, then $(f \vee (-n)) \wedge n \in E$ for every $n \in \mathbf{N}$.

If E is a complete algebra of functions on a set S, then $mE = E \cap B^*(S)$ and $\beta_E S = \nu_{mE} S$.

Property 2.1. *Let E be a b-complete algebra of functions on S. Then the operator $u : C(\nu_E S) \to B(S)$, where $u(f) = f|S$, is an isomorphism of $C(\nu_E S)$ onto E.*

Proof. Follows from Property 1.2 and the Stone-Weierstrass theorem ([10], p. 191; [24], p. 115).

Let Y be a dense subspace of a space X, $f : Y \to \mathbf{R}$ be a continuous function and $e(f, y \subseteq X) = \cup\{Z : Y \subseteq Z \subseteq X$ and there exists a continuous extension $g : Z \to \mathbf{R}$ of f over $Z\}$. If $E \subseteq C(X)$, then $e(E, Y \subseteq X) = \cap\{e(f, Y \subseteq X) : f \in E\}$.

Property 2.2. *Let E be a complete algebra of functions on a set S and $F = E \cap B^*(S)$. Then*
1. *The natural mapping $\pi = \pi(E, F) : \nu_E S \to \beta_E S = \nu_F S$ is an embedding and $\pi(\nu_E S) = e(E, S \subseteq \beta_E S)$.*
2. *$\nu_E S = \pi(\nu_E S)$ is the maximal subspace of $\beta_E S$ such that all functions on E are continuously extendable over $\nu_E S$.*
3. *$\nu_E S = \beta_E S \setminus \cup\{H \in Z_0(\beta_E S) : H \cap S = \emptyset\}$.*
4. *E is uniformly closed in $B(S)$.*
5. *If $f \in E$ and $f(x) \neq 0$ for every $x \in S$, then $1/f \in E$.*

Proof. Let $\{f_m : m \in \mathbf{N}\} \subseteq E$, $f \in B(S)$ and $\|f - f_m\| < 2^{-m}$. If $g_n = (f \wedge n) \vee (-n)$ and $g_{mn} = (f_m \wedge n) \vee (-n)$, then $\|g_n - g_{mn}\| < 2^{-m}$ and $g_{mn}, g_n \in F$ for all $n, m \in \mathbf{N}$. Hence $f \in E$. The assertion 4 is proved.

Fix $f \in E$. Suppose that $f(x) \neq 0$ for each $x \in E$ and $g = 1/f$. Let $m.n \in \mathbf{N}$ and $f_n = (f \wedge n) \vee (-n)$. Then $f_n \in F$. Consider the function $f_{mn}(x) = \min\{f_n(x), -m^{-1}\}$ if $f_n(x) < 0$ and $f_{mn}(x) = \max\{f_n(x), m^{-1}\}$ if $f_n(x) > 0$. By construction there exist the continuous extensions of functions f_n, f_{mn} on $\beta_E S$. Therefore $f_{mn} \in F$ and $g_{mn} = 1/f_{mn} \in F$ for all $m, n \in \mathbf{N}$. Let $g_m = (g \wedge m) \vee (-m)$. Then $\|g_m - g_{mn}\| \leq n^{-1}$. Hence $g_m \in F$ for every $m \in \mathbf{N}$. Therefore $g \in E$. This proves assertion 5.

By property 1.1 there exists a continuous mapping $\pi : \nu_E S \to \nu_E S$, where $\pi(x) = x$ for all $x \in S$.

Let $f \in E$ and $f_n = (f \wedge n) \vee (-n)$. By $\beta_E f$ we denote the continuous extension of f over $e(f, S \subseteq \beta_E S)$ and by $\nu_E f$ denote the continuous extension of f on $\nu_E S$. By construction, $e(f_n, S \subseteq \beta_E S) = \beta_E S$, $e(f, S \subseteq \beta_E S) = \beta_E S \setminus \cap \left\{\beta_E f_n^{-1}\{-n, n\} : n \in \mathbf{N}\right\}$ and $\nu_E f(y) = \beta_E f(\pi(y))$ for every $y \in \nu_E S$. Hence $\pi(\nu_E S) \subseteq e(E, S \subseteq \beta_E S) \subseteq \beta_E S \setminus \cup\{H \in Z_0(\beta_E S) : H \cap S = \emptyset\}$ and π is an embedding.

Let $f \in F$ and $Z(\beta_E f) \subseteq \beta_E S \setminus S$. Let $g = 1/f$. Then $e(g, S \subseteq \beta_E S) = \beta_E S \setminus Z(\beta_E f)$. Hence $e(E, S \subseteq \beta_E S) \supseteq \beta_E S \setminus \{H \in Z_0(\beta_E S) : H \cap S = \emptyset\}$. This proves assertions 1, 2 and 3. □

Corollary 2.3. *Let E be a b-complete algebra of functions on a set S. The algebra E is complete if and only if $H \cap S \neq \emptyset$ for every non-empty set $H \in Z_0(\nu_E S)$.*

Corollary 2.4. *Let X be a pseudocompact space, E be a b-complete algebra of functions on the set X and $E \subseteq C(X)$. Then E is a complete algebra of functions on a set X and the space (X, T_E) is pseudocompact.*

Corollary 2.5. *Let E be a b-complete algebra of functions on a set S and the space (S, T_E) is pseudocompact. Then E is a complete algebra of functions on the set S.*

Corollary 2.6. *Let E be a b-complete or a complete algebra of functions on a set S. Then E is a sublattice of $B(S)$.*

Corollary 2.7. *Let E be a complete algebra of functions on a set S, $m \in \mathbf{N}$, $f \in E$ and $f(x) \geq 0$ for every $x \in S$. Then $g = f^{1/m} \in E$.*

Proof. We put $f_n = (f \wedge n^m)$. Then $f_n \in F = E \cap B^*(S)$ and $g_n = (f_n)^{1/m} = g \vee n$. By virtue of Property 2.1, $g_n \in F$ for every $n \in \mathbf{N}$ and $g \in E$. □

Property 2.8. *Let E be a b-complete or a complete algebra of functions on a set S and $0 < \alpha \leq \Omega$. Then $T_{t_\alpha E}$ is a Baire topology of the space (S, T_E).*

Proof. Let $f \in E$ and $H = Z(f)$. We put $g = 1 \wedge f^2$, $g_n = g^{1/n}$ and $h = t - \lim g_n$. Then $Z(h) = Z(f) = h^{-1}(-1,1)$ and $g \in t_1 E \subseteq T_\alpha E$. Hence the G_δ-subsets of (S, T_E) are open in the topology $T_{t_\alpha E}$. It is clear that $Z(T_\Omega E) \subseteq Z_\Omega(S, T_E)$. This finished the proof. □

Property 2.9. *Let E be a complete algebra of functions on a space X and $C^*(X) \subseteq E$. Then $C(X) \subseteq E$. Moreover, if $E \cap B^*(X) = C^*(X)$, then $E = C(X)$.*

Proof. Obvious.

Property 2.10. *Let E_1 and E_2 be the complete algebras of functions on a set S. The algebra $E = E_1 \cap E_2$ is complete if and only if it separates S.*

Example 2.11. Let X be a locally compact non-compact space. Denote by αX the one-point compactification of X and $\alpha C(X) = \{f|X : f \in C(\alpha X)\}$. By construction, the algebra $\alpha C(X)$ is b-complete and $\nu_{\alpha C(X)} X = \alpha X$. We affirm that the algebra $\alpha C(X)$ is complete if and only if X is not Lindelöf.

Let X be a Lindelöf space. Then $H = \alpha X \setminus X \in Z_0(\alpha X)$. From Corollary 2.3 the algebra $\alpha C(X)$ is not complete.

Suppose that the space X is not Lindelöf. Then $H = \alpha X \setminus X$ is not G_δ-subset of a space αX and $\Phi \cap X \neq \emptyset$ for every non-empty subset $\Phi \in Z_0(\alpha X)$. By virtue of Corollary 2.3, the algebra $\alpha C(X)$ is complete.

3. Extensions of functions. Fix an infinite set S and a complete algebra E of functions on S. Let $F = S \cap B^*(S)$ and $\alpha F = t_\alpha E \cap B^*(S)$. For every $\alpha \leq \Omega$ there are continuous mappings $\pi_\alpha = \pi(t_\alpha E, E) : \nu_{t_\alpha E} S \to \nu_E S$ and $p_\alpha : \pi(\alpha F, F) : \nu_{\alpha F} S \to \nu_F S$ such that $\pi_\alpha(x) = p_\alpha(x)$ for all $x \in S$. By virtue of Property 2.2, $\nu_E S \subseteq \nu_F S$, $\nu_{t_\alpha E} S \subseteq \nu_{\alpha F} S$ and $\pi_\alpha - p_\alpha | \nu_{t_\alpha E} S$.

Consider the algebras of functions $ct_0 E = \{f \in C(\nu_E S) : f|S \in E\}$ and $ct_\alpha E = [\cup\{ct_\beta E : \beta < \alpha\}]_{t, B(\nu_E S)}$.

Theorem 3.1. *Let $\alpha \leq \Omega$. Then:*

1. *For every $f \in T_\alpha E$ there exists a unique functions $\nu_E f \in ct_\alpha E$ such that $f = \nu_E f | S$.*
2. *If $\{f, f_n : n \in \mathbf{N}\} \subseteq T_\Omega E$ and $f = t - \lim f_n$, then $\nu_E f = t - \lim \nu_E f_n$.*
3. *The mapping $\nu_\alpha : ct_\alpha E \to T_\alpha E$, where $\nu_\alpha(g) = g|S$, is a topological isomorphism relatively to the t- topology.*
4. $\pi(\nu_{\alpha t_\alpha E} S) = p_\alpha(\nu_{\alpha t_\alpha E} S) = \nu_E S$.

5. *The mapping π_α is one-to-one.*
6. *If $\alpha \geq 1$, then $\nu_{t_\alpha E}S = P\nu_E S$.*

Proof. Assertions 1 and 2 for $\alpha = 0$ follow from Property 2.2. Suppose that $\alpha \geq 1$ and the assertion 1 is true for every function $h \in \cap\{t_\beta E : \beta < \alpha\} = E_\alpha$. Fix a function $f \in t_\alpha E$. Then $f = t-$ limit $\lim g_n(y)$ does not exist. Then there exists a set $H \in Z_0(\beta_e S)$ such that $y \in H \cap \nu_E S \subseteq \{g_{-1}(g_n(y)) : n \in \mathbf{N}\} \subseteq \beta_E S \backslash S$. Hence, by virtue of Property 2.2, $y \notin \nu_E S$. Therefore there exists the extension $\nu_E f = t - \lim \nu_E f_n = t - \lim g_n$ of function f. The same argument proves that $\nu_E(t - \lim f_n) = t - \lim \nu_E f_n$ for every sequence $\{f_n \in t_\Omega E : n \in \mathbf{N}\}$.

Let $g, f \in ct_\Omega E$ and $g|S = h|S$. Suppose that $g(y) \neq h(y)$ for some $y \in \nu_E S$. Then for some $H \in Z_0(\beta_E S)$ we have $y \in H \cap \nu_E S \subseteq g^{-1}(g(y)) \cap h^{-1}(h(y))$. Hence $y \notin \nu_E S$. This completes the proof of assertions 1 and 2. The assertion 3 follows from assertions 1 and 2.

Let $x, y \in \nu_{t_\alpha E}S$, $x \neq y$ and $\pi_\alpha(x) = \pi_\alpha(y)$. Then $g(x) \neq g(y)$ for some $g \in ct_\alpha E$. Let $f = g|S$ and $h(s) = \nu_E f(\pi_\alpha(s))$ for every $s \in S$. Then $h \in ct_\alpha E$, $h(x) = h(y)$ and $h|S = g|S$. Hence π_α is one-to-one. Assertion 5 is proved.

Let $y \in \beta_E S \setminus p_\alpha(\nu_{t_\alpha E}S)$. Then there exists a set $H \in Z_0(\beta_E S)$ such that $y \in H \subseteq \beta_E S \setminus \nu_E S$. By construction, $p_\alpha^{-1}(y) \subseteq p_\alpha^{-1}(H) = W \in Z_0(\nu_{\alpha F} S)$ and $W \cap S = \emptyset$. By virtue of Property 2.2, we have $p_\alpha^{-1}(y) \cap \nu_{t_\alpha E}S = \emptyset$. Assertion 4 is proved.

Let $\alpha \geq 1$. By virtue of Property 2.8 and assertions 1 – 5, $\nu_{t_\alpha E}S = P\nu_E S$. The proof is complete. □

Corollary 3.2. *For every real compact space X the space PX is real compact.*

Corollary 3.3 (P. R. Mayer [13]). *$B_\alpha(X) = \{f|X : f \in B_\alpha(\nu x)\}$ for every $\alpha \leq \Omega$.*

Corollary 3.4. *For every space X and $\alpha \geq 1$ we have $P\nu X = r_\alpha PX$.*

4. Zero-set spaces.

Definition 4.1 (H. Gordon [12]). *Let X be a set. The collection $\mathcal{Z}$ of subsets of X is called a zero-set structure on X and $(X, \mathcal{Z})$ is called a zero-set space if $\mathcal{Z}$ satisfies the following conditions:*
1. *$\emptyset, X \in \mathcal{Z}$.*
2. *$\mathcal{Z}$ is closed under finite unions.*
3. *$\mathcal{Z}$ is closed under countable intersections.*
4. *For each pair of distinct points of X, there is a $Z \in \mathcal{Z}$ which contains one of the points.*
5. *Whenever $A, B \in \mathcal{Z}$ and $A \cap B = \emptyset$, then there are $C, D \in \mathcal{Z}$ with $A \subseteq X \setminus C$, $B \subseteq X \setminus D$ and $(X \setminus C) \cap (X \setminus D) = \emptyset$.*
6. *Wherever $Z \in \mathcal{Z}$, there exists a sequence $\{Z_n : n \in \mathbf{N}\} \subseteq \mathcal{Z}$ such that $X \setminus Z = \cup\{Z_n : n \in \mathbf{N}\}$.*

A mapping $\varphi : X \to Y$ of a zero-set space $(X, \mathcal{Z})$ into a zero-set space $(Y, \mathcal{Z}')$ is a zero-set mapping if $\varphi^{-1}(\mathcal{Z}') \subseteq \mathcal{Z}$. The mapping φ is a zero-set homeomorphism if φ is one-to-one and φ, φ^{-1} are zero-set mappings.

A function $f : X \to \mathbf{R}$ on a zero-set space $(X, \mathcal{Z})$ is a zero-set function if $f^{-1}(Z_0(\mathbf{R})) \subseteq \mathcal{Z}$. We denote by $C(X, \mathcal{Z})$ the family of all zero-set functions on $(X, \mathcal{Z})$. Let $C^*(X, \mathcal{Z}) = C(X, \mathcal{Z}) \cap B^*(X)$.

Example 4.2 ([12], section 8). Let X be a space. Then $(X, Z_\alpha(X))$ is a zero-set space and $C(X, Z_\alpha(X)) = B_\alpha(X)$ for all $\alpha \geq 0$.

Theorem 4.3. *Let $(X, \mathcal{Z})$ be a zero-set space. Then:*
1. $\mathcal{Z} = \{Z(f) : f \in C(X, \mathcal{Z})\}$.
2. $(f \wedge \alpha) \vee (-\alpha) \in C(X, \mathcal{Z})$ *for every* $f \in C(X, \mathcal{Z})$ *and a positive number* $\alpha \in \mathbf{R}$.
3. *The algebra* $C(X, \mathcal{Z})$ *is complete.*

Proof. Assertions 1 and 2 are proved in ([12], Theorems 3.5 and 3.7). By ([12], Theorem 3.5) $C^*(X, \mathcal{Z})$ is a b-complete algebra of functions on X. If $(f \wedge n) \vee (-n) \in C(X, \mathcal{Z})$ for every $n \in \mathbf{N}$, then from ([12], Theorem 3.7) we have $f \in C(X, \mathcal{Z})$. The proof is complete. □

Lemma 4.4. *Let E be a complete algebra of functions on a set $X, Y = \beta_E X$, $A \subseteq X$ and $B \subseteq X$. Then $cl_Y A \cap cl_Y B = \emptyset$ if and only if there exist functions $f, g \in E$ such that $A \subseteq Z(f)$, $B \subseteq Z(g)$ and $Z(f) \cap Z(g) = \emptyset$.*

Proof. If $cl_Y A \cap cl_Y B = \emptyset$, then from Property 2.1 there exist functions $f, g \in E$ for which $Z(f) \cap Z(g) = \emptyset$, $A \subseteq Z(f)$ and $B \subseteq Z(g)$.

Let $f, g \in E$ and $Z(f) \cap Z(g) = \emptyset$. We assume that $f = (f \wedge 1) \vee 0$ and $g = (g \wedge 1) \vee 0$. There exists a continuous function $\varphi \in C(\beta_E X)$ such that $\varphi(x) = \min\{1, f(x) + g(x)\}$ for every $x \in X$. Then $Z(\varphi) \subseteq Y \backslash X$. We put $H = cl_Y(Z(f) \cup Z(g))$. The function $\varphi_1 : H \to [0, 2]$, where $\varphi_1(x) = 2^{-2}\varphi(x)$ if $x \in cl_Y Z(f)$ and $\varphi_1(x) = (2 - 2^{-2})\varphi(x)$ if $x \in cl_Y Z(g)$, is continuous. There exists a continuous function $\varphi_2 : Y \to [0, 2]$ such that $\varphi_1 = \varphi_2 | H$. Let $\psi = \varphi_2 + 2^{-2}\varphi$. By construction, $Z(\varphi) \cup Z(\psi) \subseteq Y \backslash X$, $\psi(x) = 2^{-1}\varphi(x)$ if $x \in Z(f)$ and $\psi(x) = 2\varphi(x)$ if $x \in Z(g)$. Let $h(x) = (\varphi(x))^{-1} - (\psi(x))^{-1}$ for every $x \in X$. Then $h \in E$, $h(x) \leq -1$ if $x \in Z(f)$ and $cl_Y Z(f) \cap cl_Y Z(g) = \emptyset$. The proof is complete. □

Theorem 4.5. *Let E be a complete algebra of functions on a set X. Then:*
1. $(X, Z(E))$ *is a zero-set space.*
2. $E = C(X < Z(E))$.

Proof. Assertion 1 follows from ([12, Theorem 2.3).

Let $f \in E$, $b, c \in \mathbf{R}$ and $c \leq b$. We put $\varphi = (f - c_X) \vee 0$ and $\psi = (f - b_X) \wedge 0$. Then $\varphi, \psi \in E$ and $f^{-1}(c, b) = X \setminus (Z(\varphi) \cup z(\psi))$. Hence $f \in C(X, Z(X)) = F$, $E \subseteq F$ and there exists a continuous mapping $p : \beta_F X \to \beta_E X$ such that $p(x) = x$ for every $x \in X$. Let $Y = \beta_E X$, $S = \beta_F X$, $A \subseteq X$, $B \subseteq X$ and $cl_S A \cap cl_S B = \emptyset$. Then there exists a continuous function $f : S \to [0, 1]$ such that $A \subseteq f^{-1}(0) \in Z(E)$ and $B \subseteq f^{-1}(1) \in Z(E)$. By virtue of Lemma 4.4, $cl_Y A \cap cl_Y B = \emptyset$ and p is one-to-one. Hence $E = C(X, Z(X))$. The proof is complete. □

Corollary 4.6. *Let F be a b-complete algebra of functions on a set X. Then $E = C(X, Z(F))$ is the minimal complete algebra of functions on X such that $F \subseteq E$.*

Theorem 4.7. *Let F be a b-complete algebra of functions on a set X. The following assertions are equivalent:*
1. *If $H, L \in Z_0(\nu_F X)$ and $H \cap L \cap X = \emptyset$, then $cl_{\nu_F X}(H \cap X) \cap cl_{\nu_F X}(L \cap X) = \emptyset$.*
2. *There exists a unique complete algebra E of functions on the set X such that $F = E \cap B^*(X)$.*

Proof. Implication $2 \to 1$ follows from Lemma 4.4. The proof of the implication $1 \to 2$ is similar to that of assertion 2 of Theorem 4.5. □

For every family $\mathcal{L}$ of subsets of a set X denote by $\delta c\mathcal{L}$ the collection of all countable intersections of sets of the family $C\mathcal{L} = \{X \setminus H : H \in \mathcal{L}\}$.

Lemma 4.8. *If $\mathcal{Z}$ is a zero-set structure on a set X, then $\delta c\mathcal{Z}$ is a zero-set structure on X and $C(X, \delta c\mathcal{Z}) = t_1 C(X, \mathcal{Z})$.*

Proof. Obvious. □

Corollary 4.9. *If E is a complete algebra of functions on a set X, then for every $\alpha \leq \Omega$ the algebra $T_\alpha E$ is complete.*

Corollary 4.10. *$B_\alpha(X)$ is a complete algebra of functions on the set X for every space X.*

Theorem 4.11. *Let T be a topology of a Londelöf space X. Then for an algebra E of functions on X the following assertions are equivalent:*
1. *E is a complete algebra and $T = T_E$.*
2. *$E = C(X)$.*

Proof. The implication $2 \to 1$ is obvious. Let E be a complete algebra of functions on X and $T_E = T$. Fix the closed subsets H, L of X for which $H \cap L = \emptyset$. Let $Y = \beta_E X$, $H_1 = cl_Y H$ and $L_1 = cl_Y L$. Since X is a Lindelöf spaces, then there exists a closed G_δ-subset P of Y such that $H_1 \cap L_1 \subseteq P \subseteq Y \setminus X$. From the proof of Lemma 4.4 it follows that there are continuous functions $f, g : Y \to [0,2]$ such that $Z(f) = Z(g) = P$, $g(x) = 2^{-1} f(x)$ if $x \in H$ and $g(x) = 2f(x)$ if $x \in L$. Consider the continuous function $h(x) = (f(x))^{-1} - (g(x))^{-1}$ on X. By construction, $h \in E$, $H \subseteq h^{-1}(-\infty, -1)$ and $L \subseteq h^{-1}(2^{-1}, +\infty)$. Hence, from Theorem 4.7, $L_1 \cap H_1 = \emptyset$, $Y = \beta X$ and $E = C(X)$. The proof is complete. □

Corollary 4.12. *Every Lindelöf space admits a unique zero-set structure.*

5. Isomorphism problem for algebras of functions. Let X be a space and $f \in B_\Omega(X)$. Then νf is a continuous extension of f on νX. If E is a complete or b-complete algebra of functions on the set X and $f \in E$, then $\nu_E f$ is a continuous extension of f over $\nu_E X$ and $\beta_E f$ is a continuous extension of f over $e(f, X \subseteq \beta_E X)$.

If E is a b-complete algebra of functions on a set X, then we assume that $\beta_E X = \nu_E X$.

Theorem 5.1. *Let E be a Banach algebra of functions on a space X, F be a Banach algebra of functions on a space Y, $B_1^*(X) \subseteq E \subseteq B_\Omega^*(X)$, $B_1^*(Y) \subseteq F \subseteq B_\Omega^*(Y)$ and $\varphi : E \to F$ be a ring isomorphism. Then there exists a homeomorphism $\psi : \nu_E X \overset{\text{onto}}{\longrightarrow} \nu_F Y$ such that:*
1. $\psi(\nu X) = \nu Y$.
2. $\psi(Z(\nu_E f)) = Z(\nu_F(\varphi(f)))$ *for every* $f \in E$.

Proof. By virtue of property 2.2 and Theorem 3.1, $P\nu X$ is a subspace of $\nu_E X$ and $P\nu Y$ is a subspace of $\nu_F Y$.

A proper subring J of E is a prime ideal of E if $J \cdot E \subseteq J$ and $f \cdot g \in J$ implies $f \in J$ or $g \in J$. For every maximal prime ideal J of E there exists a unique point $x(J) \in \nu_E X$ such that $J = \{f \in E : \nu_E f(x(J)) = 0\}$. Also for every maximal prime ideal J of F there exists a unique point $y(J) \in \nu_F Y$ such that $J = \{f \in F : \nu_F f(y(J)) = 0\}$ (see [11], Chapter 4). Hence there exists a unique one-to-one mapping $\psi : \nu_E X \to \nu_F Y$ such that $\psi(x(J)) = y(\varphi(J))$ for every maximal prime ideal J of E. By construction, $Z(\nu_E f) = \{x(J) : J$ is maximal prime ideal of E and $f \in J\}$. Therefore $Z(\nu_F(\varphi(f))) = \psi(Z(\nu_E f))$ for any $f \in E$.

Since $B_1^*(X) \subseteq E \subseteq B_\Omega^*(X)$ and $B_1^*(Y) \subseteq F \subseteq B_\Omega^*(Y)$, by virtue of Property 2.2, $P\nu X = \nu_E X \setminus \cup\{H \in Z_0(\nu_E X) : H \cap P\nu X = \emptyset\}$ and $P\nu Y = \nu_F Y \setminus \cup\{H \in Z_0(\nu_F Y) : H \cap P\nu Y = \emptyset\}$. The spaces $P\nu X$ and $P\nu Y$ are P-spaces. Hence $x \in \nu_E X \setminus P\nu X$ if and only if there exists a set $H \in Z_0(\nu_E X)$ such that $x \in H$ and $intH = \emptyset$ and, similarly, $y \in \nu_F Y \setminus P\nu Y$ if and only if $y \in G$ and $\int G = \emptyset$ for some $G \in Z_0(\nu_F Y)$. Therefore $\psi(P\nu X) = P\nu Y$. The proof is complete. □

Theorem 5.2. *Let E be a b-complete or complete algebra of functions on a set X, F be a b-complete or complete algebra of functions on a set Y and $\varphi : E \overset{\text{onto}}{\longrightarrow} F$ be a ring isomorphism. Then there exists a homeomorphism $\psi : \beta_E X \overset{\text{onto}}{\longrightarrow} \beta_F Y$ such that:*
1. $\varphi(mE) = mF$, *where* $mE = E \cap B^*(X)$ *and* $mF = F \cap B^*(Y)$.
2. $\psi(\nu_E X) = \nu_F Y$.
3. $Z(\beta_F \varphi(f)) = \psi(Z(\beta_E f)$ *and* $Z(\nu_F \varphi(f)) = \psi(Z(\nu_E f))$ *for every* $f \in E$.
4. *The algebra E is complete if and only if the algebra F is complete.*

Proof. Let $E^+ = \{f \in E : f \geq 0\}$ and $F^+ = \{g \in F : g \geq 0\}$. By virtue of Property 2.7, $E^+ = \{f^2 : f \in E\}$ and $F^+ = \{g^2 : g \in F\}$. Hence $F^+ = \varphi(E^+)$. If $f, g \in E$ and $f < g$, then $f - g \in E^+$, $\varphi(f) - \varphi(g) \in F^+$ and $\varphi(f) < \varphi(g)$. Therefore φ is a lattice isomorphism. It is clear that $g = 1_X$ if and only if $f \cdot g = f$ for every $f \in E$. Hence $\varphi(1_X) = 1_Y$ and $\varphi(\lambda_X) = \lambda_Y$ for every $\lambda \in \mathbf{R}$. If $f \in mE$, then $-n_X < f < n_X$ for some $n \in \mathbf{N}$. Therefore $\varphi(mE) = mF$ and $\varphi((f) \wedge n) \vee (-n)) = (\varphi(f) \wedge n) \vee (-n)$ for all $f \in E$ and $n \in \mathbf{N}$. The assertions 1 and 4 are proved. In particular, mE and mF are b-complete algebras and $\zeta = \varphi | mE$ is a ring isomorphism of mE onto mF.

For every maximal prime ideal J of mE there exists a unique point $x(J) \in \beta_E X = \nu_{mE} X = \beta_{mE} X$ such that $J = \{f \in mE : \beta_E f(x(J)) = 0\}$ and for every maximal prime ideal J of mF there exists a unique point $y(J) \in \beta_F Y$ such that

$J = \{g \in mF : \beta_F g(y(J)) = 0\}$ (see[11], Chapter 4).

Then there exists a unique one-to-one mapping ψ of $\beta_E X$ onto $\beta_F Y$ such that $\psi(x(J)) = y(\varphi(J))$ for every maximal prime ideal J of mE. By construction, $Z(\beta_F\varphi(f)) = \psi(Z(\beta_E f))$ for any function $f \in E$.

If $E = mE$, then $F = mF$. In this case the proof is complete.

Let $E \neq mE$. Then E and F are complete algebras of functions. Fix a point $x \in \beta_E X \setminus \nu_E X$. By virtue of Property 2.2, there exists a set $H \in Z_0(\beta_E X)$ such that $x \in H \subseteq \beta_E X \setminus \nu_E X$. Fix a continuous function $f \in mE$ such that $H = Z(\beta_E f)$. From Property 2.2, $g = 1/f \in E \setminus mE$. Let $h = \varphi(f)$ and $p = \varphi(g)$. By construction, $h \cdot p = 1_Y$, $p(y) > 0$ for every $y \in Y$ and $e(p, Y \subseteq \beta_F Y) = \beta_F Y \setminus Z(\beta_F h) = \beta_F Y \setminus Z(\beta_F h) = \beta_F Y \setminus \psi(z(\beta_E f))$. Hence $\psi(x) \in \beta_F Y \setminus \nu_F y$ and $\psi(\nu_E X) = \nu_F Y$. Assertion 2 is proved. If $f \in E$ and $g = \min\{1, f^2\}$, then $Z(\beta_E f) = Z(\beta_E g)$. This completes the proof. □

Definition 5.3. *An algebra E of functions on a set S is real compact if (S, T_E) is a real compact space.*

A complete algebra E of functions on a set S is real compact if and only if $\nu_E S = (S, T_E)$.

A zero-set space $(X, \mathcal{Z})$ is real compact (see[12]) if and only if the algebra $C(X, \mathcal{Z})$ is real compact.

Corollary 5.4. *Let E be a complete real compact algebra of functions on a set X and F be a complete real compact algebra of functions on a set Y. Then for every ring isomorphism $\varphi : E \overset{\text{onto}}{\longrightarrow} F$ there exists a unique homeomorphism $\psi : \overset{X}{\longrightarrow} Y$ of the space (X, T_E) onto a space (Y, T_F) such that $Z\nu_F\varphi(f)) = \psi(Z(\nu_E f))$ for all $f \in E$.*

Corollary 5.5. *Let $(X, \mathcal{Z})$ and $(Y, \mathcal{Z}')$ be real compact zero-spaces. The algebras $C(X, \mathcal{Z})$ and $C(Y, \mathcal{Z}')$ are ring isomorphic if and only if the zero-set spaces $(X, \mathcal{Z})$ and $(Y, \mathcal{Z}')$ are zero-set homeomorphic.*

Theorem 5.6. *If E is a complete algebra of functions on a set X and F is a complete algebra of functions on a set Y, then the following assertions are equivalent:*
1. *E is ring isomorphic to F.*
2. *There exists a homeomorphism $\psi : \nu_E X \overset{\text{onto}}{\longrightarrow} \nu_F Y$ such that $\psi(\{Z(\nu_E f) : f \in E\}) = \{Z(\nu_F g) : g \in F\}$.*
3. *There exists a homeomorphism $\zeta : \beta_E X \overset{\text{onto}}{\longrightarrow} \beta_F Y$ such that $\zeta(\nu_E X) = \nu_F Y$.*

Proof. Implications $1 \to 3 \to 2$ follow from Theorem 5.2. We put $\mathcal{Z} = \{Z(\nu_E f) : f \in E\}$ and $\mathcal{Z}' = \{Z(\nu_F g) : g \in F\}$. Then ψ is a zero-set homeomorphism of a real compact zero-set space $(\nu_E X, \mathcal{Z})$ onto a real compact zero-set space $(\nu_F Y, \mathcal{Z}')$. From Theorem 4.5 it follows that $E = \{F|X : f \in C(\nu_E X, \mathcal{Z})\}$ and $F = \{g|Y : g \in C(\nu_F Y, \mathcal{Z}')\}$. Hence implication $2 \to 1$ follows from Corollary 5.5. The proof if complete. □

A mapping $f : X \to Y$ of a space X into a space Y is called:

– Baire measurable of class α if $f^{-1}(Z_0(Y)) \subseteq Z_\alpha(X)$;

– Baire homeomorphism of class (α,β) if f is one-to-one, $f^{-1}(Z_0(Y)) \subseteq Z_\alpha(X)$ and $f(Z_0(x)) \subseteq Z_\beta(Y)$;
– Baire isomorphism of class (α,β) if f is one-to-one and $f(Z_\alpha(X)) = Z_\beta(Y)$.

Every Baire isomorphism of class (α,β) is also Baire homeomorphism of class (α,β). Every Baire homeomorphism of class (α,β) is a Baire isomorphism of class (μ,η) for some $\mu \in \Omega$ and $\eta \in \Omega$.

Corollary 5.7. *If X and Y are spaces, $0 \le \alpha \le \Omega$ and $0 \le \beta \le \Omega$, then the following assertions are equivalent:*
1. *There exists a Baire isomorphism $\psi : \nu X \to \nu Y$ of class (α,β).*
2. *The zero-set spaces $(\nu X, Z_\alpha(\nu X))$ and $(\nu Y, Z_\beta(\nu Y))$ are zero-set homeomorphic.*
3. *$B_\alpha(X)$ is ring isomorphic to $B_\beta(Y)$.*

Corollary 5.8. *If X and Y are spaces and $\alpha,\beta \ge 1$, then the following assertions are equivalent:*
1. *There exists a Baire isomorphism $\psi : \nu X \to \nu Y$ of class (α,β).*
2. *The spaces $b_\alpha PX$ and $b_\beta PY$ are homeomorphic.*
3. *$B_\alpha(X)$ is ring isomorphic to $B_\beta(Y)$.*
4. *$B^*_\alpha(X)$ is ring isomorphic to $B^*_\beta(Y)$.*

Corollary 5.9 (J. E. Jayne [22]). *Let X and Y be real compact spaces and $\alpha,\beta \ge 1$. Then the following assertions are equivalent:*
1. *There exists a Baire isomorphism $\psi : X \to Y$ of class (α,β).*
2. *$B_\alpha(X)$ is ring isomorphic to $B_\beta(Y)$.*
3. *$B^*_\alpha(X)$ is ring isomorphic to $B^*_\beta(Y)$.*

Example 5.10. Fix a compact space X and $\alpha \ge 1$. Then $Y = b_\alpha PX$ is a compact space and the algebra $B^*_\alpha(X)$ is ring isomorphic to $B_0(Y) = C(Y)$. If X is an infinite countable space, then Y is a Stone-Čech compactification of a countable discrete space and Y is uncountable. Therefore the spaces X and Y are not Baire isomorphic. Hence the assumption $\alpha,\beta \ge 1$ in Corollaries 5.8 and 5.9 are essential.

Theorem 5.11. *Let X and Y be spaces with countable pseudocharacter, E be a b-complete or a complete algebra of functions on X, F be a b-complete or a complete algebra of functions on Y, $B^*_1(X) \subseteq E$ and $B^*_1(Y) \subseteq Y$. If $\varphi : E \overset{\text{onto}}{\longrightarrow} F$ is a ring isomorphism, then there exists a unique homeomorphism $\psi : \beta_E X \to \beta_F Y$ such that $\psi(X) = Y$ and $\psi(Z(\beta_E f)) = Z(\beta_F \varphi(f))$ for every $f \in E$.*

Proof. The spaces (X, T_E) and (Y, T_F) are discrete. From Theorem 5.1 there exists a homeomorphism $\psi : \beta_E X \to \beta_F Y$ such that $\psi(Z(\beta_E f) = Z(\beta_E \varphi(f))$ for every $f \in E$. If $x \in X$, then $\{x\}$ is an open subset of $\beta_E X$, $\{\psi(x)\}$ is an open subset of $\beta)FY$ and $\psi(x) \in Y$. Hence $\psi(X) = Y$. □

From Čech's theorem ([10], Problem 3.6G(b)) and Theorem 5.2 it follows:

Corollary 5.12. *Let X and Y be first countable spaces. X and Y are Baire isomorphic of class (α,β) if and only if the algebra $B^*_\alpha(X)$ is ring isomorphic to $B^*_\beta(Y)$.*

6. Isomorphism of topological algebras of functions. On the spaces of functions in this section we consider only the topology of pointwise convergence.

Theorem 6.1. *Let E be a b-complete algebra of functions on a set X, F be a b-complete algebra of functions on a set Y and $\varphi : E \xrightarrow{\text{onto}} F$ be a ring topological isomorphism. Then there exists a unique homeomorphism $\psi : \beta_E X \to \beta_F Y$ such that $\psi(X) = Y$, $\psi|Y$ is a homeomorphism of the space (X, T_E) onto the space (Y, T_F) and $Z(\varphi(f)) = \psi(Z(f))$ for any $f \in E$.*

Proof. In the proof of Theorem 5.2 we construct a unique homeomorphism $\psi : \beta_E X \to \beta_F Y$ such that $\psi(x(J)) = y(\varphi(J))$ for every maximal prime ideal J of $E = mE$ and $Z(\varphi(f)) = \psi(Z(f))$ for all $f \in E$. The maximal ideal J of E is closed in E if and only if $x(J) \in X$. Hence $\psi(X) = Y$. The proof is complete. □

Theorem 6.2. *Let E be a complete algebra of functions on a set X, let F be a complete algebra of functions on a set Y, $mE = E \cap B^*(X)$ and let $mF = F \cap B^*(Y)$. Then the following assertions are equivalent:*
1. *The topological rings E and F are topologically isomorphic.*
2. *The topological rings mE and mF are topologically isomorphic.*
3. *The zero-set spaces $(X, Z(E))$ and $(Y, Z(F))$ are zero-set homeomorphic.*

Proof. Let $\varphi : E \to F$ be a ring isomorphism of E onto F. By virtue of Theorem 5.2, we have $\varphi(mE) = mF$. Implication $1 \to 2$ is proved. Implication $2 \to 3$ follows from Theorem 6.1. Implication $3 \to 1$ follows from Theorem 4.5. □

Corollary 6.3. *If $(X, \mathcal{Z})$ and $(Y, \mathcal{Z}')$ are zero-set spaces, then the following assertions are equivalent:*
1. *The spaces $(X, \mathcal{Z})$ and $(Y, \mathcal{Z}')$ are zero-set homeomorphic.*
2. *The topological rings $C(X, \mathcal{Z})$ and $C(Y, \mathcal{Z}')$ are topologically isomorphic.*
3. *The topological rings $C^*(X, \mathcal{Z})$ and $C^*(Y, \mathcal{Z}')$ are topologically isomorphic.*

Corollary 6.4. *If X and Y are spaces and $\alpha, \beta \geq 0$, then the following assertions are equivalent:*
1. *There exists a Baire isomorphism $\psi : X \to Y$ of class (α, β).*
2. *The zero-set spaces $(X, Z_\alpha(X))$ and $(Y, Z_\beta(Y))$ are zero-set homeomorphic.*
3. *The topological rings $B_\alpha(X)$ and $B_\beta(Y)$ are topologically isomorphic.*
4. *The topological rings $B^*_\alpha(X)$ and $B^*_\beta(Y)$ are topologically isomorphic.*

7. On F-spaces. A space X is an F-space if for every subset $U \in CZ_0(X)$ the set clU is open.

A family Σ of subsets of a set S is called a field-base of subsets of S if it satisfies the following conditions:
1. If $A, B \in \Sigma$, then $S \setminus B \in \Sigma$, $A \cup B \in \Sigma$ and $A \cap B \in \Sigma$.
2. If $x, y \in S$ and $x \neq y$, then $A \cap \{x, y\} = \{x\}$ for some $A \in S$.

For any family Σ of subsets of a set S, $B^*(S, \Sigma)$ denotes the norm closed subspace of $B^*(S)$ generated by the characteristic functions of sets in Σ.

Fix a family Σ of subsets of a set S. Let $\sigma\Sigma$ be the smallest collection of subsets which contains Σ and is closed under countable unions, $\delta\Sigma = \{\cap\{H_n : n \in \mathbf{N}\} : \{H_n : n \in \mathbf{N}\} \subseteq \Sigma\}$ and $f\Sigma = \sigma\Sigma \cap \delta\Sigma$.

For every field-base Σ of subsets of a set S denote $Z_0(S,\Sigma) = \sigma\Sigma$, $CZ_0(S,\Sigma) = \sigma\Sigma$, $Z_\alpha(S,\Sigma) = \delta(\cup\{CZ_\mu(S,\Sigma) : \mu < \alpha\})$, $CZ_\alpha(S,\Sigma) = \{S \setminus H : H \in Z_\alpha(S,\Sigma)\}$ and $CZ_\alpha(S,\Sigma) \cap Z_\alpha(S,\Sigma) = A_\alpha(S,\Sigma)$.

Fix a field-base Σ of subsets of a set S.

Lemma 7.1. *$Z_\alpha(S,\Sigma) = \sigma A_\alpha(S,\Sigma)$ for all $\alpha \leq \Omega$ and $A_\alpha(S,\Sigma)$ is a field-base of subsets of a set S.*

· Proof. Obvious. □

Lemma 7.2. *$(S, Z_\alpha(S,\Sigma))$ is a zero-set space for every α.*

Proof. Let $H \in Z_\alpha(S,\Sigma)$. Then $H = \cap\{H_n : n \in \mathbf{N}\}$ for some sequence $\{H_n \in A_\alpha(S,\Sigma) : n \in \mathbf{N}\}$. Then $V_n = S \setminus H_n \in H_n \in A_\alpha(S,\Sigma)$ and $S \setminus H = \cup\{V_n : n \in \mathbf{N}\}$. □

Corollary 7.3. *If $B_\alpha(S,\Sigma) = C(S, Z_\alpha(S,\Sigma))$ and $B^*_\alpha(S,\Sigma) = C^*(S, Z_\alpha(S,\Sigma))$ then $B_\alpha(S,\Sigma)$ is a complete algebra of functions on S and $B^*_\alpha(S,\Sigma) = B^*(S, A_\alpha(S,\Sigma))$.*

Lemma 7.4. *Let $0 \leq \alpha < \beta \leq \Omega$, $E = B^*_\alpha(S,\Sigma)$ and $F = B^*_\beta(S,\Sigma)$. Then:*
1. *$E \subseteq F$, $T_E \subseteq T_F$ and $dim \nu_E S = 0$.*
2. *If $\alpha > 0$, then $T_E = T_F$ and $\delta\Sigma$ is a base of topology T_E.*
3. *If $U \in CZ_0(\nu_E S)$, then $cl_{\nu_F S}(U \cap S)$ is open in $\nu_F S$.*

Proof. Let $H \in Z_\Omega(S,\Sigma)$. Then $H = \cup\{\cap\{H(n_1,n_2,\ldots,n_k) : k \in \mathbf{N}\} : (n_1,n_2,\ldots) \in \mathbf{N}^{\mathbf{N}}\}$ for some family $\{H(n_1,n_2,\ldots,n_k) \in \Sigma : k, n_1, n_2, \ldots, n_k \in \mathbf{N}\}$. The family $\nu A_\alpha(S,\Sigma) = \{cl_{\nu_E S} H : H \in A_\alpha(S,\Sigma)\}$ is an open and closed base of a compact space $\nu_E S$. This prooves 1. and 2.

Let $U \in CZ_0(\nu_E S)$. Then there exists a sequence $\{U_n \in \nu A_\alpha(S,\Sigma) : n \in \mathbf{N}\}$ such that $U = \cup\{U_n : n \in \mathbf{N}\}$ and $U_n \cap U_m = \emptyset$ for $n < m$. Hence $U \cap S \in \sigma A_\alpha(S,\Sigma) \subseteq A_\beta(S,\Sigma)$ and $cl_{\nu_F s}(U \cap S) \in \nu A_\beta(S,\Sigma)$. Assertion 3 is proved. □

Corollary 7.5. *If $E = B^*_\Omega(S,\Sigma)$, then $\nu_E S$ is an F-space.*

Corollary 7.6. *$b_\Omega PX$ is an F-space for every space X.*

Lemma 7.7. *If $E = B^*_\alpha(S,\Sigma)$ and $\alpha > 0$, then the following assertions are equivalent:*
1. *$\nu_E S$ is an F-space.*
2. *$A_\alpha(S,\Sigma)$ is a σ-field.*

Proof. If $A_\alpha(S,\Sigma)$ is a σ-field, then $A_\alpha(S,\Sigma) = Z_\Omega(S,\Sigma)$ and by virtue of Corollary 7.5, $\nu_E S$ is an F-space.

Let $\nu_E S$ be an F-space and $U \in CZ_\alpha(S,\Sigma)$. Then the sets U and $S \setminus U$ are open in (S, T_E). By virtue of Lemma 7.4, there are open subsets V and W of

$\nu_E S$ and a sequence $\{U_n : n \in \mathbf{N}\} \subseteq A_\alpha(S,\Sigma)$ such that $V \cup \{cl_{\nu_E S} U_n : n \in \mathbf{N}\}$, $U = \cup\{U_n : n \in \mathbf{N}\} = V \cap S$ and $W \cap S = S \setminus U$. The set $H = cl_{\nu_E S} V$ is open in $\nu_E S$. By construction, $H = cl_{\nu_E S} U$, $H \cap W \cap S = \emptyset$ and $H \cap S = U$. Hence $U \in A_\alpha(S,\Sigma)$, $A_\alpha(s,\Sigma) = CZ_\alpha(S,\Sigma)$ and $A_\alpha(S,\Sigma)$ is a σ-field. The proof is complete. □

Example 7.8. *Let X be an infinite compact F-space and Σ be a family of all open and closed subsets of X. Then Σ is a field-base of subsets of X, $E = B_0(X,\Sigma) = C(X)$, $\nu_E X = X$ and $A_0(X,\Sigma)$ is not a σ-field.*

Proposition 7.9. *Let $(X, \mathcal{Z})$ be a zero-set space, E be a b-complete algebra of functions on X and $B^*(X) \cap t_1 C(X,\mathcal{Z}) \subseteq E$.*
1. *If $\{x_n \in X : n \in \mathbf{N}\}$ is a convergent sequence of space $\nu_E X$, then there exists $n \in \mathbf{N}$ such that $x_m = x_n$ for every $n > m$.*
2. *If Y is an infinite countable subset of X, then Y is closed and discrete in (X, T_E). Moreover, if $t_2^*(C(X,\mathcal{Z})) \subseteq E$, then the closure of Y in $\nu_E S$ is homeomorphic to the Stone-Čech compactification βY of the discrete space Y.*
3. *Every infinite subset Z of X contains an infinite countable subset Y such that the closure of Y in $\nu_E X$ is homeomorphic to βY of the discrete space Y.*
4. *Every pseudocompact subspace of the space (X, T_E) is finite.*
5. *If $Y \subseteq X$ and $Z = cl_{\nu_E X} Y$, then the character of every point $x \in Z \setminus Y$ in Z is uncountable.*

Proof. Let $Y = \{y_n \in X : n \in \mathbf{N}\}$ and $x_n \neq x_m$ for $n < m$.

If $\sigma_\alpha = Z(t_\alpha C(X,\mathcal{Z})) \cap CZ(t_\alpha C(X,\mathcal{Z}))$, then there exists a family $\{\Phi_n \in \Sigma_1 : n \in \mathbf{N}\}$ such that:

(i) $y_n \in \Phi_n$ for every $n \in \mathbf{N}$;

(ii) if $n < m$, then $\Phi_n \cap \Phi_m = \emptyset$.

By construction, $\{\Phi_n : n \in \mathbf{N}\}$ is a discrete family of open and closed subsets of the space (X, T_E). For every $H \subseteq \mathbf{N}$ we consider a function $f_H : X \to \mathbf{R}$ such that $f_H^{-1}(0) = \cup\{\Phi_n : n \in \mathbf{N}\}$ and $f_H^{-1}(1) = X \setminus \cap\{\Phi_n : n \in \mathbf{N}\}$. If $t_2^* C(X,\mathcal{Z}) \subseteq E$, then $f_H \in E$ for every $n \in \mathbf{N}$ and every two of disjoint subsets of Y have disjoint closures in $\nu_E X$.

Fix an infinite subset Z of X. Then there exists a sequence $\{V_n : n \in \mathbf{N}\}$ such that:

(iii) $X \setminus V_n \in \mathbf{Z}$ for any $n \in \mathbf{N}$;

(iv) $V_n \cap Z \neq 0$ for all $n \in \mathbf{N}$;

(v) if $n < m$, then $V_n \cap V_m = \emptyset$.

Fix a subset $Y = \{y_n \in Z \cap V_n : n \in \mathbf{N}\}$. Let $W_1 = X \setminus \cup\{V_n : n \in \mathbf{N}\}$ and $W_n = V_n$ for $n > 1$. Then $\{W_n : n \in \mathbf{N}\}$ is a discrete cover of the space (X, T_E). For every $H \subseteq N$ we consider the function $g_H : X \to \mathbf{R}$ such that $g_H(\cup\{W_n : n \in \mathbf{N}\}) = 0$ and $g_H(X \setminus \cup\{W_n : n \in \mathbf{N}\}) = 1$. It is clear that $g_H \in E$. Hence the closure of Y in $\nu_E X$ is homeomorphic to βY of the discrete space Y. Assertions 2 and 3 are proved. Assertions 1, 4 and 5 follow from assertion 2. The proof is complete. □

8. On $\mathcal{Z}$-perfect mappings. A subset Φ in a space X is called bounded if every real-valued continuous function on X is bounded on Φ. The set Φ is bounded in X if and only if the set $cl_{\nu X}\Phi$ is compact.

Definition 8.1. *A mapping $f : X \to Y$ is called Z-perfect if it is continuous, $f(H)$ is closed in Y for every $H \in Z_0(X)$ and the fiber $f^{-1}(y)$ is bounded in X for every point $y \in Y$.*

Every perfect mapping is Z-perfect.

Lemma 8.2. *Let $f : X \to Y$ be a Z-perfect mapping. If all fibers $f^{-1}(y)$ are compact subsets of X, then f is perfect.*

Proof. Let H be a closed subset of X and $\{H_\mu : \mu \in M\} = \{A \in Z_0(X) : H \subseteq A\}$. Then $f(H) = \cap\{H_\mu : \mu \in M\}$. □

Lemma 8.3. *Let $\nu f : \nu X \to Y$ be a continuous extension of a mapping $f : X \to Y$ of a space X onto a space Y with a countable pseudocharacter. Then the following assertions are equivalent:*
1. *f is Z-perfect.*
2. *$\nu f : \nu X \to Y$ is perfect.*

Proof. Let f be Z-perfect. Fix a point $x \in \nu X \setminus cl_{\nu X} f^{-1}(y)$. There exist sets $A, B \in Z_0(\nu X)$ such that $cl_{\nu X} f^{-1}(y) \subseteq B$, $x \in A$ and $A \cap B = \emptyset$. Then $V = Y \setminus f(A \cap X)$ is open in Y and $y \in W \subseteq V$ for some $W \in Z_0(Y)$. If $x \in \nu f^{-1}(y)$, then $x \in P = A \cap \nu f^{-1} W \in Z_0(\nu X)$ and $P \cap X = \emptyset$. Hence $\nu f^{-1}(y) = cl_{\nu X} f^{-1}(y)$ and f is a compact mapping.

Fix $H \in Z_0(\nu X)$. If $y \in \nu f(H) \setminus f(X \cap H)$, then $H \setminus \nu f^{-1}(X \cap H) = P$ is a G_δ-set of νX and $P \cap X = \emptyset$. Hence $f(H \cap X) = \nu f(H)$ and νf is Z-perfect. By virtue of Lemma 8.2, νf is perfect.

Suppose that νf is perfect. Then f is bounded, i.e. the fiber $f^{-1}(y)$ are bounded in X. Let $H \in Z_0(\nu X)$ and $\nu f(H) \neq f(X \cap H)$. If $y \in \nu f(H) \setminus f(X \cap H)$, then $f^{-1}(y) \in X \setminus H$, $\{y\}$ is a G_δ-subset of Y, $P = H \cap \nu f^{-1}(y) \subseteq \nu X \setminus X$ and P is a G_δ-set in νX. Hence $P = \emptyset$ and $\nu f(H) = f(H \cap X)$ for every $H \in Z_0(\nu X)$. It follows from equality $Z_0(X) = \{X \cap H : H \in Z_0(\nu)\}$ that the mapping f is Z-perfect. The proof is complete. □

Corollary 8.4 ([3], Proposition 1.1). *Every continuous mapping $f : X \to Y$ of a pseudocompact space X into a metric or a first countable real compact space Y is Z-perfect.*

Lemma 8.5. *Let $f : X \to Y$ be a Z-perfect mapping into a separable metric space Y and $G : X \to S$ be a continuous mapping onto a separable space S. Then the diagonal mapping $\varphi = \Delta\{f, g\} : X \to Y \times S$, where $\varphi(x) = (f(x), g(y))$, is Z-perfect.*

Proof. It is clear that $\nu\varphi = \Delta\{\nu f, \nu g\} : \nu X \to Y_1 = \varphi(X)$ and Y_1 is a closed subset of $Y \times S$. The mapping νf is perfect. From Theorem 3.7.9 [10] it follows that $\nu\varphi$ is perfect. By Lemma 8.3 the mapping φ is Z-perfect. The proof is complete. □

Remark 8.6. The implication $1 \to 2$ in Lemma 8.3 is correct for every space Y.

Example 8.7. Let S be a pseudocompact non-compact space and $X = (\beta S \times [0,1]) \setminus ((\beta S \setminus S) \times \{0\})$. Consider the continuous mapping $f : X \to Y = \beta S$ onto a compact space Y, where $f(x,y) = x$ for every point $(x,y) \in X$. By construction, $\beta X = \nu X = \beta S \times [0,1]$, $H = X \times \{0\} \in Z_0(X)$ and $f(H) = S \subseteq \beta S$. Hence $\nu f : \nu X \to Y$ is a perfect mapping and the mapping f is not Z-perfect.

9. Factorization theorem. A space X is called a paracompact p-space if there exists a perfect mapping of X onto a metric space [1]. Every Čech complete paracompact space is a paracompact p-space (Z. Frolic [10], Problem 5.5.9).

Proposition 9.1. *Let $f : X \to Y$ be a Baire measurable mapping of class α. Then there exist a space S, a continuous mapping $\varphi : X \to S$ and a mapping $\psi : S \to Y$ such that:*
1. *$f = \psi\varphi$ and $w(S) = w(Y)$.*
2. *If X is a Lindelöf p-space, i.e. X admits a perfect mapping onto a separable metric space [1], then the mapping φ is perfect.*
3. *If Y is a Lindelöf space, then ψ is a Baire measurable mapping of class α.*
4. *If Y is a separable metric space and f is a Z-perfect mapping, then the mapping φ is Z-perfect.*

Proof. Fix a closed base $\{H_\mu \in Z_0(Y) : \mu \in M\}$ of a space Y, where $|M| = w(Y)$. Then for every $\mu \in M$ there exist a separable metric space Y_μ and a continuous mapping $\varphi_\mu : X \to S_\mu$ such that $\varphi_\mu^{-1}(\varphi_\mu(f^{-1}H_\mu)) = f^{-1}H_\mu \in Z_\alpha(X)$ and $\varphi_\mu(f^{-1}H_\mu) \in Z_\alpha(S_\mu)$ (see [2], Lemma 1.2). Consider the diagonal mapping $\varphi = \Delta\{\varphi_\mu : \mu \in M\} : X \to S = \varphi(S) \subseteq \Pi\{S_\mu : \mu \in M\}$, where $\varphi(x) = \{\varphi_\mu(x) : \mu \in M\}$. The mapping φ is continuous and $w(Z) = |M| = w(Y)$. If $x \in S$, then $\psi(x) = f(\varphi^{-1}(x))$ is a one-point subset of Y. Hence $\psi : S \to Y$ is a single-valued mapping and $f(X) = \psi(\varphi(x))$ for every point $x \in X$. Assertion 1 is proved.

If $g : X \to Y_1$ is a perfect or a Z-perfect mapping onto a separable metric space Y_1, then we assume that $Y_1 = S_\mu$ and $g = \varphi_\mu$ for some $\mu \in M$. If the mapping g is perfect, then by Theorem 3.7.9 [10] the mapping φ is perfect. If Y is a separable metric space and g is a Z-perfect mapping, then from Lemma 8.5 it follows that the mapping φ is Z-perfect. The assèrtions 2 and 4 are proved.

Consider the projections $\pi_\mu : S \to S_\mu$. By construction,

$$\psi_\mu^{-1}(H_\mu) = \pi_\mu^{-1}(\varphi_\mu(f^{-1}(H_\mu)))$$

and $\psi^{-1}(H_\mu) \in Z_\alpha(S)$ for every μ. Let Y be a Lindelöf space and $H \in Z_0(Y)$. Then there exists a countable subset $A \subseteq M$ such that $H = \cap\{H_\mu : \mu \in A\}$. Therefore $\psi^{-1}H = \cap\{\psi^{-1}H_{mu} : \mu \in A\} \in Z_\alpha(S)$. This proves assertion 3. □

Corollary 9.2. *Let X and Y be Baire homeomorphic Linelöf p-spaces. Then* $w(X) = w(Y)$.

Corollary 9.3 ([4], Section 12). *Let X and Y be Baire homeomorphic compact spaces. Then $w(X) = w(Y)$.*

10. Extension of mappings.

Theorem 10.1. *Let $\varphi : X \to Y$ be a Baire measurable mapping of class α. Then:*
1. *There exists a Baire measurable mapping $\nu\varphi : \nu X \to \nu Y$ of class α such that $\varphi = \nu\varphi|X$.*
2. *If $\psi : \nu X \to \nu Y$ is a Baire measurable mapping and $\varphi = \psi|X$ then $\psi = \nu\varphi$.*

Proof. We assume that $Y \subseteq R^{C(Y)} = \Pi\{R_g : g \in C(Y)\}$ and $y = \{g(y) : g \in C(Y)\}$ for every $y \in Y$. Then $g : Y \to \mathbf{R} = R_g$ is a continuous projection. If $f \in C(Y)$, then $g_f : X \to \mathbf{R} = R_f$, where $g_f(x) = f(\varphi(x))$, is a Baire measurable mapping of class α. By Theorem 3.1 there exists a Baire measurable function $\varphi_f \in B_\alpha(\nu X)$ such that $g_f = \varphi_f|X$ and $g_f(X) = \varphi_f(X)$. Consider the mapping $\nu\varphi = \Delta\{\varphi_f : f \in C(Y)\} : \nu X \to S = \nu\varphi(\nu X) \subseteq \mathbf{R}^{C(Y)}$, where $\nu\varphi(x) = \{\varphi_f(x) : f \in C(Y)\}$. Suppose there exists a point $b = \{b_f : f \in C(Y)\} \in S \setminus \nu Y$. The set νY is closed in $\mathbf{R}^{C(Y)}$. There exists a finite subset A of $C(Y)$ such that $W = \{z = \{z_f : f \in C(Y)\} \in \mathbf{R}^{C(Y)} : z_f = b_f$ for every $f \in A\} \in Z_0(\mathbf{R}^{C(Y)})$, $b \in W$ and $W \cap \nu Y = \emptyset$. Then $H = \nu\varphi^{-1}(W) = \cap\{\varphi_f^{-1}(b_f) \in Z_\alpha(\nu X)$ and $\nu\varphi^{-1}(b) \in H \subseteq \nu X \setminus X$. Therefore $S \subseteq \nu Y$. By construction, $\nu\varphi^{-1}(Z(\nu f)) = Z(\varphi_f) \in Z_\alpha(\nu X)$ and $\varphi = \nu\varphi|X$. In particular, $\nu\varphi$ is a Baire measurable mapping of class α.

Let $\psi : \nu X \to \nu Y$ be a Baire measurable mapping and $\varphi = \psi|X$. Suppose that $x \in \nu X$ and $\psi(x) \neq \nu\varphi(x)$. There exist H, $B \in Z_0(\varphi Y)$ such that $\psi(x) \in H$, $\nu\varphi(x) \in B$ and $H \cap B = \emptyset$. Then $x \in P = \psi^{-1}H \cap \nu\varphi^{-1}B \subseteq \nu X \setminus X$ and P is a Baire set. This contradiction finishes the proof. □

Corollary 10.2. *Let $\varphi : X \to Y$ be a Baire homeomorphism of class (α, β). Then:*
1. *There exists a unique Baire homeomorphism $\nu\varphi : \nu X \to \nu Y$ of class (α, β) such that $\varphi = \nu\varphi|X$.*
2. *If φ is a Baire isomorphism of class (α, β), then $\nu\varphi$ is a Baire isomorphism of class (α, β).*
3. *The space X is real compact if and only if Y is real compact.*

11. Equivalence of families of sets. Non existence of ring isomorphism. A family $\mathcal{Z}$ of subsets of a set X is equivalent to a family $\mathcal{L}$ of subsets of a set Y if there exists a one-to-one mapping $\varphi : X \to Y$ such that $\varphi(\mathcal{L}) = \mathcal{Z}$. The mapping φ is called an $(\mathcal{L}, \mathcal{Z})$-isomorphism. This notation was introduced by E. Spilrain [25, 26] (see [7]).

Theorem 11.1 ([20, 21] for complete separable metric spaces). *Let X and Y be separable metric spaces, X or Y contains some non-empty compact perfect subset and $\alpha < \Omega$. If $\mathcal{L} \subseteq CZ_\alpha(X)$ and $Z_\alpha(Y) \subseteq \mathcal{Z}$, then the families $\mathcal{L}$ and $\mathcal{Z}$ are not equivalent.*

Proof. Let $\varphi : X \to Y$ be an $(\mathcal{L}, \mathcal{Z})$-isomorphism. Then $\varphi^{-1}(Z_\alpha(Y)) \subseteq CZ_\alpha(X)$ and φ is a Baire measurable mapping. If F is an uncountable compact subset of X, then there exists an uncountable subset H of F such that $\varphi|H$ is a topological embedding (see [16], p. 32). The space $\Phi = \varphi(H)$ is compact and there exists a set $P \in (\mathcal{Z} \cap Z_\alpha(\Phi) \setminus CZ_\alpha(Y)$. By premise $\varphi^{-1}P \in \mathcal{L} \subseteq CZ_\alpha(X)$ and, by construction, $\varphi^{-1}P \in Z_\alpha(H) \setminus CZ_\alpha(X)$. This contradiction completes the proof. □

Theorem 11.2. *Let X admit a Z-perfect mapping onto a separable metric space and Y contain a compact subset $H \in Z_0(Y)$ such that there exists a continuous mapping of H onto $[0,1]$. Then:*

1. *For every Baire homeomorphism $h : \nu X \to \nu Y$ of class (α,β) there are: separable metric spaces Z and S, an uncountable compact subset Φ of Z, a perfect mapping $\varphi : \nu X \overset{\text{onto}}{\longrightarrow} Z$, a continuous mapping $\psi : \nu Y \overset{\text{onto}}{\longrightarrow} S$ and a Baire homeomorphism $g : Z \to S$ of class (α,β) such that $g|\Phi$ is a topological embedding, $\psi^{-1}(g(\Phi)) \subseteq H$ and $\psi(h(x)) = g(\varphi(x))$ for every $x \in X$.*
2. *If $\alpha < \Omega$, $\mathcal{L} \subseteq CZ_\alpha(\nu X)$ and $Z_\alpha(\nu Y) \subseteq \mathcal{Z} \subseteq Z_\Omega(\nu Y)$ or $Z_\alpha(\nu X) \subseteq \mathcal{L} \subseteq Z_\Omega(\nu X)$ and $\mathcal{Z} \subseteq CZ_\alpha(\nu Y)$, then $\mathcal{L}$ is not equivalent to $\mathcal{Z}$.*
3. *If $\alpha < \Omega$, $\mathcal{L} \subseteq CZ_\alpha(X)$ and $Z_\alpha(Y) \subseteq \mathcal{Z} \subseteq Z_\Omega(Y)$ or $Z_\alpha(X) \subseteq \mathcal{L} \subseteq Z_\Omega(X)$ and $\mathcal{Z} \subseteq CZ_\alpha(Y)$, then $\mathcal{L}$ is not equivalent to $\mathcal{Z}$.*
4. *If $\alpha \neq \beta$, then $Z_\alpha(X)$ is not equivalent to $Z_\beta(Y)$ and $Z_\alpha(\nu X)$ is not equivalent $Z_\beta(\nu Y)$.*
5. *If $\alpha \neq \beta$, then $B_\alpha(X)$ is not ring isomorphic to $B_\beta(Y)$.*
6. *If $0 \neq \alpha \neq \beta \neq 0$, then $B^*_\alpha(X)$ is not ring isomorphic to $B^*_\beta(Y)$.*

Proof. Let $h : \nu X \to \nu Y$ be a Baire homeomorphism of class (α,β). There exist a separable metric space Z_1 and a continuous mapping $\psi_1 : \nu Y \to Z_1$ such that $\psi^{-1}(\psi_1(H)) = H$ and $\psi_1(H)$ is an uncountable compact. The mapping $\varphi_1 : \nu X \to Z_1$, where $\varphi_1(x) = \psi_1(h(x))$ is Baire measurable of class α. By Proposition 9.1 there exist a separable metric space Z_2, a perfect mapping $\varphi_2 : \nu X \to Z_2$ and a Baire measurable mapping $h_1 : Z_2 \to Z_1$ of class α such that $\varphi_1(x) = h_1(\varphi_2(x))$ for every $x \in \nu X$. The mapping $\psi_2 = \varphi_2 \circ h^{-1} : \nu Y \to Z_2$ is Baire measurable of class β. Then there exist a separable metric space Z_3, a continuous mapping $\psi_3 : \nu Y \to Z_3$ and a Baire measurable mapping $h_2 : Z_3 \to Z_2$ of class β such that $\psi_2 = h_2 \circ \psi_3$. Therefore there exist a sequence $\{Z_n : n \in \mathbf{N}\}$ of separable metric spaces and mappings $\{\varphi_n : \nu X \to Z_n, \psi_n : \nu Y \to Z_n, h_n : Z_{n+1} \to Z_n : n \in \mathbf{N}\}$ such that:

1. $\varphi_n = \psi_n \circ h$, $\psi_n = \varphi_n \circ h^{-1}$, $\varphi_n = h_n \circ \varphi_{n+1}$ and $\psi_n = h_n \circ \psi_{n+1}$ for every $n \in \mathbf{N}$;
2. The mapping φ_{2n} is perfect, the mappings φ_{2n-1}, h_{2n-1} are Baire measurable of class α, the mappings ψ_{2n}, h_{2n} are Baire measurable of class β and the mapping ψ_{2n-1} is continuous for every $n \in \mathbf{N}$.

The mapping $\varphi : \nu X \to Z = \varphi(Z) \subseteq \Pi\{Z_{2n} : n \in \mathbf{N}\}$, where $\varphi(x) = \{\varphi_{2n}(x) : n \in \mathbf{N}\}$ is perfect and the mapping $\psi : \nu Y \to S = \psi(S) \subseteq \Pi\{Z_{2n-1} : n \in \mathbf{N}\}$, where $\psi(y) = \{\psi_{2n-1}(y) : n \in \mathbf{N}\}$, is continuous. By construction, $\varphi^{-1}(\varphi H) = H$ and the compact $\varphi(H)$ is uncountable. The mapping $g : Z \to S$, define through $g(\{z_{2n}\}) = \{h_{2n-1}(z_{2n}\}$ and $g^{-1}(\{z_{2n+1}\}) = \{h_{2n}(z_{2n+1})\}$ is a Baire homeomorphism of class (α,β). Then there exists a non-empty compact perfect subset Φ of Z such that $g(\Phi) \subseteq \psi(H)$ and $g|\Phi$ is a topological embedding (see [16]). For every $\mu < \Omega$ we fix the set $U_\mu \in Z_{mu}(\Phi) \setminus CZ_\mu(\Phi)$. By virtue of the J. Saint-Raymond theorem [23, 22] and the results of paper ([6], Section 3) we have

$$V_\mu = \varphi^{-1} U_\mu \in Z_\mu(\nu x) \setminus CZ_\mu(\nu X) \text{ and}$$

$$W_\mu = \psi^{-1}(gU_\mu) \in Z_\mu(H_\mu) \setminus CZ_\mu(H) \subseteq Z_\mu(\nu Y) \setminus CZ_\mu(\nu Y).$$

By construction, $h(V_\mu) = W_\mu$ for every μ. From Theorem 3.1 it follows that $V_\mu \cap X \in Z_\mu(X) \setminus CZ_\mu(X)$. This proves Assertions 1, 2, 3 and 4. Assertion 5 follows from Theorem 5.2 and assertion 4. Assertion 6 follows from Assertion 5 and Corollary 5.8.

Corollary 11.3. *Let X and Y be Lindelöf p-spaces, and X or Y contain a non-empty compact perfect subset. Then the conclusions of Theorem* 11.2 *hold true.*

Corollary 11.4. *Let X and Y be pseudocompact spaces and βX or βY contain a non-empty perfect subset. Then the conclusions of Theorem* 11.2 *hold true.*

Corollary 11.5. *Let X and Y be compact spaces and X or Y contain a non-empty perfect subset. Then:*
1. *The assertions of Theorem* 11.2 *hold true.*
2. *If $0 < \alpha < \beta$, then $B^*_\alpha(X)$ is not ring isomorphic to $B^*_\beta(Y)$.*

Theorem 5 in [14] (see [22], Theorem 6.3.2) states that Assertion 2 of Corollary 11.5 holds true for all $0 \leq \alpha < \beta$. This is in contradiction with Example 5.10. Hence the proof in [14, 22] is not quite correct.

Corollary 11.6 (F. K. Dashiell [8] for compact metric spaces). *Let X and Y be first countable Lindelöf p-spaces and X or Y contain a non-empty compact perfect subset. If $\alpha \neq \beta$, then $B_\alpha(X)$ is not ring isomorphic to $B_\beta(Y)$ and $B^*_\alpha(X)$ is not ring isomorphic to $B^*_\beta(Y)$.*

Theorem 11.7. *Let $0 \leq \alpha < \beta \leq \Omega$, Y be a compact space, containing a non-empty perfect subset and X be a compact space satisfying one of the following conditions:*
1. *X contains no subspaces homeomorphic to βN.*
2. *X is first countable space.*
3. *X is sequential space.*
4. *X is sequentially compact.*
5. *Every closed infinite subset of X contains a non-trivial convergent sequence.*
6. *X is scattered.*
7. *The tightness $t(X) < 2^{\aleph_0}$.*
8. *X is a hereditarily normal space.*
9. *X is a hereditarily separable space.*

Then $B^*_\alpha(X)$ is not ring isomorphic to $B^*_\beta(Y)$.

Proof. Implications $2 \to 3 \to 4 \to 5 \to 1$, $6 \to 5 \to 1$, $8 \to 1$ and $9 \to 7 \to 1$ are obvious.

If $0 < \alpha < \beta$, then Theorem 11.7 follows from Theorem 11.2.

Let $0 = \alpha < \beta$ and $\psi : B_0^*(X) = C(X) \overset{\text{onto}}{\longrightarrow} B_\beta^*(Y)$ be a ring isomorphism. Then there exists some homeomorphism h of the compact space X onto the compact space $b_\beta PY$. By Proposition 7.9 there exists a countable subset Z of Y such that the closure Φ of Z in $b_\beta PY$ is homeomorphic to the Stone-Čech compactification βN of the discrete space N. Hence βN is embedded in X. This contradiction complete the proof. □

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