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DIFFERENTIAL GAME DESCRIBED BY A HYPERBOLIC SYSTEM. ε–MAXIMINS AND POSITIONAL STRATEGIES

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Abstract. The present paper deals with an antagonistic differential game of hyperbolic type with a vector pay-off function. A solution has been defined by a differential game — ε–Slater minimax (maximin) strategy. An existence theorem and some properties of ε–Slater maximins (minimaxes) from the theory of multicriterial classic antagonistic games have been proved for the case studied in the present paper. Thus for the set of ε–Slater maximins (minimaxes) the inward and outward stability is proved. The assertion for the relation between the ε–Slater saddle point and the ε–Slater maximin and minimax strategies, is given. An example is considered which shows that in this assertion some conditions of regularity cannot be reduced. Also games with a separable pay-off function have been considered. For these games, a simple description of the set of ε–Slater saddle points consisting of program strategies is given. It is shown through an example that for this case such a description cannot be given when the corresponding strategies of the ε–Slater saddle point are non-programm. The same example is used also to prove that the property of separability for the linear problem is not valid for non-program strategies.

Introduction. The following multicriterial antagonistic differential game with a vector pay-off function is considered:

$$\langle \Xi, \{\mathcal{U}, \mathcal{V}\}, \{\rho_i(h(T))\}_{i\in\mathbf{N}}\rangle, \tag{0.1}$$

where $\mathbf{N} = \{1, \dots, N\}$, $N \geqq 1$ is the number of criteria.

The controllable system Ξ is described by the boundary-value problem of hyperbolic type

$$\frac{\partial^2 y}{\partial t^2} = Ay + b_1 u_1 + c_1 v_1 + f_1 \text{ in } G = (t_0, T) \times \Omega, \tag{0.2}$$

$$y_{|t=t_0} = y_0, \ \frac{\partial y}{\partial t}_{|t=t_0} = y_1 \ \text{ in } \Omega \tag{0.3}$$

$$\sigma_1 \frac{\partial y}{\partial \nu_A} + \sigma_2 y = b_2 u_2 + c_2 v_2 + f_2 \text{ in } \Sigma = (t_0, T) \times \Gamma, \tag{0.4}$$

where $\sigma_i \in \{0,1\}$, $i = 1,2$, $\sigma_1 + \sigma_2 \geqq 1$.

The solution y of (0.2)-(0.4) is taken as in [4, p. 116, Theorem 4.1], see also [12, Lemma 1]. Let $h(t) = (y(t), \partial y(t)/\partial t)$. The exact description of the participant symbols of (0.2) - (0.4) is given in the Introduction further on.

The properties of ε-Slater minimaxes (maximins) of game (0.1) will be considered in the present paper. Let $\rho = (\rho_1, \dots, \rho_N)$ be the vector pay-off function of a multicriterial problem. It is known that a set B is inward stable if there exist no two elements $x' \in B$ and $x'' \in B$ for which $\rho(x') > \rho(x'')$, i.e. $\rho_i(x') > \rho_i(x'')$ for $\forall i = 1, \dots, N$. One of the disadvantages with the use of vector ε-saddle points is the lack of inward stability. This means that there can exist two vector ε-Slater saddle points for which the values of all components of the vector $\rho(\cdot)$ at one of them are greater than the respective components of the other ε-saddle point. Therefore another solution of game (0.1) - ε-Slater minimax is introduced in the present paper and it possesses "the guarantee" properties of the ε-saddle point [5, Ch. I] together with the inward stability.

In Section 1 all these properties of ε-Slater minimaxes (maximins) are considered: the existence theorem and the assertions for inward and outward stability are given too. The assertion for the relation between the ε-Slater saddle points and the ε-Slater minimax (maximin) strategies (Lemma 1.3) is proved. An example is given so as to show that Lemma 1.3 does not hold if the regularity conditions (1.5) are not satisfied.

In Section 2 the games with a separable pay-off function are considered. Such classic (non-differential) games are well studied and they have comparatively simple description with respect to the set of saddle (ε-saddle) points (for example see [5, p.81-91]). The description of the set of ε-Slater saddle points (Lemma 2.2) and a property of separability for the linear problem - (2.6) can be given only for program strategies. Theorem 2.1 and Theorem 2.2 show that Lemma 2.2 and equality (2.6) are not valid if the corresponding strategies are non-program.

Some results from multicriterial optimization are used in the present paper. Multicriterial classic problems (games) are studied in [5], [7], [2], [8]. Multicriterial dynamic problems (games) described by a system of ordinary differential equations are considered in [1], [6], and the corresponding examples are given there as well. The present paper deals with some problems of positional control of systems described by means of a hyperbolic boundary-value problem. Such control problems arise in distributed parameter systems, with motion of oscillated character [11, p. 39]. Problems of that type are the control of fading fluctuations of gas streams (liquid, electricity or other physical substances) into long pipelines; fading (or generating) of waves into experimental pools or electromagnetic fluctuations into wavelines and resonance accelerators, etc., see [11]. The present paper is a continuation of [12] and the results of [5] and [7] for non-differential games are used.

It is supposed that in problem (0.2) – (0.4), the set $\Omega \neq \emptyset$ is bounded and open in $\mathbf{R}^n$ with boundary $\Gamma = \partial\Omega$; Ω and Γ satisfy the following conditions of regularity [10, p. 212, 222, Conditions 1), 2) and $\mathcal{R}$]:

1) Ω is a strictly Lipschitz domain [10, p. 30-31];

2) For almost all points $x^0 \in \Gamma$ (in the sense of measure of Γ) there exist a plane tangential to Γ, such that the equation of Γ in a small neighbourhood of $x^0 \in \Gamma$ in a local Descartes coordinate system is of the form $\eta_n = \omega(\eta_1, \dots, \eta_{n-1})$, where the axis η_n is directed to the outside normal to Γ in x^0 and the axes $\eta_1, \dots, \eta_{n-1}$ lie in the plane tangential to Γ in x^0. It is supposed that there exist all partial derivatives of ω of second order and that the eigenvalues $\mu_1(x^0), \dots, \mu_n(x^0)$ of the quadratic form $\sum_{k,l=1}^{n-1} \frac{\partial^2\omega}{\partial\eta_k\partial\eta_l}\xi_k\xi_l$ in x^0 satisfy the condition $\sup_{k,x^0\in\Gamma}\{\mu_k(x^0)\} \leqq K$ for some constant $K \geqq 0$;

$\mathcal{R}$) If $\sigma_1 = 0$, then the problem $\widetilde{\Delta} z = \varphi$, $z_{|\Gamma} = 0$ in Ω is solvable in $H^2_{2,0}(\Omega) \stackrel{\text{def}}{=} H^1_0(\Omega) \cap H^2_2(\Omega)$ for some dense in $L_2(\Omega)$ subset of functions φ, (where $\widetilde{\Delta}$ is the Laplace operator in Ω).

It is supposed that the coefficients of boundary-value problem (0.2)-(0.4) satisfy the conditions: $y_0 = y_0(x) \in L_2(\Omega)$, $y_1 = y_1(x) \in (H^1_2(\Omega))^*$, $f_1 = f_1(x,t) \in L_2(G)$, $f_2 = \sum_{j=1}^m f^{(1)}_{2j}(t) f^{(2)}_{2j}(x)$, where $f^{(1)}_{2j}(t) \in L_\infty(t_0, T)$, $f^{(2)}_{2j}(x) \in L_2(\Gamma)$, $j = 1, \dots, m$; $H^s_p(\Omega) = W^s_p(\Omega)$, $L_2(\Omega) = H^0_2(\Omega)$, $H^1_0(\Omega) = \mathring{W}^1_2(\Omega)$ etc., see [3,4,10,12], are the corresponding Sobolev spaces, H^* is the dual functional space of H (for example $H^{-1}_2(\Omega) = (H^1_0(\Omega))^*$, etc.). The functions $b_1 = b_1(x,t)$ and $c_1 = c_1(x,t)$($b_2 = b_2(x)$ and $c_2 = c_2(x)$) are measurable, bounded in $G(\Gamma)$ and take values in $\mathbf{R}^{r_1}$ and $\mathbf{R}^{m_1}$($\mathbf{R}^{r_2}$ and $\mathbf{R}^{m_2}$) respectively. The operator A is of the form: $A[.] = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial[.]}{\partial x_j}\right) - a(x)[.]$, where $a_{ij}(x) = a_{ji}(x)$, $a(x) \geqq a_0 = \text{const} > 0$, $\partial a_{ij}(x)/\partial x_k$, $i,j,k = 1, \dots, n$ are functions which are measurable (in the Lebesgue sense), bounded in Ω and there exist constants $\alpha > 0$ and $\beta > 0$ such that for each $x \in \Omega$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, the following inequalities hold:

$$\alpha \sum_{j=1}^n \xi_j^2 \leqq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqq \beta \sum_{j=1}^n \xi_j^2;$$

$\frac{\partial[.]}{\partial\nu_A} = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial[.]}{\partial x_j}\cos(\nu, x_i)$ is the conormal derivative, corresponding to the self-adjoint elliptic operator A of second order and ν is the exterior normal to Γ.

Next the sets of strategies will be described. The following sets $P(t) = P_1(t) \times P_2(t)$ ($Q(t) = Q_1(t) \times Q_2(t)$), $t \in [t_0, T]$, $0 \leqq t_0 < T$ where $P_1(t) \subset L_2(\Omega; \mathbf{R}^{r_1})$ and

$P_2(t) \subset \mathbf{R}^{r_2}$ ($Q_1(t) \subset L_2(\Omega; \mathbf{R}^{m_1})$ and $Q_2(t) \subset \mathbf{R}^{m_2}$) are given. These sets are convex, closed (in the respective spaces), measurable and uniformly bounded with respect to t, $\forall t \in [t_0, T]$. The vector-functions $u = (u_1, u_2) \in P(t)$ and $v = (v_1, v_2) \in Q(t)$ are called program strategies.

It was proved in [12, Lemma 1] that all the conditions of [4, p. 116,Theorem 4.1] are satisfied and hence the solution of (0.2) – (0.4) is defined as in [4]. Let $\mathcal{H} = L_2(\Omega) \times (H_2^1(\Omega))^*$ for $\sigma_1 = 1$ and $\mathcal{H} = H_2^{-1}(\Omega) \times (H_{2,0}^2(\Omega))^*$, (where $H_{2,0}^2(\Omega) = H_0^1(\Omega) \cap H_2^2(\Omega)$) for $\sigma_1 = 0$. It was proved in [12, Theorem 1] that $h(.) = (y(.), \partial y(.)/\partial t) \in C([t_0, T], \mathcal{H})$.

The present paper uses the formalization of a differential game described by a hyperbolic system. The solution of the initial boundary-value problem is treated as in [4] in one space and the controllable process obtained is considered for another space. The respective objects are linked by one and the same Fourier series.

The result of game (0.1) is evaluated by criteria, given by the functionals ρ_i in $\mathcal{H}$, $i \in \mathbf{N}$; $\rho(h(T)) = (\rho_1(h(T)), \ldots, \rho_N(h(T)))$ is called a vector pay-off function of game (0.1). It is supposed that the functionals ρ_i are strongly continuous (s.-continuous) in $\mathcal{H}$. The first player choosing the strategy $U \in \mathcal{U}$ strives to smaller possible values of all criteria $\rho_i(h(T))$, $i \in \mathbf{N}$; the second player using a strategy $V \in \mathcal{V}$ strives to their maximization. Each player chooses a strategy of his own which is independent of the other player's strategy.

As in [12], we are going to use the following notations: $\mathbf{R}^N_> = \{\rho = (\rho_1, \ldots, \rho_N) \in \mathbf{R}^N \mid \rho_i > 0, \forall i \in \mathbf{N}\}$, $\mathbf{R}^N_{\geqq} = \{\rho = (\rho_1, \ldots, \rho_N) \in \mathbf{R}^N \mid \rho_i \geqq 0, \forall i \in \mathbf{N}\}$, $\mathbf{R}^N_{\geq} = \{\rho = (\rho_1, \ldots, \rho_N) \in \mathbf{R}^N \mid \rho_i \geqq 0, \forall i \in \mathbf{N}, \rho \neq 0_N\}$, where 0_N is the zero-vector in $\mathbf{R}^N$, $\rho^{(1)} > \rho^{(2)} \iff \rho^{(1)} - \rho^{(2)} \in \mathbf{R}^N_> \iff \rho^{(2)} < \rho^{(1)}$, $\rho^{(1)} \not> \rho^{(2)} \iff \rho^{(1)} - \rho^{(2)} \notin \mathbf{R}^N_>$. The other relations are introduced analogously. For example $\rho^{(1)} \not\geq \rho^{(2)}$ if and only if the relation $\rho^{(1)} \geq \rho^{(2)}$ is not satisfied, i.e. if and only if $\exists i_0 \in \mathbf{N} : \rho^{(1)}_{i_0} < \rho^{(2)}_{i_0}$ or $\rho^{(1)} = \rho^{(2)}$.

Let $\Delta \in \boldsymbol{\Delta}$ be an arbitrary partition of the interval $[t_0, T]$ by the points $t_0 = \tau_0 < \tau_1 < \ldots < \tau_{m(\Delta)} = T$, ($\boldsymbol{\Delta}$ is the set of these partitions). Then $\delta(\Delta) \stackrel{\text{def}}{=} \max\{(\tau_{j+1} - \tau_j) \mid j = 0, 1, \ldots, m(\Delta) - 1\}$. The set of all sequences of partitions $\{\Delta^{(k)}\}_1^\infty \subset \boldsymbol{\Delta}$ with the property $\lim\limits_{k \to \infty} \delta(\Delta^{(k)}) = 0$ is denoted by Π. Also we denote

$$\underline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U, V]) =$$

$$= \left(\lim_{\delta \to 0} \inf_{h_\Delta[.], \delta(\Delta) \leqq \delta} \rho_1(h_\Delta[T; p_0, U, V]), \ldots, \lim_{\delta \to 0} \inf_{h_\Delta[.], \delta(\Delta) \leqq \delta} \rho_N(h_\Delta[T; p_0, U, V]) \right),$$

$$\overline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U, V]) =$$

$$= \left(\lim_{\delta \to 0} \sup_{h_\Delta[.], \delta(\Delta) \leqq \delta} \rho_1(h_\Delta[T; p_0, U, V]), \ldots, \lim_{\delta \to 0} \sup_{h_\Delta[.], \delta(\Delta) \leqq \delta} \rho_N(h_\Delta[T; p_0, U, V]) \right);$$

The sets of positional strategies of the first (second) player, related to $P(t)$ $(Q(t))$ are denoted by $\mathcal{U}(\mathcal{V})$ etc. All other notations and concepts as positional strategy, step motion, ε-Slater saddle point etc., and the corresponding assertions can be found in [12].

1. ε−Slater minimaxes.

I. The differential game (0.1) is considered. Let $U \in \mathcal{U}$ be an arbitrary strategy of the first player. The set $R[U]$ is defined as follows:

The vector $\rho[U] \in R[U]$ if there exists a sequence of partitions $\{\Delta^{(k)}\}_1^\infty \in \Pi$ and a corresponding sequence of step motions $h_{\Delta^{(k)}}[.] \in h_{\Delta^{(k)}}[.;p_0,U]$ such that $\rho[U] = \lim_{\delta(\Delta^{(k)})\to 0} \rho(h_{\Delta^{(k)}}[T])$. The latter supposes that the corresponding strategy $V \div v(t) \in Q(t_0,T)$ is fixed, where $h_{\Delta^{(k)}}[.] = h_{\Delta^{(k)}}[.;p_0,U,v]$. Thus for a given strategy U, different (fixed on $\Delta^{(k)}$) functions v cause sets of elements of limits $\rho[U]$ described above, whose union is $R[U]$ when v is changed in the admissible set $Q(t_0,T)$. For each strategy $U \in \mathcal{U}, R[U]$ is a bounded set since the set $D(T;p_0)$ is a compact subset of $\mathcal{H}$ [12, Theorem 1] and the functional ρ is s.-continuous in $\mathcal{H}$.

Let $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_N) \in \mathbf{R}^N_{\geq}$ be a fixed vector. For each strategy $U \in \mathcal{U}$ we define the set

$$R^{(\varepsilon)}[U] = \{\rho^{(\varepsilon)}[U] \in R[U] \mid \rho^{(\varepsilon)}[U] \not< \rho[U] - \varepsilon, \forall\rho[U] \in R[U]\}. \tag{1.1}$$

Thus $\rho^{(\varepsilon)}[U]$ is an ε−Slater maximal vector with respect to the set $R[U]$, [5, p.11].

Definition 1.1. *The strategy $U^{(\varepsilon)} \in \mathcal{U}$ is called an ε−Slater minimax strategy of game (0.1) if there exists a vector $\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \in R^{(\varepsilon)}[U^{(\varepsilon)}]$ such that*

$$\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \not> \rho^{(\varepsilon)}[U] + \varepsilon, \ \forall U \in \mathcal{U}, \ \rho^{(\varepsilon)}[U] \in R^{(\varepsilon)}[U]. \tag{1.2}$$

The vector $\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}]$ is called an ε−Slater minimax of game (0.1) and the set of these vectors is denoted by $\widehat{R}^{(\varepsilon)}$.

The sets $R^{(\varepsilon)}[U]$ and $\widehat{R}^{(\varepsilon)}$ will be bounded since the sets $R[U]$ and $\rho(D(T;p_0)) = \{\rho(h) \mid h \in D(T;p_0)\}$ are bounded [12, Theorem 1].

Lemma 1.1. *Let $\overline{A}$ be a bounded and non-empty subset of $\mathbf{R}^N$ and $\varepsilon \in \mathbf{R}^N_{\geq}$. Then there exists an ε−Slater minimal (maximal) vector in $\overline{A}$.*

Proof. Let us point out that an analogous assertion for the case when $\overline{A}$ is a compact subset of $\mathbf{R}^N$ and $\varepsilon = 0_N$ is proved in [7, p. 142]. Since $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_N) \in \mathbf{R}^N_{\geq}$, let for example $\varepsilon_1 > 0$ and $\overline{A}_1 = \mathrm{Pr}_1\,\overline{A} = \{\overline{a}_1 \in \mathbf{R} | (\overline{a}_1,\ldots,\overline{a}_N) = \overline{a} \in \overline{A}\}$. But $\overline{A}$ is a bounded set and $\varepsilon_1 > 0$, hence $\inf \overline{A}_1 = \overline{a}_1^0 > -\infty$ and $\exists \overline{a}_1^{(\varepsilon)} \in \overline{A}_1 : \overline{a}_1^{(\varepsilon)} - \overline{a}_1^0 \leqq \varepsilon_1$. Then $\overline{a}_1^{(\varepsilon)} \leqq \overline{a}_1^0 + \varepsilon_1 \leqq \overline{a}_1 + \varepsilon_1$, $\forall \overline{a}_1 \in \overline{A}_1 \Rightarrow \overline{a}^{(\varepsilon)} \not> \overline{a} + \varepsilon$, $\forall \overline{a} \in \overline{A}$, where $\overline{a}^{(\varepsilon)} \in \overline{A}$ is such that $\mathrm{Pr}_1\,\overline{a}^{(\varepsilon)} = \overline{a}_1^{(\varepsilon)}$. The vector $\overline{a}^{(\varepsilon)}$ thus defined is ε−Slater minimal in $\overline{A}$.

Lemma 1.2. *The sets $R[U]$ and $R^{(\varepsilon)}[U]$ are non-empty compacts of $\mathbf{R}^N$, i.e. $R[U]$ and $R^{(\varepsilon)}[U] \in comp.\mathbf{R}^N$ for each strategy $U \in \mathcal{U}$.*

Proof. As it is shown above, $R[U]$ and $R^{(\varepsilon)}[U]$ are bounded subsets of the bounded set $\rho(D(T;p_0))$. From [12, Theorem 1], for every sequence $\{\Delta^{(k)}\}_1^\infty \in \Pi$ with $\lim_{k\to\infty} \delta(\Delta^{(k)}) = 0$ and the corresponding sequence of step motions $h_{\Delta^{(k)}}[.] \in h_{\Delta^{(k)}}[.;p_0,U]$, there exists a subsequence $\{\Delta^{(k_j)}\}$ such that $\exists \lim_{j\to\infty} \rho(h_{\Delta^{(k_j)}}[T])$ and this limit is bounded by a constant, which does not depend on the choice of the corresponding sequences. Hence $R[U] \neq \emptyset$. Let $r_0 = \lim_{k\to\infty} r_k$ and r_k be an arbitrary vector in $R[U]$. Then for each natural number k, there exists a natural number $j = j(k) > k$, a partition $\Delta_k^{(j)} \in \Pi$, which is the "j" term on the "k" sequence $\{\Delta_k^{(i)}\}_{i=1}^\infty \in \Pi$, (i.e. $\lim_{i\to\infty} \delta(\Delta_k^{(i)}) = 0$, $\forall k = 1,2,\ldots$) and the corresponding step motion $h_{\Delta_k^{(j)}}[.] = h_{\Delta_k^{(j)}}[.;p_0,U,v^{(k)}]$ such that $\mid r_k - \rho(h_{\Delta_k^{(j)}}[T]) \mid < \varepsilon_k$, where $\lim_{k\to\infty} \varepsilon_k = \lim_{k\to\infty} \delta(\Delta_k^{(j(k))}) = 0$. Since $P(t)$ and $Q(t)$ are bounded sets of the corresponding functional spaces (see the Introduction), then there exists a subsequence $\{v^{(k_p)}\}_{p=1}^\infty \subset \{v^{(k)}\}_{k=1}^\infty$ such that $\lim_{p\to\infty} v^{(k_p)} = v^{(0)} \in Q(t)$ in the weak* topology in $L_\infty([t_0,T], L_2(\Omega;\mathbf{R}^{m_1}) \times \mathbf{R}^{m_2})$. From [12, Theorem 1], it follows that $r_0 = \lim_{p\to\infty} \rho(h_{\Delta_{k_p}^{(j(k_p))}}[T;p_0,U,v^{(0)}]) \in R[U]$ and hence $R[U]$ is a closed subset of $\mathbf{R}^N$, i.e. $R[U] \in comp.\mathbf{R}^N$. Now the multicriterial problem $\Gamma(U) = < R[U], \rho(h[T]) >$ is considered. From (1.1) and Lemma 1.1 $R^{(\varepsilon)}[U]$ is the set of all $\varepsilon-$Slater maximal (ε-weak effective) solutions of $\Gamma(U)$, hence $R^{(\varepsilon)}[U]$ is a non-empty compact of $\mathbf{R}^N$, see also [7,p.142]. The lemma is proved.

By analogy with the definition of $R[U]$ we can define the set $R[V]$ for an arbitrary strategy $V \in \mathcal{V}$ of the second player. By analogy with Definition 1.1 we give

Deffinition 1.2. *The strategy $V^{(\varepsilon)} \in \mathcal{V}$ is called an $\varepsilon-$Slater maximin strategy of game (0.1) if there exists a vector $\widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}] \in R_{(\varepsilon)}[V^{(\varepsilon)}]$ such that*

$$\widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}] \not< \rho_{(\varepsilon)}[V] - \varepsilon,\ \forall V \in \mathcal{V},\ \rho_{(\varepsilon)}[V] \in R_{(\varepsilon)}[V], \tag{1.3}$$

where $R_{(\varepsilon)}[V] = \{\rho_{(\varepsilon)}[V] \in R[V] \mid \rho_{(\varepsilon)}[V] \not> \rho[V] + \varepsilon,\ \forall \rho[V] \in R[V]\}$. The vector $\widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}]$ is called $\varepsilon-$Slater maximin of game (0.1) and the set of these vectors is denoted by $\widehat{R}_{(\varepsilon)}$.

By analogy with Lemma 1.2 we can prove that the sets $R[V]$ and $R_{(\varepsilon)}[V]$ are non-empty compacts of $\mathbf{R}^N$.

Further the results only for $\varepsilon-$Slater minimax strategies will be given. For $\varepsilon-$Slater maximin strategies, analogous properties are valid as well.

The strategy $U^{(\varepsilon)} \in \mathcal{U}$ is called an $\varepsilon-$minimax strategy of game (0.1) if $N = 1$, i.e. game (0.1) is with a scalar pay-off function and the signs $\not<$ and $\not>$ in (1.1) and (1.2) are replaced by $\geq$ and $\leq$ respectively [5]. Thus the given definitions for an $\varepsilon-$Slater minimax (maximin) strategy includes the concept of an ε–minimax (maximin) strategy of game (0.1) with a scalar pay-off function (for $N = 1$) as a particular case.

From Lemma 1.2 and [12, Theorem 1], the set $R^{(\varepsilon)} \stackrel{def}{=} R^{(\varepsilon)}[U] \mid U \in \mathcal{U}\}$ of all ε–Slater maximal vectors is a non-empty and bounded subset of $\mathbf{R}^N$. According to Lemma 1.1, there exists an ε–Slater minimal vector with respect to $R^{(\varepsilon)}$ and this vector is denoted by $\widehat{\rho}^{(\varepsilon)} \in \widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}]$. From Definition 1.1, the vector $\widehat{\rho}^{(\varepsilon)}$ thus defined is ε–Slater minimax in game (0.1) and the corresponding strategy $U^{(\varepsilon)} \in \mathcal{U}$ is an ε–Slater minimax strategy of this game. Thus, the following theorem is proved:

Theorem 1.1 (Existence theorem). *For each choice of the initial position $p_0 \in [t_0, T] \times \mathcal{H}$ and each $\varepsilon \in \mathbf{R}^N_{\geq}$ there exists an ε-Slater minimax (maximin) strategy of game* (0.1).

II. Stability. Let us consider a multicriterial antagonistic game with a vector pay-off function ρ and the sets of strategies X (Y) of the first (second) player.

Let the vector $\varepsilon \in \mathbf{R}^N_{\geq}$ be given. At first, the following definitions from the general theory of games [5, 8] are given:

Definition 1.3. *A subset $\widehat{R} \subseteq R = \{\rho(x,y) \mid x \in X, y \in Y\} = \rho(X,Y)$ is called ε–inward stable, if for each $\rho^{(1)} \in \widehat{R}$ and $\forall \rho^{(2)} \in \widehat{R}$ it follows that $\rho^{(1)} \not> \rho^{(2)} + \varepsilon$.*

Definition 1.4. *The subset $R^{(1)}$, $(R^{(1)} \subseteq R^{(2)} \subseteq R = \rho(X,Y) = \{\rho(x,y) \mid x \in X, y \in Y\})$ is called ε-outward stable with respect to $R^{(2)} \subseteq R$, if for each $\rho^{(2)} \in R^{(2)}$, there exists $\rho^{(1)} \in R^{(1)}$ such that $\rho^{(1)} \leqq \rho^{(2)}$.*

The given definitions are generalized for the case of differential antagonistic games with a vector pay-off function, if we take $R = \{R[U] \mid U \in \mathcal{U}\}$ or $R = \{R[V] \mid V \in \mathcal{V}\}$.

Theorem 1.2. *Let the vector $\varepsilon \in \mathbf{R}^N_{\geq}$ be given and $R \stackrel{\rm def}{=} \{R[U] \mid U \in \mathcal{U}\}$. The set $\widehat{R}^{(\varepsilon)}$ of all ε-Slater minimaxes of game* (0.1) *is ε-inward stable and $\widehat{R}^{(\varepsilon)}$ is ε-outward stable with respect to $R^{(\varepsilon)} = \{R^{(\varepsilon)}[U] \mid U \in \mathcal{U}\}$. The sets $\widehat{R}_{(\varepsilon)}$ and $R_{(\varepsilon)} = \{R_{(\varepsilon)}[V] \mid V \in \mathcal{V}\}$ have analogous properties.*

Proof. From (1.2) and (1.3) (Definition 1.1 and Definition 1.2) the ε–inward stability of the sets $\widehat{R}^{(\varepsilon)}$ and $\widehat{R}_{(\varepsilon)}$ is proved.

Now we shall prove the assertion about the ε-outward stability. The proof will be given by using the method of [7, p. 158]. Let $\rho^{(\varepsilon)}$ be an arbitrary element of $R^{(\varepsilon)}$. Let us consider the set $R_1^{(\varepsilon)} = \{\rho \in R^{(\varepsilon)} \mid \rho \leqq \rho^{(\varepsilon)}\}$. Clearly $R_1^{(\varepsilon)} \subseteq R^{(\varepsilon)}$ is a bounded and non-empty subset of $\mathbf{R}^N$, since $\rho^{(\varepsilon)} \in R_1^{(\varepsilon)}$. From Lemma 1.1 there exists an ε–Slater minimal vector $\widehat{\rho} \in R_1^{(\varepsilon)}$ for the set $R_1^{(\varepsilon)}$. We shall prove that $\widehat{\rho}$ is an ε–Slater minimal vector with respect to $R^{(\varepsilon)}$ as well.

Suppose the contrary, i.e. that there exists $\widetilde{\rho} \in R^{(\varepsilon)}$ such that $\widehat{\rho} > \widetilde{\rho} + \varepsilon$. But $\widehat{\rho} \in R_1^{(\varepsilon)} \Rightarrow \widehat{\rho} \leqq \rho^{(\varepsilon)} \Rightarrow \rho^{(\varepsilon)} \geqq \widehat{\rho} > \widetilde{\rho} + \varepsilon \Rightarrow \widetilde{\rho} < \rho^{(\varepsilon)}$. Hence $\widetilde{\rho} \in R_1^{(\varepsilon)}$ and

$$\widehat{\rho} > \widetilde{\rho} + \varepsilon, \quad \text{where } \widehat{\rho} \in R_1^{(\varepsilon)}, \quad \widetilde{\rho} \in R_1^{(\varepsilon)}. \tag{1.4}$$

Relation (1.4) shows that $\widehat{\rho}$ is not an ε–Slater minimal vector in $R_1^{(\varepsilon)}$, which contradicts the choice of $\widehat{\rho}$. Therefore $\widehat{\rho}$ is an ε–Slater minimal vector in $R^{(\varepsilon)}$ and there exists a strategy $U^{(\varepsilon)} \in \mathcal{U}$ such that $\widehat{\rho} \in \widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \in \widehat{R}^{(\varepsilon)}$ (see the proof of Theorem 1.1). But $\widehat{\rho} \in R_1^{(\varepsilon)}$, i.e. for an arbitrary element $\rho^{(\varepsilon)} \in R^{(\varepsilon)}$ we found the vector $\widehat{\rho}$ such that $\widehat{\rho} \leqq \rho^{(\varepsilon)}$ and $\widehat{\rho}$ is the corresponding ε–Slater minimax ($\widehat{\rho} \in \widehat{R}^{(\varepsilon)}$).

By analogy we can prove the ε–outward stability of the set $\widehat{R}_{(\varepsilon)}$ with respect to $R_{(\varepsilon)}$, i.e. that $\forall \rho \in R_{(\varepsilon)}$, $\exists \widehat{\rho} \in \widehat{R}_{(\varepsilon)} : \widehat{\rho} \geqq \rho$, i.e. the inequality is contrary to the corresponding inequality of Definition 1.4. Thus the theorem is proved.

The relation between the ε–Slater saddle points and the ε–Slater minimax and maximin strategies is given by the following

Lemma 1.3. *Let us assume that the situation $(U^\varepsilon, V^\varepsilon) \in \mathcal{U} \times \mathcal{V}$ is an ε–Slater saddle point in game* (0.1) *and there exist partitions $\{\Delta^{(k)}\}_1^\infty \in \Pi$ and $\{\overline{\Delta}^{(k)}\}_1^\infty \in \Pi$ and corresponding step motions $h_{\Delta^{(k)}}[.] \in h_{\Delta^{(k)}}[.; p_0, U^\varepsilon, V^\varepsilon]$ and $h_{\overline{\Delta}^{(k)}}[.] \in h_{\overline{\Delta}^{(k)}}[.; p_0, U^\varepsilon, V^\varepsilon]$ such that*

$$\begin{aligned} &\lim_{\delta(\Delta^{(k)}) \to 0} \rho(h_{\Delta^{(k)}}[T]) = \underline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0.U^\varepsilon, V^\varepsilon]) \\ &\lim_{\delta(\overline{\Delta}^{(k)}) \to 0} \rho(h_{\overline{\Delta}^{(k)}}[T]) = \overline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0.U^\varepsilon, V^\varepsilon]). \end{aligned} \tag{1.5}$$

Then there exist ε–Slater maximin and minimax strategies $U^{(\varepsilon)} \in \mathcal{U}$ and $V^{(\varepsilon)} \in \mathcal{V}$ with the corresponding ε–Slater minimax $\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}]$ and maximin $\widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}]$ such that the following inequalities are valid

$$\begin{aligned} &\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \leqq \underline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U^\varepsilon, V^\varepsilon]) \\ &\leqq \overline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U^\varepsilon, V^\varepsilon]) \leqq \widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}]. \end{aligned} \tag{1.6}$$

Proof. From (1.5) (using the proof of Lemma 1.2) it follows that the right-hand sides of (1.5) belong respectively to the sets $R[U^\varepsilon]$ and $R[V^\varepsilon]$, and from [12, Definition 2], (1.1), Definition 1.1 and Definition 1.2 we obtain:

$$\begin{aligned} &\underline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U^\varepsilon, V^\varepsilon]) \in R^{(\varepsilon)}[U^\varepsilon] \subseteq R^{(\varepsilon)} \\ &\overline{\mathrm{LIM}}_{\delta \to 0}\, \rho(h_\Delta[T; p_0, U^\varepsilon, V^\varepsilon]) \in R_{(\varepsilon)}[V^\varepsilon] \subseteq R_{(\varepsilon)}. \end{aligned} \tag{1.7}$$

From (1.7) and Theorem 1.2 it follows that there exist an ε–Slater minimax strategy $U^{(\varepsilon)}$ and a corresponding vector $\widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \in R^{(\varepsilon)}[U^{(\varepsilon)}]$ such that the first inequality of (1.6) is valid. By analogy, the other inequality of (1.6) is proved, using the assertion of ε-outward stability of the set $\widehat{R}_{(\varepsilon)}$ with respect to $R_{(\varepsilon)}$ (see Theorem 1.2). The proof is completed.

Remark. We should note that the assertions, analogous to Lemma 1.2, Lemma 1.3, Theorem 1.1 and Theorem 1.2 for the case of classic games, $\varepsilon = 0_N$ and compact sets of strategies, are proved in [5] and [8].

Lemma 1.3 is not true if the relations (1.5) are not satisfied. This assertion is proved through the following example.

Example 1.1. It is supposed that the controllable system Ξ for game (0.1) is described by the following boundary-value problem

$$\partial^2 y/\partial t^2 - \partial^2 y/\partial x^2 = (u_1(t) + v_1(t))\sin x \text{ in } G = (0,1) \times (0,\pi)$$

$$y(x,0) = (\partial y/\partial t)(x,0) = 0 \quad \text{for } x \in \Omega = (0,\pi) \tag{1.8}$$

$$y(0,t) = y(\pi,t) = 0 \text{ for } t \in (0,1), \qquad y \in C([t_0,T], H_0^1(0,\pi)).$$

Program and positional strategies will be used, where $P_1(t) = Q_1(t) = [0,1]$ and $P_2(t) = Q_2(t) = \emptyset$. The set of strategies of the first (second) player is denoted by $\mathcal{U}(\mathcal{V})$ as well.

The vector pay-off function has two components $\rho(h(T)) = (\rho_1(y(1)), \rho_2(y(1)))$ and it is of the form

$$\rho_1(y(1)) = \int_0^\pi y(x,1)dx, \quad \rho_2(y(1)) = -\int_0^\pi y(x,1)dx.$$

The unique solution $y(x,t)$ of (1.8) [3, p.320-327] for fixed measurable and bounded functions $u_1(t) \in [0,1]$ and $v_1(t) \in [0,1]$, $\forall t \in [0,1]$, is:

$$y(x,t) = \textstyle\int_0^t (u_1(s) + v_1(s))\sin(t-s)ds.\sin x.$$

Clearly the function $y(x,t) \geqq 0$ is strictly increasing with respect to $u_1(t)$ and $v_1(t)$, $\forall (x,t) \in G = (0,1) \times (0,\pi)$ as in [12, Lemma 6] and $0 \leqq y(x,t) \leqq \int_0^t 2\sin(t-s)ds.\sin x = 2(1-\cos t)\sin x$. Hence $0 \leqq \rho_1(y[1]) \leqq 4(1-\cos 1)$ and the minimum of $\rho_1(y[1])$ is reached for $U^{(0)} \div u_1^{(0)}(t) \equiv 0$, $V^{(0)} \div v_1^{(0)}(t) \equiv 0$, and the maximum — for $U^{(1)} \div u_1^{(1)}(t) \equiv 1$, $V^{(1)} \div v_1^{(1)}(t) \equiv 1$, $\forall t \in [0,1]$. The corresponding values of the vector pay-off function are respectively $\rho(y_\Delta[1;0,0,0,U^{(0)},V^{(0)}]) = (0,0)$ and $\rho(y_\Delta[1;0,0,0,U^{(1)},V^{(1)}]) = (\widehat{c}, -\widehat{c})$ for every partition Δ of the interval [0,1], where $\widehat{c} = 4(1-\cos 1) > 0$.

Next let us proceed by constructing the positional strategies U^* and V^*. Let us remind that a positional strategy is said to be a mapping assigning a function $u \in P(t_1,t_2]$ ($v \in Q(t_1,t_2]$) to any triplet $(t_1,t_2,h(t_1)) \in [0,1] \times [0,1] \times \mathcal{H}$, ($\forall t_1, t_2 \in [0,1]$: $t_1 < t_2$; $h(t) = (y(t), y'(t)) \in \mathcal{H}$, $\forall t \in [0,1]$) [9,12], (here $\mathcal{H} = X = H_0^1(0,\pi) \times L_2(0,\pi)$, see [9, Example 3.1]). Let the set $S \subset [0,1]$, ($0 \in S, 1 \in S$) be such that the sets S and $[0,1]\backslash S$ are dense in the interval [0,1]. The strategy U^* is defined for $t \in (t_1,t_2]$

(independently of h) as follows:

$$U^* \div u_1^* = \begin{cases} 1 & \text{if } t_1 \in S \text{ and } t_2 \in S \\ 0 & \text{if } t_1 \notin S \text{ or } t_2 \notin S \end{cases}$$

The strategy V^* is defined in the same way. The following equality is valid:

$$\begin{aligned} &\underline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) = \\ &= \Big(\lim_{\delta\to 0} \inf_{h_\Delta[.],\delta(\Delta)\leqq\delta} \rho_1(y_\Delta[1;0,0,0,U^*,V^*]), \\ &\lim_{\delta\to 0} \inf_{h_\Delta[.],\delta(\Delta)\leqq\delta} \rho_2(y_\Delta[1;0,0,0,U^*,V^*])\Big) = \\ &= \left(\lim_{\delta(\Delta^{(k)})\to 0} \rho_1(y_{\Delta^{(k)}}[1]),\ \lim_{\delta(\overline{\Delta}^{(k)})\to 0} \rho_2(y_{\overline{\Delta}^{(k)}}[1])\right) = (0,-\widehat{c}). \end{aligned} \tag{1.9}$$

Here the first sequence of partitions $\{\Delta^{(k)}\}_1^\infty \in \Pi$ has the property that all numbers of the partitions $\tau_j^{(k)} \notin S$, $\forall j = 1,2,\dots,m(\Delta^{(k)})-1$, $\forall k = 1,2,\dots$, and we have $\overline{\tau}_j^{(k)} \in S$, $\forall j = 1,2,\dots,m(\overline{\Delta}^{(k)})-1$, $\forall k = 1,2,\dots$ for the second sequence of partitions $\{\overline{\Delta}^{(k)}\}_1^\infty \in \Pi$. By analogy it is proved that

$$\overline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) = (\widehat{c},0). \tag{1.10}$$

At the same time the set of values R of the vector pay-off function in the example considered is of the form:

$$R = \{(\alpha,-\alpha) \mid \alpha \in [0,\widehat{c}]\}. \tag{1.11}$$

The following properties of the situation (U^*,V^*) will be noted:

1. The situation $(U^*,V^*) \in \mathcal{U}\times\mathcal{V}$ is an ε–Slater saddle point [12, Definition 2] for $\varepsilon = (\frac{\widehat{c}}{2},\frac{\widehat{c}}{2}) \in \mathbf{R}^2_>$ in the differential game studied. Really, the following relations are valid

$$\rho(y_{\Delta^{(1)}}[1]) = (\alpha_1,-\alpha_1) \not> \underline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) + \varepsilon$$

$$= (0,-\widehat{c}) + (\frac{\widehat{c}}{2},\frac{\widehat{c}}{2}) = (\frac{\widehat{c}}{2},-\frac{\widehat{c}}{2}),$$

and by analogy

$$\rho(y_{\Delta^{(2)}}[1]) = (\alpha_2,-\alpha_2) \not< \overline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) - \varepsilon = (\frac{\widehat{c}}{2},-\frac{\widehat{c}}{2}),$$

(where $\alpha_1 \in [0,\widehat{c}]$ and $\alpha_2 \in [0,\widehat{c}]$) for arbitrary partitions $\Delta^{(1)}$ and $\Delta^{(2)}$ and the corresponding arbitrary step motions

$$h_{\Delta^{(1)}}[.] \in h_{\Delta^{(1)}}[.;0,0,0,U^*] \quad \text{and} \quad h_{\Delta^{(2)}}[.] \in h_{\Delta^{(2)}}[.;0,0,0,V^*].$$

2. Equations (1.5) are not valid for the situation (U^*, V^*), since for each sequence of partitions $\{\Delta^{(p)}\}_1^\infty \in \Pi$ and each corresponding sequence of step motions $h_{\Delta^{(p)}}[.] \in h_{\Delta^{(p)}}[.;0,0,0,U^*]$, $\lim\limits_{\delta(\Delta^{(p)})\to 0} \rho(y_{\Delta^{(p)}}[1]) = (\alpha_3, -\alpha_3) \in R$, (if this limit exists), where the number $\alpha_3 \in [0,\widehat{c}]$ (1.11). At the same time, the right-hand sides of (1.5) do not belong to R for $U^\varepsilon = U^*$ and $V^\varepsilon = V^*$, see (1.9), (1.10), (1.11).

3. From Definition 1.1, $\widehat{\rho}^{(\varepsilon)}[U^*] = (\alpha, -\alpha) \not> \rho^{(\varepsilon)}[U] + \varepsilon = (\beta, -\beta) + \varepsilon$, $\forall U \in \mathcal{U}$, $\forall \rho^{(\varepsilon)}[U] \in R^{(\varepsilon)}[U]$, $\forall \varepsilon \in \mathbf{R}^2_{\geqq}$ and $\alpha, \beta \in [0,\widehat{c}]$. Hence U^* is an ε–Slater minimax strategy, $\forall \varepsilon \in \mathbf{R}^2_{\geqq}$. By analogy it is proved that V^* is an ε–Slater maximin strategy, $\forall \varepsilon \in \mathbf{R}^2_{\geqq}$. The corresponding ε–Slater minimaxes and maximins form the set R (1.11), i.e. $\widehat{R}^{(\varepsilon)} = \widehat{R}_{(\varepsilon)} = R \ \ \forall \varepsilon \in \mathbf{R}^2_{\geqq}$ and

$$\underline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) = (0,-\widehat{c}) \leq \widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}], \forall \widehat{\rho}^{(\varepsilon)}[U^{(\varepsilon)}] \in R,$$

$$\overline{\mathrm{LIM}}_{\delta\to 0}\, \rho(y_\Delta[1;0,0,0,U^*,V^*]) = (\widehat{c},0) \geq \widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}], \quad \forall \widehat{\rho}_{(\varepsilon)}[V^{(\varepsilon)}] \in R$$

see (1.11) for each ε–Slater minimax (maximin) strategy $U^{(\varepsilon)}$ ($V^{(\varepsilon)}$). Therefore relations (1.6) are not valid for the example under consideration.

2. Games with a separable linear pay-off function. In this section it is supposed in addition that every component of the vector pay-off function $\rho(h[T]) = (\rho_1(h[T]), \ldots, \rho_N(h[T]))$ of game (0.1) is a linear s.-continuous functional in $\mathcal{H}$. It will be shown that the controllable system Ξ of (0.1) is described by the following boundary-value problems:

$$(2.1) \qquad \left| \begin{array}{l} \dfrac{\partial^2 y^{(1)}}{\partial t^2} = Ay^{(1)} + b_1 u_1 + 0.5 f_1 \text{ in } G = (t_0, T) \times \Omega, \\[2mm] y^{(1)}\Big|_{t=t_0} = 0.5 y_0, \quad \dfrac{\partial y^{(1)}}{\partial t}\Big|_{t=t_0} = 0.5 y_1, \quad \text{in } \Omega \\[2mm] \sigma_1 \dfrac{\partial y^{(1)}}{\partial \nu_A} + \sigma_2 y^{(1)} = b_2 u_2 + 0.5 f_2 \text{ in } \Sigma = (t_0, T) \times \Gamma, \end{array} \right.$$

and

$$(2.2)\qquad \left|\begin{array}{l} \dfrac{\partial^2 y^{(2)}}{\partial t^2} = Ay^{(2)} + c_1 v_1 + 0.5 f_1 \text{ in } G = (t_0, T) \times \Omega, \\ y^{(2)}_{|t=t_0} = 0.5 y_0, \quad \dfrac{\partial y^{(2)}}{\partial t}_{|t=t_0} = 0.5 y_1, \quad \text{in } \Omega \\ \sigma_1 \dfrac{\partial y^{(2)}}{\partial \nu_A} + \sigma_2 y^{(2)} = c_2 v_2 + 0.5 f_2 \text{ in } \Sigma = (t_0, T) \times \Gamma. \end{array}\right.$$

Let us remind that all the coefficients of (2.1) and (2.2) satisfy all the conditions given in the Introduction.

The following two multicriterial dynamic problems will correspond to differential game (0.1)

$$(2.3)\qquad \langle \Xi^{(2)}, \mathcal{V}, \{\rho_i(h^{(2)}(T))\}_{i\in\mathbf{N}}\rangle \qquad \text{and}$$

$$(2.4)\qquad \langle \Xi^{(1)}, \mathcal{U}, \{-\rho_i(h^{(1)}(T))\}_{i\in\mathbf{N}}\rangle, h^{(j)} = (y^{(j)}, \partial y^{(j)}/\partial t), j = 1, 2.$$

Setting $\mathcal{U} = \{0\}$, the following definition is obtained from [12, Definition 2]

Definition 2.1. *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \mathbf{R}^N_{\geq}$ be a fixed vector. The strategy $V^\varepsilon \in \mathcal{V}$ is called an ε-Slater maximal strategy of problem* (2.3) *if there exists a constant $\delta(\varepsilon) > 0$ such that for $\forall V \in \mathcal{V}$ and for $\forall h^{(2)}_\Delta[.] \in h^{(2)}_\Delta[.; p_0, V]$ with $\delta(\Delta) \leqq \delta(\varepsilon)$, the following vector inequality is valid:*

$$(2.5)\qquad \rho(h^{(2)}_\Delta[T]) - \varepsilon \not> \underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho(h^{(2)}_\Delta[T; p_0, V^\varepsilon]).$$

The vector $\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho(h^{(2)}_\Delta[T; p_0, V^\varepsilon])$ is called an ε-Slater maximum of problem (2.3).

From [12, Theorem 3], there exist an ε-Slater maximal strategy and an ε-Slater maximum of problem (2.3), $\forall \varepsilon \in \mathbf{R}^N_{\geq}$.

Now it is supposed that the sets $\mathcal{U}$ and $\mathcal{V}$ consist only of program strategies.

The following assertion is formulated:

Lemma 2.1. *Let $y(.)$, $y^{(j)}(.)$, $j = 1, 2$ be the solutions of problems* (0.2) – (0.4), (2.1) *and* (2.2) *for $u = (u_1, u_2) \in P(t)$ and $v = (v_1, v_2) \in Q(t)$. Let $h(.) = (y(.), \partial y(.)/\partial t)$ and $h^{(j)}(.) = (y^{(j)}(.), \partial y^{(j)}(.)/\partial t)$, $j = 1, 2$. Then $h = h^{(1)} + h^{(2)}$.*

Let us remind that the solutions of the above problems are used in sense of [4, Theorem 4.1] and $h(.)$ and $h^{(j)}(.)$, $j = 1, 2$ belong to $C([t_0, T], \mathcal{H})$, [12, Theorem 1].

Let $h_\Delta[.; p_0, U, V]$, $h^{(1)}_\Delta[.; t_0, 0.5y_0, 0.5y_1, U]$ and $h^{(2)}_\Delta[.; t_0, 0.5y_0, 0.5y_1, V]$, (where $p_0 = (t_0, y_0, y_1)$) be the step motions corresponding to (0.2)–(0.4), (2.1) and (2.2), and caused by the program strategies $U \div u = (u_1, u_2)$ and $V \div v = (v_1, v_2)$[12]. Then from Lemma 2.1 the following equality holds:

$$\begin{aligned} h_\Delta[T; p_0, U, V] &= h_\Delta^{(1)}[T; t_0, 0.5y_0, 0.5y_1, U]+ \\ &+h_\Delta^{(2)}[T; t_0, 0.5y_0, 0.5y_1, V]. \end{aligned} \tag{2.6}$$

From [12, Definition 2], Definition 2.1 and (2.6), the following assertion is proved by analogy with [5, p. 81, 82]

Lemma 2.2. *Let the fixed vector* $\varepsilon \in \mathbf{R}^N_>$ *be given and let* $U^\varepsilon \in \mathcal{U}$ *and* $V^\varepsilon \in \mathcal{V}$ *be program strategies. Then the situation* $(U^\varepsilon, V^{\bar\varepsilon})$ *is an* ε*–Slater saddle point of* (0.1) *if and only if* U^ε *and* V^ε *are* ε*–Slater maximal strategies of problems* (2.4) *and* (2.3) *respectively.*

If the positional (non-program) strategies are used for the construction of step motions, then equality (2.6) and Lemma 2.2 are not valid. This fact is proved by the following example.

Example 2.1 (Continuation of [12, Example]). Let us remind that the controllable system Ξ for game (0.1) is described by the following boundary-value problem

$$\begin{aligned} &\partial^2 y/\partial t^2 = \partial^2 y/\partial x^2 \ \text{ in } \ G = (0,1) \times (0,\pi), \\ &y(x,0) = (\partial y/\partial t)(x,0) = 0 \ \text{ for } x \in \Omega = (0,\pi), \\ &-(\partial y/\partial x)(0,t) = u(t) + v(t),\ (\partial y/\partial x)(\pi,t) = 0 \ \text{ for } \ t \in (0,1). \end{aligned} \tag{2.7}$$

Program and positional strategies will be used, where $P_2(t) = Q_2(t) = [0,1]$ and $P_1(t) = Q_1(t) = \emptyset$. The set of strategies of the first (second) player is denoted by $\mathcal{U}(\mathcal{V})$ as well.

The vector pay-off function has two components $\rho(h(T)) = (\rho_1(y(1)),\ \rho_2(y(1))) = (\int_0^\pi y(x,1)dx,\ -\int_0^\pi y(x,1)dx)$. We should note that these components are linear and s.-continuous functionals in $\mathcal{H} = L_2(0,\pi) \times (H_2^1(0,\pi))^*$.

Now we shall remind the constructing of the strategies $U^{(j)}$ and $V^{(j)}, j = 0,1,2$. We take:

$$\begin{aligned} &U^{(0)} \div u^{(0)}(t) \equiv 0, && V^{(0)} \div v^{(0)}(t) \equiv 0, \forall t \in [0,1], \\ &U^{(1)} \div u^{(1)}(t) \equiv 1, && V^{(1)} \div v^{(1)}(t) \equiv 1, \forall t \in [0,1]. \end{aligned}$$

Next let us proceed by constructing the positional strategies $U^{(2)}$ and $V^{(2)}$.Let the set S satisfy the conditions of Example 1.1. Then it is said that the triplet (t_1, t_2, h) satisfies Condition **A**, if t_1 and t_2 belong to S and $h(t_1) = (0,0)$. The strategies $U^{(2)}$ and $V^{(2)}$ are defined as follows: if the triplet (t_1, t_2, h) satisfies Condition **A**, then $U^{(2)} \div u^{(2)} = 0$ and $V^{(2)} \div v^{(2)} = 1$; otherwise $U^{(2)} \div u^{(2)} = 1$ and $V^{(2)} \div v^{(2)} = 0$.

In the Example of [12] it was shown that the situation $(U^{(2)}, V^{(2)})$, which consists of non-program strategies, is an ε–Slater saddle point, $\forall\varepsilon \in \mathbf{R}^2_>$. At the same time we shall prove the following

Theorem 2.1. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_>$ and $\varepsilon_1 + \varepsilon_2 < c$, where*

$$c = \frac{1}{2}\rho_1(y_{\Delta'}[1;0,0,0,U^{(1)},V^{(1)}]) = \int_0^1 \int_0^\tau d\xi d\tau. \tag{2.8}$$

Then the strategy $U^{(2)} \in \mathcal{U}$ is not an ε–Slater maximal strategy of multicriterial problem (2.4).

Proof. Let us point out that c is the same constant, defined in [12,Lemma 6].

Suppose the contrary. Then relation (2.5) for the strategy $U^{(2)}$ in problem (2.4) can be formulated in the following equivalent form:

There exists a constant $\delta(\varepsilon) > 0$ such that for every strategy $U \in \mathcal{U}$ and for $\forall h^{(1)}_\Delta[.] \in h^{(1)}_\Delta[.;t_0, 0.5y_0, 0.5y_1, U]$ with $\delta(\Delta) \leqq \delta(\varepsilon)$, at least one of the following two inequalities

$$\rho_1(y^{(1)}_\Delta[1]) + \varepsilon_1 \geqq \lim_{\delta\to 0} \sup_{y^{(1)}_\Delta[.],\delta(\Delta)\leqq\delta} \rho_1(y^{(1)}_\Delta[1;t_0, 0.5y_0, 0.5y_1, U^{(2)}]) \tag{2.9}$$

or

$$\rho_2(y^{(1)}_\Delta[1]) + \varepsilon_2 \geqq \lim_{\delta\to 0} \sup_{y^{(1)}_\Delta[.],\delta(\Delta)\leqq\delta} \rho_2(y^{(1)}_\Delta[1;t_0, 0.5y_0, 0.5y_1, U^{(2)}]) \tag{2.10}$$

are valid. Inequalities (2.9) and (2.10) are obtained from (2.5) (Definition 2.1), taking into account the sign - in the criteria of (2.4). Since $\rho_2 = -\rho_1$, multiplying (2.10) by -1 we obtain that (2.10) can be written in the form

$$\rho_1(y^{(1)}_\Delta[1]) - \varepsilon_2 \leqq \lim_{\delta\to 0} \inf_{y^{(1)}_\Delta[.],\delta(\Delta)\leqq\delta} \rho_1(y^{(1)}_\Delta[1;0,0,0,U^{(2)}]). \tag{2.11}$$

Now we take into account the example in [12]. Thus we prove that inequalities (2.9)–(2.11) take the form

$$\rho_1(h^{(1)}_\Delta[1]) + \varepsilon_1 \geqq c \quad \text{or} \quad \rho_1(h^{(1)}_\Delta[1]) - \varepsilon_2 \leqq 0, \tag{2.12}$$

where U is an arbitrary strategy and $h^{(1)}_\Delta[.] \in h^{(1)}_\Delta[.;0,0,0,U]$ is an arbitrary element with $\delta(\Delta) \leqq \delta(\varepsilon)$. Inequality (2.12) is analogous to [12, (25)]. Using a suitable construction as in [12, Example] we conclude that for each a and b such that $0 \leqq a < b \leqq c$ there exist $\bar{c} \in (a,b)$ and $\Delta^{(\bar{c})}$ with $\delta(\Delta^{(\bar{c})}) < \delta(\varepsilon)$ so that the following equality holds

$$\rho_1(h^{(1}_{\Delta(\bar{c})}[1;0,0,0,U^{(2)}]) = \bar{c}. \tag{2.13}$$

Relation (2.13) is analogous to [12, (26)]. Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_>$ and $\varepsilon_1 + \varepsilon_2 < c$. Then, the number $\overline{c}$ can be chosen so that $\overline{c} \in (\varepsilon_2, c - \varepsilon_1)$ and for the step motion $h_{\Delta(\overline{c})}[.] = (y_{\Delta(\overline{c})}[\cdot], y'_{\Delta(\overline{c})}[\cdot])$, corresponding to (2.13), none of inequalities (2.12) is satisfied. This shows, that relation (2.5) is not valid, i.e. $U^{(2)}$ is not an ε–Slater maximal strategy, $\forall \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_> : \varepsilon_1 + \varepsilon_2 < c$. The proof is completed.

Theorem 2.2. *Equality* (2.6) *is not valid for the positional situation* $(U^{(2)}, V^{(2)}) \in \mathcal{U} \times \mathcal{V}$.

Proof. Let us take the partition Δ' so that $m(\Delta') = 2$ and the partition Δ' is defined by the points $\tau_0 = 0$, $\tau_1 \in (0,1)$ is an arbitrary point of S and $\tau_2 = 1$. Then for the step motions $h_{\Delta'}[.; 0,0,0, U^{(2)}, V^{(2)}]$, $h^{(1)}_{\Delta'}[.; 0,0,0,U^{(2)}]$ and $h^{(2)}_{\Delta'}[.; 0,0,0,V^{(2)}]$, corresponding to problems (0.2)–(0.4), (2.1) and (2.2), the following inequality will be proved:

$$y_{\Delta'}[1; 0,0,0,U^{(2)}, V^{(2)}] \neq y^{(1)}_{\Delta'}[1; 0,0,0,U^{(2)}] + y^{(2)}_{\Delta'}[1; 0,0,0,V^{(2)}]. \tag{2.14}$$

Let the constant c be defined as in (2.8). Using the proof of [12, Lemma 6], we obtain:

$$\rho_1(y_{\Delta'}[1; 0,0,0,U^{(2)}, V^{(2)}]) = c, \tag{2.15}$$

since for every partition $\Delta' \in \boldsymbol{\Delta}$, $u^{(2)}(t) + v^{(2)}(t) = 1$, $\forall t \in (0,1]$, where the functions $u^{(2)}(t)$ and $v^{(2)}(t)$ correspond to the strategies $U^{(2)}$ and $V^{(2)}$. The points τ_0, τ_1 and τ_2 belong to S and hence

$$\rho_1(y^{(1)}_{\Delta'}[1; 0,0,0,U^{(2)}]) = \rho_1(h_{\Delta'}[1; 0,0,0,U^{(2)}, V^{(0)}]) = 0, \tag{2.16}$$

since Condition **A** is satisfied for the triplets $(\tau_0 = 0, \tau_1, h)$ and $(\tau_1, \tau_2 = 1, h)$ and then $u^{(2)}(t) = 0, \forall t \in (0,1]$.

Now we shall prove that

$$\rho_1(y^{(2)}_{\Delta'}[1; 0,0,0,V^{(2)}]) = c', \quad \text{where} \quad 0 < c' < c. \tag{2.17}$$

The solution $y(t)$ of (2.7) for fixed functions $u(t)$ and $v(t)$ is of the form

$$\begin{aligned} y(t) &= \pi^{-1} \int_0^t \int_0^\tau [u(\xi) + v(\xi)] d\xi d\tau + \\ &+ \sum_{j=1}^{\infty} j^{-1} \omega_j(0) \int_0^t [u(\tau) + v(\tau)] \sin j(t-\tau) d\tau \omega_j(x), \\ &\text{where } \omega_j(x) = \sqrt{2/\pi} \ \cos jx, [12]. \end{aligned} \tag{2.18}$$

Now we take into account that Condition **A** is satisfied for the triplet $(0, \tau_1, h)$. Then

$$u^{(2)}(t) = 0, \quad v^{(2)}(t) = 1, \quad \forall t \in (0, \tau_1] \tag{2.19}$$

and using (2.18), (2.19) and the proof of [12, Lemma 6], it follows that

$$\int_0^{\pi} y_{\Delta'}[\tau_1; 0,0,0,U^{(0)},V^{(2)}]dx = \int_0^{\tau_1}\int_0^{\tau} d\xi d\tau > 0, \tag{2.20}$$

since $\int_0^{\pi} \omega_j(x)dx = 0$, see [12, Example]. From (2.20) we obtain that

$$y_{\Delta'}[\tau_1] \stackrel{\text{def}}{=} y_{\Delta'}[\tau_1; 0,0,0,U^{(0)},V^{(2)}] \neq 0 \tag{2.21}$$

and then Condition **A** is not satisfied for the triplet $(\tau_1, 1, h = (y_{\Delta'}[.], y'_{\Delta'}[.]))$.

Thus, from (2.19) and (2.21), it follows that

$$v^{(2)}(t) = 1,\ \forall t \in (0,\tau_1] \text{ and } v^{(2)}(t) = 0,\ \forall t \in (\tau_1, 1]. \tag{2.22}$$

Using (2.18), (2.22), (2.8) and the proof of [12, Lemma 6] we obtain:

$$\begin{aligned} &\rho_1(y^{(2)}_{\Delta'}[1;0,0,0,V^{(2)}]) = \rho_1(y_{\Delta'}[1;0,0,0,U^{(0)},V^{(2)}]) = \\ &= \int_0^1\int_0^{\tau} v^{(2)}(\xi)d\xi d\tau = c' < c = \int_0^1\int_0^{\tau} d\xi d\tau, \end{aligned} \tag{2.23}$$

where $v^{(2)}(.)$ is defined from (2.22), i.e. (2.17) is proved. From (2.15), (2.16) and (2.17), the following relations are obtained:

$$\begin{aligned} &\rho_1(y_{\Delta'}[1;0,0,0,U^{(2)},V^{(2)}]) = c \neq \rho_1(y^{(1)}_{\Delta'}[1;0,0,0,U^{(2)}]) + \\ &+\rho_1(y^{(2)}_{\Delta'}[1;0,0,0,V^{(2)}]) = c', \end{aligned} \tag{2.24}$$

since $0 < c' < c$, see (2.23). From (2.24) it follows that (2.14) is proved, which shows that equality (2.6) is not valid for the positional strategies $U^{(2)}$ and $V^{(2)}$. The theorem is proved.

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