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SOME RESULTS ON THE COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

ADEM KILIÇMAN, BRIAN FISHER

ABSTRACT. Let f, g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$, $g_n = g * \delta_n$, where $\{\delta_n\}$ is a certain sequence converging to the Dirac delta-function. The neutrix product $f \square g$ is said to exist and be equal to h if

$$\mathrm{N-}\lim_{n\to\infty}\langle f_n g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$. Neutrix products of the form $\ln x_+ \square \delta^{(s)}(x)$ and $x_+^{-s} \square \delta^{(s)}(x)$ are evaluated from which further neutrix products are obtained.

The following definition of a neutrix was given by van der Corput [1]:

Difinition 1. *Let N be an additive group of functions defined on a set N' with values in an additive group N'' with the property that the only constant function in N is the zero function. Then N is said to be a neutrix and the functions in N are said to be negligible.*

Example 1. Let $N' = N'' = R$, the real numbers and let N be the set of real-valued functions of the form

$$N = \{a\sin x + b\cos x : a, b \in R\}.$$

Then N is a neutrix.

Now suppose N' is a subspace of a topological space X having an accumulation point y which is not in N'. Let $N'' = R$ (or C the complex numbers). Let N be an additive group of real (or complex) valued functions defined on N', with the property that if N contains a function $\nu(x)$ which converges to a finite limit c as x tends to y, then $c = 0$. Then N is a neutrix, since if f is in N and $f(x) = c$ for all x in N', then $\lim_{x\to y} f(x) = c$ implies $c = 0$.

This leads us to the following definition:

Difinition 2. *Let f be a real (or complex) valued function on N' and suppose there exists c in R (or C) such that $f(x)-c$ is in N. Then c is called the neutrix limit of $f(x)$ as x tends to y and we write*

$$\mathrm{N-}\lim_{x\to y} f(x) = c.$$

Notice that if a neutrix limit c exists then it is unique, since if $f(x)-c$ and $f(x)-c'$ are in N, then

$$c - c' \in N \Rightarrow c = c'.$$

Also notice that if N is a neutrix containing the set of all functions which converge to zero in the normal sense as x tends to y, then

$$\lim_{x\to y} f(x) = c \Rightarrow \mathrm{N-}\lim_{x\to y} f(x) = c.$$

From now on, the neutrix N we will use will have domain the positive integers, range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \quad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

Example 2. The Gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

but more generally we have

$$\Gamma^{(r)}(x) = \mathrm{N-}\lim_{n\to\infty} \int_{1/n}^\infty t^{x-1} \ln^r t e^{-t}\, dt$$

for $x \neq 0, -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$, see [7].

Example 3. The Beta function $B(x,y)$ is defined for $x, y > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt,$$

but more generally, if

$$B_{r,s}(x,y) = \frac{\partial^{r+s}}{\partial^r x \partial^s y} B(x,y),$$

we have

$$B_{r,s}(x,y) = \mathrm{N-}\lim_{n\to\infty} \int_{1/n}^{1-1/n} t^{x-1} \ln^r t (1-t)^{y-1} \ln^s(1-t)\, dt$$

for $x, y \neq 0, -1, -2, \ldots$ and $r, s = 0, 1, 2, \ldots$, see [8].

Example 4. The distribution x_+^λ is defined

$$\langle x_+^\lambda, \phi(x)\rangle = \int_0^\infty x^\lambda \phi(x)\, dx$$

for $x > -1$ and by

$$\langle x_+^\lambda, \phi(x)\rangle = \int_0^\infty x^\lambda \Big[\phi(x) - \sum_{i=0}^{m-1} \frac{x^i}{i!}\phi^{(i)}(0)\Big]\, dx$$

for $-m-1 < \lambda < -m$ and arbitrary ϕ in $\mathcal{D}$, but more generally,

$$\langle x_+^\lambda \ln^r x, \phi(x)\rangle = \mathrm{N}\!-\!\lim_{n\to\infty} \int_{1/n}^\infty x^\lambda \ln^r x\phi(x)\, dx$$

for $\lambda \neq -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$, see [6].

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^1 \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution f.

The following definition for the product of two distributions was given in [3].

Difinition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the neutrix product $f \square g$ of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \langle f_n g_n, \phi\rangle = \langle h, \phi\rangle$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a,b). If

$$\lim_{n\to\infty}\langle f_n g_n, \phi\rangle = \langle h, \phi\rangle,$$

we simply say that the product $f.g$ exists and equals h, see [2].

This definition of the neutrix product is clearly commutative. A non–commutative neutrix product, denoted by $f \circ g$, was considered in [5].

We now prove the following theorem.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f\Box g^{(i)}$ exist on the interval (a,b) for $i = 0,1,\dots,r$. Then the neutrix products $f^{(k)}\Box g$ exist on the interval (a,b) and*

$$f^{(k)}\Box g = \sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\Box g^{(i)}]^{(k-i)} \tag{1}$$

$k = 1,2,\dots,r.$

Proof. Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a,b) and suppose that the neutrix products $f\Box g^{(i)}$ exist on the interval (a,b) for $i = 0,1,\dots,r$. Put

$$f_n = f * \delta_n, \quad g_n = g * \delta_n.$$

Then

$$\begin{aligned}\langle f\Box g, \phi\rangle &= \operatorname*{N-lim}_{n\to\infty}\langle f_n, g_n\phi\rangle,\\ \langle f\Box g', \phi\rangle &= \operatorname*{N-lim}_{n\to\infty}\langle f, g_n'\phi\rangle.\end{aligned}$$

Further

$$\begin{aligned}\langle (f\Box g)', \phi\rangle &= -\langle f\Box g, \phi'\rangle = -\operatorname*{N-lim}_{n\to\infty}\langle f_n, g_n\phi'\rangle\\ &= -\operatorname*{N-lim}_{n\to\infty}\langle f_n, (g_n\phi)' - g_n'\phi\rangle\\ &= \operatorname*{N-lim}_{n\to\infty}\langle f_n', g_n\phi\rangle + \operatorname*{N-lim}_{n\to\infty}\langle f_n, g_n'\phi\rangle\end{aligned}$$

and so

$$\operatorname*{N-lim}_{n\to\infty}\langle f_n', g_n\phi\rangle = \langle (f\Box g)', \phi\rangle - \langle f\Box g', \phi\rangle.$$

This proves that the neutrix product $f'\Box g$ exists and satisfies equation (1) for the case $k = 1$. Thus

$$(f\Box g)' = f'\Box g + f\Box g'. \tag{2}$$

Now suppose that equation (1) holds for some $k < r$. Then by our assumption, the neutrix product $f^{(k)}\Box g$ exists and using equation (2) we have

$$\begin{aligned}[f^{(k)}\Box g]' &= f^{(k+1)}\Box g + f^{(k)}\Box g' \\ &= f^{(k+1)}\Box g + \sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\Box g^{(i+1)}]^{(k-i)} \\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\Box g^{(i)}]^{(k-i+1)}.\end{aligned}$$

Thus

$$\begin{aligned}f^{(k+1)}\Box g &= \sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\Box g^{(i)}]^{(k-i+1)} + \sum_{i=1}^{k+1}\binom{k}{i-1}(-1)^i[f\Box g^{(i)}]^{(k-i+1)} \\ &= \sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i[f\Box g^{(i)}]^{(k-i+1)}.\end{aligned}$$

Equation (1) now follows by induction.

The following two theorems hold, see [4] and [12] respectively.

Theorem 2. *The neutrix product $x_+^r\Box\delta^{(s)}(x)$ exists and*

$$x_+^r\Box\delta^{(s)}(x) = \frac{(-1)^r s!}{2(s-r)!}\delta^{(s-r)}(x), \tag{3}$$

for $r = 0, 1, 2, \ldots$ and $s = r+1, r+2, \ldots$.

Theorem 3. *The neutrix product $x^{-r}\Box\delta^{(s)}(x)$ exists and*

$$x^{-r}\Box\delta^{(s)}(x) = c_{rs}\delta^{(r+s)}(x), \tag{4}$$

where

$$c_{rs} = \frac{(-1)^{s-1}}{(r-1)!(r+s)!}\int_{-1}^{1} v^{r+s}\rho^{(s)}(v)\int_{-1}^{1}\ln|v-u|\rho^{(r)}(u)\,du\,dv,$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ In particular

$$x^{-r}.\delta^{(r-1)}(x) = \frac{(-1)^r r!}{(2r)!}\delta^{(2r-1)}(x), \tag{5}$$

for $r = 1, 2, \ldots$. Further,

$$\frac{(-1)^s}{(s-1)!}x^{-r}\Box\delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!}x^{-s}\Box\delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{6}$$

for $r, s = 1, 2, \ldots$.

Note that in the following, the distributions x_+^{-r} and x_-^{-r} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)}$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov [9].

The neutrix product $x_+^{-r} \square \delta^{(r-1)}(x)$ was considered in [3] where it was proved that

$$x_+^{-r} \square \delta^{(r-1)}(x) = \frac{(-1)^r r!}{2(2r)!}\delta^{(2r-1)}(x)$$

for $r = 1, 2, \ldots$

We now prove the following generalization of this result.

Theorem 4. *The neutrix products* $\ln x_+ \square \delta^{(s)}(x)$, $\ln x_- \square \delta^{(s)}(x)$, $\ln|x| \square \delta^{(s)}(x)$, $x_+^{-r} \square \delta^{(s)}(x)$ *and* $x_-^{-r} \square \delta^{(s)}(x)$ *exist and*

$$\begin{aligned} \ln x_+ \square \delta^{(s)}(x) &= b_s \delta^{(s)}(x) \\ &= (-1)^s \ln x_- \square \delta^{(s)}(x) \\ &= \tfrac{1}{2} \ln|x| \square \delta^{(s)}(x), \end{aligned}$$

where

$$b_s = \frac{1}{s!}\int_{-1}^{1} v^s \rho^{(s)}(v) \int_{-1}^{v} \ln(v-u)\rho(u)\, du\, dv$$

for $s = 0, 1, 2, \ldots$ *and*

$$\begin{aligned} x_+^{-r} \square \delta^{(s)}(x) &= \tfrac{1}{2} c_{rs} \delta^{(r+s)}(x) \\ &= (-1)^r x_-^{-r} \square \delta^{(s)}(x) \end{aligned}$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots$. *In particular*

$$x_+^{-r} \square \delta^{(r-1)}(x) = \frac{(-1)^r r!}{2(2r)!}\delta^{(2r-1)}(x), \tag{12}$$

for $r = 1, 2, \ldots$ *Further,*

$$\frac{(-1)^s}{(s-1)!} x_+^{-r} \square \delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!} x_+^{-s} \square \delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{13}$$

for $r, s = 1, 2, \ldots$

P r o o f. We put

$$(\ln x_+)_n = \ln x_+ * \delta_n(x) = \int_{-1/n}^{x} \ln(x-t)\delta_n(t)\,dt$$

on the interval $(-1/n, 1/n)$. Then

$$\int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^i\,dt = \int_{-1/n}^{1/n} x^i \delta^{(s)}(x) \int_{-1/n}^{x} \ln(x-t)\delta_n(t)\,dt\,dx$$
$$= n^{s-i}\int_{-1}^{1} v^i\rho^{(s)}(v)\int_{-1}^{v}\ln(v-u)\rho(u)\,du\,dv - n^{s-i}\ln n\int_{-1}^{1} v^i\rho^{(s)}(v)\int_{-1}^{v}\rho(u)\,du\,dv,$$

on making the substitutions $nt = u$ and $nx = v$, for $i = 0, 1, 2, \ldots$

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^i\,dx = 0, \tag{14}$$

for $i = 0, 1, 2, \ldots, s-1$ and

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^s\,dx = \int_{-1}^{1} v^s\rho^{(s)}(v)\int_{-1}^{v}\ln(v-u)\rho(u)\,du\,dv = s!b_s, \tag{15}$$

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^{s+1}\,dx = 0. \tag{16}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x) = \sum_{i=0}^{s} \frac{\phi^{(i)}(0)}{i!}x^i + \frac{\phi^{(s+1)}(\xi x)}{(s+1)!}x^{s+1},$$

where $0 < \xi < 1$. Using equations (14), (15) and (16), it follows that

$$\operatorname*{N-lim}_{n\to\infty}\langle(\ln x_+)_n\delta_n^{(s)}(x), \phi(x)\rangle = b_s\phi^{(s)}(0) = b_s\delta^{(s)}(x),$$

proving equation (7) for $s = 0, 1, 2, \ldots$

Equation (8) follows on replacing x by $-x$ in equation (7) and equation (9) then follows on noting that $\ln|x| = \ln x_+ + \ln x_-$.

Theorem 1 now shows us that the neutrix product $x_+^{-r}\Box\delta^{(s)}(x)$ exists and

$$\begin{aligned} x_+^{-r}\Box\delta^{(s)}(x) &= \sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r+i-1}}{(r-1)!}b_{s+i}\delta^{(r+s)}(x) \\ &= (-1)^r x_-^{-r}\Box\delta^{(s)}(x) \end{aligned}$$

on replacing x by $-x$. From equation (4) we have

$$x^{-r}\Box\delta^{(s)}(x) = x_+^{-r}\Box\delta^{(s)}(x) + (-1)^r x_-^{-r}\Box\delta^{(s)}(x) = c_{rs}\delta^{(r+s)}(x).$$

Equations (10), (11), (12) and (13) now follow and further we have

$$c_{rs} = 2\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r+i-1}}{(r-1)!}b_{s+i}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ In particular

$$\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r+i-1}}{(r-1)!}b_{r+i-1} = \frac{(-1)^r r!}{2(2r)!},$$

for $r = 1, 2, \ldots$, since

$$c_{r,r-1} = \frac{(-1)^r r!}{2(2r)!}.$$

Thus each b_{2s+1} can be solved as a linear sum of $b_0, b_2, \ldots, b_{2s}$ and so each c_{rs} is a linear sum of $b_0, b_2, \ldots, b_{2s}, \ldots$

Theorem 5. *The neutrix products $x_+^{-r}\Box x_-^s$ and $x_-^{-r}\Box x_+^s$ exist and*

$$\begin{aligned} x_+^{-r}\Box x_-^s &= \sum_{i=s+1}^{r}\frac{(-1)^{r-s+i}s!}{(r-1)!}b_{i-s-1}\delta^{(r-s-1)}(x) \\ &= (-1)^{r-s-1}x_-^{-r}\Box x_+^s, \end{aligned}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

Proof. The product of $\ln x_+$ and x_-^s is a straightforward product of locally summable functions, see [2], and

$$\ln x_+ . x_-^s = 0 \tag{19}$$

for $s = 0, 1, 2, \ldots$ Putting $g(x) = x_-^s$, we have

$$g^{(i)}(x) = \begin{cases} \dfrac{(-1)^i s!}{(s-i)!}x_-^{s-i}, & 0 \le i \le s, \\ (-1)^{s+1}s!\delta^{(i-s-1)}(x), & i > s. \end{cases}$$

Thus, by equation (19) we have

$$\ln x_+ . g^{(i)}(x) = 0$$

for $i = 0, 1, \dots, s$ and by equation (7) we have

$$\ln x_+ \Box g^{(i)}(x) = (-1)^{s+1} s! b_{i-s-1} \delta^{(i-s-1)}(x)$$

for $i = s+1, s+2, \dots$ It now follows from equation (1) that

$$\begin{aligned} (\ln x_+)^{(r)} \Box g(x) &= (-1)^{r-1}(r-1)! x_+^{-r} \Box x_-^s \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^i [\ln x_+ \Box g^{(i)}(x)]^{(r-i)} \\ &= \sum_{i=s+1}^{r} \binom{r}{i} (-1)^{s+i-1} s! b_{i-s-1} \delta^{(r-s-1)}(x). \end{aligned}$$

Equation (17) follows immediately and equation (18) follows on replacing x by $-x$.

Theorem 6. *The neutrix products $x_+^{-r} \Box x_+^s$, $x_-^{-r} \Box x_-^s$, $x^{-r} \Box x_+^s$ and $x^{-r} \Box x_-^s$ exist and*

$$\begin{aligned} x_+^{-r} \Box x_+^s &= x_+^{-r+s} - (-1)^{r+s} \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x) + \\ &\quad - \sum_{i=s+1}^{r} \frac{(-1)^{r+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x), \\ x_-^{-r} \Box x_-^s &= x_-^{-r+s} + \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x) + \\ &\quad + \sum_{i=s+1}^{r} \frac{(-1)^{s+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x), \\ x^{-r} \Box x_+^s &= x_+^{-r+s} - (-1)^{r+s} \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x) + \\ &\quad - \sum_{i=s+1}^{r} \frac{2(-1)^{r+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x), \\ x^{-r} \Box x_-^s &= (-1)^r x_-^{-r+s} + (-1)^r \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x) + \\ &\quad + \sum_{i=s+1}^{r} \frac{2(-1)^{r+s+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x), \end{aligned}$$

for $r = 1, 2, \dots$ and $s = 0, 1, \dots, r-1$.

Proof. The product of x_+^{-r} and the infinitely differentiable function x^s is given by

$$x_+^{-r} . x^s = x_+^{-r+s} - (-1)^{r+s} \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x),$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$, see [9]. It follows that

$$\begin{aligned} x_+^{-r} \Box x_+^s &= x_+^{-r}.x^s - (-1)^s x_+^{-r} \Box x_-^s \\ &= x_+^{-r+s} - (-1)^{r+s} \frac{\psi(r-s-1)+\psi(r-1)}{(r-s-1)!} \delta^{(r-s-1)}(x) + \\ &\quad - \sum_{i=s+1}^{r} \frac{(-1)^{r+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x) \end{aligned}$$

proving equation (20).

Equation (21) now follows from equation (20) on replacing x by $-x$. Equation (22) follows from equations (18) and (20) and then equation (23) follows from equation (22) on replacing x by $-x$.

Theorem 7. *The neutrix products* $(x_+^r \ln x_+) \Box \delta^{(s)}(x)$, $(x_-^r \ln x_-) \Box \delta^{(s)}(x)$ *and* $(x^r \ln |x|) \Box \delta^{(s)}(x)$ *exist and*

$$\begin{aligned} (x_+^r \ln x_+) \Box \delta^{(s)}(x) &= \binom{s}{r} (-1)^r r! b_0 \delta^{(s-r)}(x) + \\ &\quad + \sum_{i=r+1}^{s} \binom{s}{i} \tfrac{1}{2} (-1)^r (i-r-1)! c_{i-r,0} \delta^{(s-r)}(x), \\ (x_-^r \ln x_-) \Box \delta^{(s)}(x) &= \binom{s}{r} r! b_0 \delta^{(s-r)}(x) + \sum_{i=r+1}^{s} \tfrac{1}{2}\, r! c_{i-r,0} \delta^{(s-r)}(x), \\ (x^r \ln |x|) \Box \delta^{(s-r)}(x) &= \binom{s}{r} (-1)^r r! b_0 \delta^{(s-r)}(x) + \\ &\quad + \sum_{i=r+1}^{s} (-1)^r (i-r-1)! c_{i-r,0} \delta^{(s-r)}(x), \end{aligned}$$

for $r = 0, 1, 2, \ldots$ *and* $s = r, r+1, \ldots$

P r o o f. We define the function $f(x_+, r)$ by

$$f(x_+, r) = \frac{x_+^r \ln x_+ - \psi(r) x_+^r}{r!}$$

and it follows easily by induction that

$$f^{(i)}(x_+, r) = f(x_+, r-i),$$

for $i = 0, 1, \ldots, r$. In particular,

$$f^{(r)}(x_+, r) = \ln x_+,$$

so that

$$f^{(i)}(x_+, r) = (-1)^{i-r-1}(i-r-1)!x_+^{-i+r},$$

for $i = r+1, r+2, \dots$ Now $f^{(i)}(x_+, r)$ is a continuous function which is zero at the origin for $i = 0, 1, \dots, r-1$ and so

$$f^{(i)}(x_+, r).\delta(x) = 0, \tag{27}$$

for $r = 0, 1, \dots, r-1$. Using equation (7) we have

$$f^{(r)}(x_+, r)\Box\delta(x) = b_0\delta(x) \tag{28}$$

and using equation (10) we have

$$f^{(i)}(x_+, r)\Box\delta(x) = -\tfrac{1}{2}(-1)^{i-r-1}(i-r-1)!c_{i-r,0}\delta^{(i-r)}(x) \tag{29}$$

for $i = r+1, r+2, \dots$

Using equations (1), (27), (28) and (29) we have

$$\begin{aligned} f((x_+, r)\Box\delta^{(s)}(x) &= \sum_{i=0}^{s}\binom{s}{i}(-1)^i[f^{(i)}(x_+, r)\Box\delta(x)]^{(s-i)} \\ &= \sum_{i=r}^{s}\binom{s}{i}(-1)^i[f^{(i)}(x_+, r)\Box\delta(x)]^{(s-i)} \\ &= \binom{s}{r}(-1)^r b_0\delta^{(s-r)}(x) + \\ &\qquad + \sum_{i=r+1}^{s}\binom{s}{i}\tfrac{1}{2}(-1)^r(i-r-1)!c_{i-r,0}\delta^{(r-s)}(x). \end{aligned}$$

Thus

$$(x_+^r \ln x_+)\Box\delta^{(s)}(x) = r!f(x_+, r)\Box\delta^{(s)}(x) + \psi(r)x_+^r\Box\delta^{(s)}(x)$$

and equation (24) follows on using equation (3).

Equation (25) now follows from equation (24) on replacing x by $-x$ and equation (26) follows on noting that

$$x^r \ln|x| = x_+^r \ln x_+ + (-1)^r x_-^r \ln x_-.$$

For further related results, see Gramchev [10], and for a survey of recent results and theories in the product of distributions, see Oberguggenberger [11].

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Department of Mathematics and
Computer Science
University of Leicester,
Leicester, LE1 7RH
England

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