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BOUNDEDNESS AND RANGE OF $\mathcal{H}$–TRANSFORMATION ON CERTAIN WEIGHTED $\mathcal{L}_p$ SPACES

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Abstract. In this paper we study the behaviour of an integral transformation containing in its kernel a Fox $\mathcal{H}$–function on certain weighted $\mathcal{L}_p$ spaces denoted by $\mathcal{L}_{\gamma,r}$. Boundedness, representation and range of $\mathcal{H}$–transformation on $\mathcal{L}_{\gamma,r}$ are analyzed.

1. Introduction. In a series of papers P.G. Rooney ([13],[14],[15],[16]) and P.G. Rooney and P. Heywood ([5],[6]) have investigated a lot of integral transformations on certain weighted $\mathcal{L}_p$ spaces. The above authors employed a procedure in which the multipliers for the Mellin integral transformation play an important role. Also A.C. McBride and W.J. Spratt ([8],[9],[10]) have studied Mellin multiplier transforms on Fréchet spaces with seminorms of $\mathcal{L}_p$ type.

In [16] P.G. Rooney studied those integral transformations τ that formally have the form:

$$(\tau f)(x) = \int_0^\infty \mathcal{G}^{m,n}_{p,q}\left(xt\middle|^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right) f(t)dt, \tag{1}$$

where $\mathcal{G}^{m,n}_{p,q}\left(z\middle|^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right)$ denotes the Meijer function [4] on the spaces $\mathcal{L}_{\gamma,r}$ introduced by him [15] and constituted, for every $\gamma \in \mathbb{R}$ and $1\leq r < \infty$, by those Lebesgue–measurable complex–valued functions f defined on $(0,\infty)$ such that

$$\|f\|_{\gamma,r} = \left\{\int_0^\infty |x^\gamma f(x)|^r \frac{dx}{x}\right\}^{\frac{1}{r}} < \infty.$$

He obtained results on the boundedness, representation and range for the transformation under consideration. Moreover, some cases in which (1) has an inverse in the same formal form were analyzed.

In this paper we analyze integral transformations formally given by

$$(Tf)(x) = \int_0^\infty \mathcal{H}^{m,n}_{p,q}\left(xt\middle|^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}\right) f(t)\,dt \tag{2}$$

on the spaces $\mathcal{L}_{\gamma,r}$. Here $\mathcal{H}^{m,n}_{p,q}\left(z\middle|^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}\right)$ represents the Fox function [4]. So, for the sake of simplicity we will denote this function by $\mathcal{H}$(z). The method used to study transformations (2) is analogous to that employed by P.G. Rooney in [16] but our results can be seen as an extension of the one obtained in [16] because the Fox function reduces to the Meijer function when all the α's and the β's are equal to 1. Transformations (2) are the so called $\mathcal{H}-$ transformations. We say that an integral transformation T is formally defined by (2) when

$$(\mathcal{M}Tf)(s) = \mathfrak{H}^{m,n}_{p,q}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| s\right)(\mathcal{M}f)(1-s)$$

for some $s \in C$ and f being in a certain $\mathcal{L}_{\gamma,r}$. Here $\mathcal{M}$ denotes as usual the Mellin transformation ([16], § 2) and

$$\mathfrak{H}^{m,n}_{p,q}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| s\right) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}$$

where $0\le m\le q$, $0\le n\le p$, $\alpha_j > 0$ and $a_j\in R$ $j = 1,\ldots,p$ and $\beta_j > 0$ and $b_j\in R$ $j = 1,\ldots,q$. Here empty products are unity by convention, we abbreviate $\mathfrak{H}^{m,n}_{p,q}\left(\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| s\right)$ to $\mathfrak{H}(s)$ when there is no possibility of confusion.

Our work is organized as follows. In Section 2 we present some properties of functions $\mathcal{H}$ and $\mathfrak{H}$. Transformations (2) in $\mathcal{L}_{\gamma,2}$ are investigated in Section 3. Sections 4 and 5 are dedicated to the study of the boundedness and the range of the $\mathfrak{H}$–transformations on $\mathcal{L}_{\gamma,r}$ with $1 < r < \infty$. Finally in Section 6 we present some interesting special cases for the developed theory.

Throughout this paper we will denote by r' as usual the conjugate to r, that is $r' = \frac{r}{r-1}$, when $1 < r < \infty$. C_0 will represent the set constituted by all those continuous functions ϕ on $(0,\infty)$ having compact support. We will denote by $\mathcal{A}$ the class of multipliers introduced by P.G. Rooney [14]. We will say that $m\in\mathcal{A}$ if there are extended real numbers $\alpha(m)$ and $\beta(m)$, with $\alpha(m) < \beta(m)$, such that:

(a) m(s) is holomorphic in the strip $\alpha(m) < \Re(s) < \beta(m)$;

(b) in every closed substrip $\sigma_1\le\Re(s)\le\sigma_2$, where $\alpha(m) < \sigma_1\le\sigma_2 < \beta(m)$, m(s) is bounded; and

(c) for $\alpha(m) < \sigma < \beta(m)$ $\left|\frac{d}{dt}m(\sigma+it)\right| = 0\left(|t|^{-1}\right)$ as $|t|\to\infty$.

2. Some properties of function $\mathcal{H}$ and $\mathfrak{H}$. In this section we establish some properties for functions $\mathcal{H}$ and $\mathfrak{H}$ that will be very useful in the sequel.

Let $m, n, p, q \in N$, being $0\le m\le q$, $0\le n\le p$, and $p+q \ge 1$; and let $\alpha_j > 0$, $a_j \in R$, $(j = 1, \dots, p)$ and $\beta_j > 0$, $b_j \in R$, $(j = 1, \dots, q)$. We define the function

$$(3) \quad \mathfrak{H}(s) = \mathfrak{H}^{m,n}_{p,q}\left(\left.{}^{(a_1,\alpha_1),\dots,(a_p,\alpha_p)}_{(b_1,\beta_1),\dots,(b_q,\beta_q)}\right| s\right) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod\limits_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}.$$

Here empty products, if any, must be understood as 1.

Some real parameters must be introduced to analyze the main properties of $\mathfrak{H}$-function. Next such parameters are given

$$\alpha = \begin{cases} \max\left\{-\frac{b_j}{\beta_j},\ j=1,\dots,m\right\}, & \text{for} \quad m>0 \\ -\infty, & \text{for} \quad m=0 \end{cases},$$

$$\beta = \begin{cases} \min\left\{\frac{1-a_j}{\alpha_j},\ j=1,\dots,n\right\}, & \text{for} \quad n>0 \\ +\infty, & \text{for} \quad n=0 \end{cases},$$

$$\delta = m+n-\frac{1}{2}(p+q),$$

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

$$\mu_1 = \sum_{j=m+1}^{q}\beta_j - \sum_{j=1}^{n}\alpha_j,$$

$$\mu_2 = \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p}\alpha_j,$$

$$\nu = \sum_{j=1}^{q}b_j - \sum_{j=1}^{p}a_j,$$

$$\xi = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j,$$

$$\eta = \prod_{j=1}^{p} {\alpha_j}^{\alpha_j} \prod_{j=1}^{q} {\beta_j}^{-\beta_j}.$$

Note that $\mathfrak{H}_{p,q}^{m,n}(s)$ is holomorphic in the strip $\alpha < \Re(s) < \beta$.

In the following proposition boundedness properties for $\mathfrak{H}$ are proved.

Proposition 1. *As* $|t| \to \infty$

$$(4) \qquad |\mathfrak{H}(\sigma + it)| \sim (2\pi)^{\delta} \eta^{-\sigma} \prod_{j=1}^{p} \alpha_j^{-a_j+\frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j-\frac{1}{2}} |t|^{\nu+\mu\sigma+\frac{p-q}{2}} \exp\left(-\frac{\pi}{2}\xi|t|\right)$$

uniformly in σ *for* σ *in any bounded subset of* R. *Also, as* $|t| \to \infty$,

$$(5) \qquad \frac{d}{dt}\mathfrak{H}(\sigma + it) = i\mathfrak{H}(\sigma + it)\times$$

$$\times \left\{ \mu \log|t| - \log\eta + i\frac{\pi}{2}\xi \ sgnt + \frac{\nu + \mu\sigma + \frac{p-q}{2}}{it} + O(t^{-2}) \right\}$$

for every $\sigma \in R$, *with* $\alpha < \sigma < \beta$.

P r o o f. By using the formula [[3],1.18(6)]

$$(6) \qquad |\Gamma(x + iy)| \sim (2\pi)^{\frac{1}{2}} |y|^{x-\frac{1}{2}} e^{-\pi\frac{|y|}{2}}, \text{ as } |t| \to \infty$$

uniformly in x for x in any bounded subset of R, we can prove (4) without difficulty.

Moreover for every $s \in C$ with $\alpha < \Re(s) < \beta$ one has

$$\frac{d}{ds}\mathfrak{H}(s) = \mathfrak{H}(s) \left[\sum_{j=1}^{m} \beta_j \Psi(b_j + \beta_j s) - \sum_{j=1}^{n} \alpha_j \Psi(1 - a_j - \alpha_j s) + \right.$$

$$\left. + \sum_{j=m+1}^{q} \beta_j \Psi(1 - b_j - \beta_j s) - \sum_{j=n+1}^{p} \alpha_j \Psi(a_j + \alpha_j s) \right],$$

where as usual $\Psi(s) = \dfrac{\Gamma'(s)}{\Gamma(s)}$. Hence since for every $c \in C$, $\sigma \in R$ being $\alpha < \Re(c) + \sigma < \beta$ ([[3], 1, 18(7)])

$$(7) \qquad \Psi(c + \sigma \pm it) = \log(\pm it) \pm \frac{c + \sigma - \frac{1}{2}}{it} + O(t^{-2}), \text{ as } |t| \to \infty$$

we can conclude that (5) holds. □

As in [16] in our next study the exceptional set of $\mathfrak{H}$ that consists of all those γ such that $\alpha < 1-\gamma < \beta$ and $\mathfrak{H}(s)$ has zeroes on the line $\Re(s) = 1-\gamma$ and plays an important role.

The function $\mathcal{H}$ was introduced by Ch. Fox [4]. Now we specify some special cases for which $\mathcal{H}$ is the Mellin transformation of function $\mathfrak{H}$.

Proposition 2. *Let $\alpha < \gamma < \beta$. If either:*

(i) $\xi > 0$ *or*

(ii) $\xi = 0$, $\mu \neq 0$ *and* $\nu + \mu\gamma - \frac{1}{2}(q-p) \leq 0$

holds, then

$$\mathcal{H}(x) = \mathcal{H}^{m,n}_{p,q}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right| x\right) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} x^{-s}H(s)ds, \tag{8}$$

for $x > 0$. *Moreover if* $\xi = 0$, $\mu = 0$ *and* $\nu - \frac{1}{2}(q-p) < 0$, *then (8) holds for every* $x > 0$ *except for* $x = \eta^{-1}$.

P r o o f. Our assertion in this Proposition can be proved in a similar way to Lemma 3.2 [16] by using the boundedness properties established in Proposition 1.□

By taking into account again Proposition 1 we obtain as an immediate consequence of Proposition 2 the following

Corollary 1. *Let $\alpha < \gamma < \beta$. If either*

(i) $\xi > 0$ *or*

(ii) $\xi = 0$, $\mu \neq 0$ *and* $\nu + \mu\gamma - \frac{1}{2}(q-p) < -1$

holds, then

$$\mathcal{H}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s}\mathfrak{H}(s)ds \tag{9}$$

and

$$|\mathcal{H}(x)| \leq A_\gamma x^{-\gamma} \tag{10}$$

for $x > 0$, *where* A_γ *is a suitable positive constant. Moreover, if* $\xi = 0$, $\mu = 0$ *and* $\nu - \frac{1}{2}(q-p) < -1$ *then,* (9) *and* (10) *hold for every* $x > 0$ *except for* $x = \eta^{-1}$.

3. The $\mathcal{H}$ - transformation on the spaces $\mathcal{L}_{\gamma,2}$. Now, we study the behaviour of the $\mathcal{H}$- transformation on the spaces $\mathcal{L}_{\gamma,2}$.

Proposition 3. *Let $\alpha < 1-\gamma < \beta$. If either:*

(i) $\xi > 0$ *or*

(ii) $\xi = 0$ *and* $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) \leq 0$

holds, then there exists a one to one transformation $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ *such that for every* $f \in \mathcal{L}_{\gamma,2}$

$$(11) \qquad (\mathcal{M}Tf)(s) = \mathfrak{H}^{m,n}_{p,q}\left(\left.{(a_1,\alpha_1),\dots,(a_p,\alpha_p) \atop (b_1,\beta_1),\dots,(b_q,\beta_q)}\right| s\right)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma.$$

Moreover T is onto provided that $\xi = 0$, $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) = 0$ and γ does not belong to the exceptional set of $\mathfrak{H}$. For every $f, g \in \mathcal{L}_{\gamma,2}$ we have

$$(12) \qquad \int_0^\infty (Tf)(x)g(x)dx = \int_0^\infty f(x)(Tg)(x)dx.$$

Also if $f \in \mathcal{L}_{\gamma,2}$ then the transform Tf of f is given by

$$(13)\ \ (Tf)(x) = x^{-\lambda}\frac{d}{dx}x^{\lambda+1}\int_0^\infty \mathcal{H}^{m,n+1}_{p+1,q+1}\left(\left.{(-\lambda,1),(a_1,\alpha_1),\dots,(a_p,\alpha_p) \atop (b_1,\beta_1),\dots,(b_q,\beta_q),(-1-\lambda,1)}\right| xt\right) f(t)dt,\ \ a.e.\ x > 0$$

provided $\lambda > -\gamma$, and

$$(14)\ \ (Tf)(x) = -x^{-\lambda}\frac{d}{dx}x^{\lambda+1}\int_0^\infty \mathcal{H}^{m+1,n}_{p+1,q+1}\left(\left.{(a_1,\alpha_1),\dots,(a_p,\alpha_p)(-\lambda,1) \atop (-1-\lambda,1)(b_1,\beta_1),\dots,(b_q,\beta_q)}\right| xt\right) f(t)dt,\ \ a.e.\ x > 0$$

when $\lambda < -\gamma$. Finally, the transformation T is not depending on γ in the following sense: if γ_i is such that $\alpha < 1-\gamma_i < \beta$ and $\nu + \mu(1-\gamma_i) - \frac{1}{2}(q-p) \leq 0$, i=1,2, and T_i represents the transformation associated to γ_i, i=1,2, then $T_1 f = T_2 f$, for each $f \in \mathcal{L}_{\gamma_1,2} \cap \mathcal{L}_{\gamma_2,2}$.

Proof. Let $\alpha < 1-\gamma < \beta$. Define $\omega(t) = \mathfrak{H}(1-\gamma+it)$, $t \in R$, ω is continuous on R because $\alpha < 1-\gamma < \beta$. Moreover, according to Proposition 1 $\omega(t) = O(1)$, as $|t| \to \infty$ when either (i) or (ii) holds. Hence $\omega \in \mathcal{L}_\infty(R)$ and by virtue of Lemma 4.1(b)[16] there exists a transformation $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ such that for every $f \in \mathcal{L}_{\gamma,2}$

$$(\mathcal{M}Tf)(1-\gamma+it) = \omega(t)(\mathcal{M}f)(\gamma-it), \ \ t \in R.$$

Since $\mathfrak{H}(s)$ is holomorphic on $\alpha < \Re(s) < \beta$ and $\mathfrak{H}(s)$ is not identically zero, then the zeros of $\mathfrak{H}(s)$ are isolated and $\omega(t) \neq 0$ a.e. $t \in R$. This fact implies that T is one to one.

On the other hand $\omega(t) \neq 0$ for every $t \in R$, provided γ is not in the exceptional set of $\mathfrak{H}$. From (4) we deduce that if $\xi = 0$ and $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) = 0$, then $\frac{1}{\omega} \in \mathcal{L}_\infty(R)$. Lemma 4.1(c) [16] allows us to conclude that T is onto. Equality (12) follows also from Lemma 4.1(c) [16].

To prove the representation formula (13) we consider the function $\omega_1(t) = \dfrac{\mathfrak{H}(1-\gamma+it)}{\lambda+\gamma-it}$, $t \in R$, with $\lambda > -\gamma$. It is clear that $\omega_1(t) \in \mathcal{L}_2(R)$. Since $\mathcal{M}$ is an unitary transformation from $\mathcal{L}_{1-\gamma,2}$ onto $L_2(R)$, the function

$$k(x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{1-\gamma-iR}^{1-\gamma+iR} x^{-s} \frac{\mathfrak{H}(s)}{\lambda+1-s} ds \tag{15}$$

where the limit is understood in the space $\mathcal{L}_{1-\gamma,2}$, is in $\mathcal{L}_{1-\gamma,2}$. Also we can write

$$\frac{\mathfrak{H}(s)}{\lambda+1-s} = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\Gamma(1-(-\lambda)-s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\Gamma(1-(-1-\lambda)-s)\prod\limits_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)} =$$

$$= \mathfrak{H}^{m,n+1}_{p+1,q+1}\left(\left.\begin{smallmatrix}(-\lambda,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q),(1-\lambda,1)\end{smallmatrix}\right| s\right)$$

The parameters associated to the last function denoted by primes, are related to the corresponding ones for the function $\mathfrak{H}^{m,n}_{p,q}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right| s\right)$ through: $\delta' = \delta$, $\mu' = \mu$, $\nu' = \nu - 1$, $\xi' = \xi$, $\alpha' = \alpha$, $\beta' = min\{\beta, 1+\lambda\}$, $n' = n+1$, $m' = m$, $p' = p+1$ and $q' = q+1$. Hence by invoking Proposition 2 we obtain

$$\mathcal{H}^{m,n+1}_{p+1,q+1}\left(\left.\begin{smallmatrix}(-\lambda,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q),(-1-\lambda,1)\end{smallmatrix}\right| x\right) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{1-\gamma-iR}^{1-\gamma+iR} x^{-s} \frac{\mathfrak{H}(s)}{\lambda+1-s} ds \tag{16}$$

for every $x > 0$ except, at most, for $x = \eta^{-1}$.

Now combining (15) and (16) we conclude that

$$k(x) = \mathcal{H}^{m,n+1}_{p+1,q+1}\left(\left.\begin{smallmatrix}(-\lambda,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q),(-1-\lambda,1)\end{smallmatrix}\right| x\right)$$

and Lemma 4.1(b) [16] leads to the representation (13) for T.

To establish (14) it is sufficient to write

$$\frac{\mathfrak{H}(s)}{\lambda+1-s} = -\frac{\mathfrak{H}(s)\Gamma(s-\lambda-1)}{\Gamma(s-\lambda)}$$

with $\lambda < -\gamma$, and to proceed as in the above case. Finally, assume that $\alpha < 1-\gamma_i < \beta$ and $\nu + \mu(1-\gamma_i) - \frac{1}{2}(q-p) \leq 0, i = 1, 2$. For $\lambda > \max\{-\gamma_1, -\gamma_2\}$, T_i admits the representation (13) on $\mathcal{L}_{\gamma_i,2}, i = 1, 2$. Hence, since the right hand side of (13) does not depend on γ, if $f \in \mathcal{L}_{\gamma_1,2} \cap \mathcal{L}_{\gamma_2,2}$, then $T_1 f = T_2 f$. $\square$

Note that the conditions $\alpha < 1-\gamma < \beta$ and $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) \leq 0$ are compatible provided that one of the three conditions

(i) $\mu = 0$ and $\nu \leq \frac{1}{2}(q-p)$

(ii) $\mu > 0$ and $\alpha < \frac{1}{\mu}(\frac{1}{2}(q-p) - \nu)$

(iii) $\mu < 0$ and $\beta > \frac{1}{\mu}(\frac{1}{2}(q-p) - \nu)$

holds. Hence transformation T can be defined on $\mathcal{L}_{\gamma,2}$ for some $\gamma \in R$ when $\alpha < \beta$ and some of the above three conditions are satisfied.

From Proposition 3 can be deduced the following

Corollary 2. *Let $\alpha < 1-\gamma < \beta$. If either*

(i) $\xi > 0$ *or*

(ii) $\xi = 0$ *and* $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) < -1$

holds, then Tf is given by (2) *for $f \in \mathcal{L}_{\gamma,2}$.*

Proof. It is easy to see that

$$\frac{d}{dx}\left[x^{\lambda+1}\mathcal{H}^{m,n+1}_{p+1,q+1}\left(\left.\begin{smallmatrix}(-\lambda,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q),(-1-\lambda,1)\end{smallmatrix}\right|x\right)\right] =$$

$$= x^{\lambda}\mathcal{H}^{m,n}_{p,q}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right|x\right) \quad \text{a.e. } x > 0$$

with $\lambda > \alpha - 1$. From (13) one deduces

$$(Tf)(x) = \int_0^\infty \mathcal{H}^{m,n}_{p,q}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right|xt\right) f(t)dt, \quad f \in \mathcal{L}_{\gamma,2}.$$

Differentiation under the integral sign is justified because according to Corollary 1 and using Hölder's inequality for every $\gamma_i, i = 1, 2$, being $\alpha < \gamma_1 < 1-\gamma < \gamma_2 < \beta$ we have

$$\int_0^\infty \left|\mathcal{H}^{m,n}_{p,q}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right|xt\right) f(t)\right| dt \leq$$

$$\leq M_1 x^{-\gamma_1}\int_0^{\frac{1}{x}} t^{-\gamma_1}|f(t)|\,dt + M_2 x^{-\gamma_2}\int_{\frac{1}{x}}^\infty t^{-\gamma_2}|f(t)|\,dt \leq$$

$$\leq \left[M_1 x^{-\gamma_1} \left\{ \int_0^{\frac{1}{x}} t^{2(1-\gamma-\gamma_1)-1} dt \right\}^{\frac{1}{2}} + M_2 x^{-\gamma_2} \left\{ \int_{\frac{1}{x}}^{\infty} t^{2(1-\gamma-\gamma_2)-1} dt \right\}^{\frac{1}{2}} \right] \|f\|_{\gamma,2} < \infty$$

for $f \in \mathcal{L}_{\gamma,2}$. $\square$

4.Boundedness and range of the $\mathcal{H}$-transformation on $\mathcal{L}_{\gamma,r}$ for $\xi = 0$. In the previous Section we established that there exists a transformation $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ satisfying (11) provided that $\alpha < 1-\gamma < \beta$, $\xi = 0$ and $\nu + \mu(1-\gamma) - \frac{1}{2}(q-p) \leq 0$. We now prove that T can be extended to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ for $1 < r < \infty$ and for suitable $1 < s < \infty$. Moreover the range $T(\mathcal{L}_{\gamma,r})$ of T is described. Such description is made in terms of the following operators:

$$(M_\xi f)(x) = x^\xi f(x) , \quad \xi \in C$$

$$(N_a f)(x) = f(x^a) , \quad a \in R, \;\; a \neq 0$$

$$(D_a f)(x) = f(ax) , \quad a > 0$$

$$(Rf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right) .$$

$$(\mathcal{I}_{a,b,c} f)(x) = \frac{a x^{-a(c+b-1)}}{\Gamma(b)} \int_0^x (x^a - t^a)^{b-1} t^{ac-1} f(t) dt , \; a > 0, \;\; \Re(b) > 0, \;\; c \in C$$

$$(\mathcal{J}_{a,b,c} f)(x) = \frac{a x^{ac}}{\Gamma(b)} \int_x^\infty (t^a - x^a)^{b-1} t^{-a(b+c-1)-1} f(t) dt , \; a > 0, \;\; \Re(b) > 0, \;\; c \in C$$

$$(h_{a,b} f)(x) = \int_0^\infty (xt)^{\frac{1}{a}-\frac{1}{2}} J_b\left(|a|(xt)^{\frac{1}{a}}\right) f(t) dt , \; \Re(b) > -1 , \; a \neq 0$$

where J_b denotes the Bessel function of first kind and order b. The behavior of these operators on $\mathcal{L}_{\gamma,r}$ was investigated in [16] (after Definition 2.2 and Theorem 5.1). We will divide our study in several cases. In the sequel T will denote the transformation defined in Proposition 3.

Proposition 4. *Let $\alpha < 1-\gamma < \beta$, $\xi = 0$, $\mu = 0$, $\nu - \frac{1}{2}(q-p) = 0$ and $1 < r < \infty$. Then $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ can be extended to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,r}]$. T is one to one when either $1 < r \leq 2$ or γ is not in the exceptional set of $\mathfrak{H}$. Moreover if γ does not belong to the exceptional set of $\mathfrak{H}$, then $T(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{1-\gamma,r}$.*

Equality (12) *holds for every* $f \in \mathcal{L}_{\gamma,r}$ *and* $g \in \mathcal{L}_{\gamma,r'}$. *For each* $f \in \mathcal{L}_{\gamma,r}$ Tf *is given by* (13) *(respectively,* (14)*) provided* $\lambda > -\gamma$ *(respectively,* $\lambda < -\gamma$*).*

Proof. By virtue of Proposition 3 $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ and for every $f \in \mathcal{L}_{\gamma,2}$

$$(\mathcal{M}Tf)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma.$$

We introduce the function $L(s) = \eta^s \mathfrak{H}(s)$. It is clear that L is holomorphic on $\alpha < \Re(s) < \beta$. From Proposition 1 we infer that

$$|L(\sigma+it)| \sim (2\pi)^\delta \prod_{j=1}^{p} \alpha_j^{-a_j+\frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j-\frac{1}{2}} \quad, \text{as} \quad |t| \to \infty$$

uniformly in σ from a bounded subset of R, and

$$\frac{d}{dt} L(\sigma+it) = iL(\sigma+it) O\left(t^{-2}\right) = O\left(t^{-2}\right), \quad \text{as} \quad |t| \to \infty$$

for every $\alpha < \sigma < \beta$. Hence $L \in \mathcal{A}$ being $\alpha(L) = \alpha$ and $\beta(L) = \beta$. According to Theorem 2.1 [16] there exists $\mathfrak{L} \in [\mathcal{L}_{\gamma,r}]$ for every $\alpha < \gamma < \beta$ and $1 < r < \infty$. Also if $\alpha < \gamma < \beta$, $1 < r \leq 2$, then for every $f \in \mathcal{L}_{\gamma,r}$

$$(\mathcal{M}\mathfrak{L}f)(s) = L(s)(\mathcal{M}f)(s), \quad \Re(s) = \gamma,$$

and $\mathfrak{L}$ is one to one in $\mathcal{L}_{\gamma,r}$.

We define $\mathfrak{L}_1 = D_\eta \, \mathfrak{L} \, R$. By using the Remark after Definition 2.2 in [16] we conclude that $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,r}]$ when $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$, and for every $f \in \mathcal{L}_{\gamma,r}$ with $1 < r \leq 2$

$$(\mathcal{M}\mathfrak{L}_1 f)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma.$$

In particular $(\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}Tf)(s)$, for $\Re(s) = 1-\gamma$ and $f \in \mathcal{L}_{\gamma,2}$ .Therefore $T = \mathfrak{L}_1$ on $\mathcal{L}_{\gamma,2}$ and $\mathfrak{L}_1$ is the unique extension of T to $\mathcal{L}_{\gamma,r}$. In the sequel we will denote $\mathfrak{L}_1$ also by T. Since D_η and R are one to one T is one to one on $\mathcal{L}_{\gamma,r}$ when $1 < r < \infty$ and $\alpha < 1-\gamma < \beta$.

The abscissae of the zeroes of $L(s)$, that are the same ones of $\mathfrak{H}(s)$, are reals and they divide the interval (α,β) in a finite numbers of intervals, let (α_1,β_1) be one of such intervals. It is easy to see that $\frac{1}{L(s)}$ is holomorphic in $\alpha_1 < \Re(s) < \beta_1$,

$$\left|\frac{1}{L(\sigma+it)}\right| \sim (2\pi)^{-\delta} \prod_{j=1}^{p} a_j{}^{a_j-\frac{1}{2}} \prod_{j=1}^{q} b_j{}^{-b_j+\frac{1}{2}}, \quad \text{as} \quad |t| \to \infty,$$

uniformly in σ when σ is in a bounded subset of R and

$$\frac{d}{dt}\frac{1}{L(\sigma+it)} = O\left(t^{-2}\right), \quad \text{as} \quad |t| \to \infty$$

for every $\alpha_1 < \sigma < \beta_1$. By invoking again Theorem 2.1 [16] we obtain that for every $1 < r < \infty$ and $\alpha_1 < \gamma < \beta_1$ T is one to one and $\mathfrak{L}(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{\gamma,r}$. Hence, since $R(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{1-\gamma,r}$ and $D_\eta(\mathcal{L}_{1-\gamma,r}) = \mathcal{L}_{1-\gamma,r}$ for every $1 < r < \infty$ and $\gamma \in R$, $T(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{1-\gamma,r}$ provided $1 < r < \infty$, $\alpha < 1-\gamma < \beta$ and γ is not in the exceptional set of $\mathfrak{H}$.

We now prove the equality (12). If $f \in \mathcal{L}_{\gamma,r}$ and $g \in \mathcal{L}_{\gamma,r'}$, then Hölder's inequality leads to

$$\left|\int_0^\infty (Tf)(x)g(x)dx\right| \leq \int_0^\infty \frac{|(Tf)(x)x^{1-\gamma}|\,|g(x)x^\gamma|}{x^{\frac{1}{r}} \quad x^{\frac{1}{r'}}}dx \leq$$

$$\leq \|Tf\|_{1-\gamma,r}\|g\|_{\gamma,r'} \leq C_{r,\gamma}\|f\|_{\gamma,r}\|g\|_{\gamma,r'}$$

where $C_{r,\gamma}$ denotes the norm of T as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,r}]$. Therefore the mapping

$$\begin{array}{rccl} P: & \mathcal{L}_{\gamma,r} \times \mathcal{L}_{\gamma,r'} & \longrightarrow & C \\ & (f,g) & \longrightarrow & \int_0^\infty (Tf)(x)g(x)dx \end{array}$$

is bounded.

In a similar way we can prove that the mapping

$$\begin{array}{rccl} \mathfrak{P}: & \mathcal{L}_{\gamma,r} \times \mathcal{L}_{\gamma,r'} & \longrightarrow & C \\ & (f,g) & \longrightarrow & \int_0^\infty f(x)(Tg)(x)dx \end{array}$$

is bounded. By virtue of Proposition 3 $P(f,g) = \mathfrak{P}(f,g)$, for every $f, g \in C_0$. Hence $P = \mathfrak{P}$ because C_0 is a dense subset of $\mathcal{L}_{\gamma,r}$.

Assume that $\lambda > -\gamma$ and consider for every $x > 0$ the function

$$g_x(t) = \begin{cases} t^\lambda, & 0 < t < x \\ 0, & t > x \end{cases}$$

Note that $g_x \in \mathcal{L}_{\gamma,r}$, $1 < r < \infty$. By using (13) we obtain

$$(Tg_x)(y) = y^{-\lambda}\frac{d}{dy}y^{\lambda+1}\int_0^x \mathcal{H}^{m,n+1}_{p+1,q+1}\left(\begin{matrix}(-\lambda,1),(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q),(-1-\lambda,1)\end{matrix}\middle| yt\right) g_x(t)dt =$$

$$= y^{-\lambda}\frac{d}{dy}\int_0^{xy} \mathcal{H}^{m,n+1}_{p+1,q+1}\left(\begin{matrix}(-\lambda,1),(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q),(-1-\lambda,1)\end{matrix}\middle| t\right) t^\lambda dt =$$

$$= x^{\lambda+1}\mathcal{H}^{m,n+1}_{p+1,q+1}\left(\begin{matrix}(-\lambda,1),(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q),(-1-\lambda,1)\end{matrix}\middle| xy\right), \quad \text{a.e.} \quad y > 0.$$

In view of (12) for every $f \in \mathcal{L}_{\gamma,r}$ one has

$$\int_0^x t^\lambda (Tf)(t)dt = x^{\lambda+1}\int_0^\infty \mathcal{H}^{m,n+1}_{p+1,q+1}\left(\begin{matrix}(-\lambda,1),(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q),(-1-\lambda,1)\end{matrix}\middle| xt\right) f(t)dt, \quad x > 0$$

and by differentiating with respect to x we conclude (13) for $f \in \mathcal{L}_{\gamma,r}$.

To prove (14) for$f \in \mathcal{L}_{\gamma,r}$ it is sufficient to proceed as in the previous case by considering the functions

$$h_x(t) = \begin{cases} 0, & 0 < t < x \\ t^{\lambda}, & t > x \end{cases}$$

with $x > 0$ and $\lambda < -\gamma$ instead of $g_x(t)$. $\square$

Before stating the following Proposition we remark that $m + n > 0$ provided $\xi = 0$. In effect if $m = n = 0$, then $\xi = -\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j < 0$ because $\alpha_j > 0$, $j = 1, \dots, p$, and $\beta_j > 0$, $j = 1, \dots, q$.

Proposition 5. *Assume that* $\xi = \mu = 0$, $\nu - \frac{1}{2}(q-p) < 0$, $\alpha < 1-\gamma < \beta$ *and* $1 < r < \infty$. *Transformation* $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ *can be extended to* $\mathcal{L}_{\gamma,r}$ *as a member of* $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ *provided that* $r \leq s$ *and* $\frac{1}{s} > \frac{1}{r} + \nu - \frac{1}{2}(q-p)$. *If either* $1 < r \leq 2$ *or* γ *is not in the exceptional set of* $\mathfrak{H}$, *then* T *is one to one. Moreover when* γ *does not belong to the exceptional set of* $\mathfrak{H}$ *then,*

$$T\left(\mathcal{L}_{\gamma,r}\right) = \mathcal{J}_{1,-\nu+\frac{1}{2}(q-p),-\alpha}\left(\mathcal{L}_{1-\gamma,r}\right), \quad for \quad \alpha > -\infty, \tag{17}$$

and

$$T\left(\mathcal{L}_{\gamma,r}\right) = \mathcal{I}_{1,-\nu+\frac{1}{2}(q-p),\beta}\left(\mathcal{L}_{1-\gamma,r}\right), \quad for \quad \beta < +\infty \tag{18}$$

while if γ *is in the exceptional set of* $\mathfrak{H}$, $T\left(\mathcal{L}_{\gamma,r}\right)$ *is a subset of the set on the right hand side of* (17) *and* (18). *Equality* (12) *holds for every* $f \in \mathcal{L}_{\gamma,r}$ *and* $g \in \mathcal{L}_{\gamma,\lambda}$ *provided that* $\lambda' \geq r$ *and* $\frac{1}{r} + \frac{1}{\lambda} < 1 - \nu + \frac{1}{2}(q-p)$. T *admits the representation* (13) *(respectively* (14)*) when* $\lambda > -\gamma$ *(respectively* $\lambda < -\gamma$*) for* $f \in \mathcal{L}_{\gamma,r}$. *If* $\nu - \frac{1}{2}(q-p) < -1$, *then* T *is given by* (2), *for* $f \in \mathcal{L}_{\gamma,r}$.

P r o o f. Assume firstly that $m > 0$ or, equivalently, that $\alpha > -\infty$. We define the function

$$L(s) = \eta^s \; \frac{\Gamma\left(s - \alpha - \nu + \frac{1}{2}(q-p)\right)}{\Gamma(s-\alpha)} \; \mathfrak{H}(s).$$

It is clear that L is holomorphic on $\alpha < \Re(s) < \beta$. Moreover according to Proposition 1, we can obtain

$$|L(\sigma + it)| \sim (2\pi)^{\delta} \prod_{j=1}^{p} \alpha_j^{-a_j + \frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j - \frac{1}{2}}, \text{ as } |t| \to \infty,$$

uniformly in σ when σ is in a bounded subset of R, and

$$\frac{d}{dt} L(\sigma + it) = O\left(t^{-2}\right), \text{ as } |t| \to \infty,$$

for every $\alpha < \sigma < \beta$. Hence $L \in \mathcal{A}$ and by virtue of Theorem 2.1 [16] there exists $\mathfrak{L} \in [\mathcal{L}_{\gamma,r}]$ for every $1 < r < \infty$ and $\alpha < \gamma < \beta$, such that for each $1 < r \leq 2$ and $\alpha < \gamma < \beta$

$$(\mathcal{M}\mathfrak{L}f)(s) = L(s)(\mathcal{M}f)(s), \quad \Re(s) = \gamma$$

when $f \in \mathcal{L}_{\gamma,r}$.

We now introduce the operator

$$\mathfrak{L}_1 = \mathcal{J}_{1,-\nu+\frac{1}{2}(q-p),-\alpha}\, D_\eta\, \mathfrak{L}\, R.$$

By taking into account the boundedness of the operators R, D_η, and $\mathcal{J}_{1,-\nu+\frac{1}{2}(q-p),-\alpha}$ on the spaces $\mathcal{L}_{\gamma,r}$ (Theorem 5.1(b)[16]) one establishes that $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ provided that $1 < r < \infty$, $\alpha < 1-\gamma < \beta$, $\nu - \frac{1}{2}(q-p) < 0$, $\xi = \mu = 0$, $s \geq r$ and $\frac{1}{s} > \frac{1}{r} + \nu - \frac{1}{2}(q-p)$. Also, for every $f \in \mathcal{L}_{\gamma,r}$ with $1 < r \leq 2$

$$\begin{aligned}(\mathcal{M}\mathfrak{L}_1 f)(s) &= \frac{\Gamma(s-\alpha)}{\Gamma\left(s-\alpha-\nu+\frac{1}{2}(q-p)\right)}\, \mathcal{M}\left[D_\eta\, \mathfrak{L}\, R\, f\right](s) = \\ &= \frac{\Gamma(s-\alpha)}{\Gamma\left(s-\alpha-\nu+\frac{1}{2}(q-p)\right)}\, \eta^{-s}\, \mathcal{M}\left[\mathfrak{L}\, R\, f\right](s) = \\ &= \mathfrak{H}(s)\, \mathcal{M}\left[R\, f\right](s) = \mathfrak{H}(s)\mathcal{M}[f](1-s), \qquad \Re(s) = 1-\gamma.\end{aligned}$$

In view of Proposition 3 for every $f \in \mathcal{L}_{\gamma,2}$ $(\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}Tf)(s)$, $\Re(s) = 1-\gamma$. Then $\mathfrak{L}_1 f = Tf$, for every $f \in \mathcal{L}_{\gamma,2}$, and $\mathfrak{L}_1$ is the unique extension of T to $\mathcal{L}_{\gamma,r}$ when the conditions are satisfied. In the sequel T also will denote the above mentioned extension $\mathfrak{L}_1$.

The one to one property and the range of the transformation T can be studied now in a way analogous as they were analyzed in Proposition 4.

We now aim to prove (12). Let $f \in \mathcal{L}_{\gamma,r}$ and $g \in \mathcal{L}_{\gamma,l}$ with $l' \geq r$ and $\frac{1}{r} + \frac{1}{l} < 1 - \nu + \frac{1}{2}(q-p)$. By using Hölder's inequality we can write

$$\left|\int_0^\infty (Tf)(x)g(x)dx\right| \leq \int_0^\infty \frac{\left|(Tf)(x)x^{1-\gamma}\right| \left|g(x)x^\gamma\right|}{x^{\frac{1}{l'}} \quad x^{\frac{1}{l}}} dx \leq$$

$$\leq \|Tf\|_{1-\gamma,l'} \|g\|_{\gamma,l} \leq C\ \|f\|_{\gamma,r} \|g\|_{\gamma,l}$$

where C denotes the norm of T as a member of $\left[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,l'}\right]$. Hence the bilinear mapping

$$\begin{array}{rccc} P: & \mathcal{L}_{\gamma,r} \times \mathcal{L}_{\gamma,l} & \longrightarrow & C \\ & (f,g) & \longrightarrow & \int_0^\infty (Tf)(x)g(x)dx \end{array}$$

is bounded. In a similar way we can prove that the bilinear mapping

$$\begin{array}{rccc} \mathfrak{P}: & \mathcal{L}_{\gamma,r} \times \mathcal{L}_{\gamma,l} & \longrightarrow & C \\ & (f,g) & \longrightarrow & \int_0^\infty f(x)(Tg)(x)dx \end{array}$$

is also bounded. Since C_0 is a dense subset of $\mathcal{L}_{\gamma,r}$ and since (12) holds when f and g are in C_0, equality (12) also is true for $f \in \mathcal{L}_{\gamma,r}$ and $g \in \mathcal{L}_{\gamma,l}$.

Representation formulas (13) and (14) can be seen as the corresponding ones in Proposition 4.

Finally (2) follows from (13) and (14) when $\nu - \frac{1}{2}(q-p) < -1$ by differentiating under the integral sign as in Corollary 1.

To prove (18) we can proceed as in the proof of (17) by considering the function

$$\omega(s) = \frac{\Gamma\left(\beta - \nu + \frac{1}{2}(q-p) - s\right)}{\Gamma(\beta - s)}\, \eta^s\, \mathfrak{H}(s)$$

instead of $L(s)$. □

Proposition 6. *Let $\xi = 0$, $\mu < 0$, $1 < r < \infty$ and $\alpha_1 < 1 - \gamma < \beta_1$ where*

$$\alpha_1 = \max\left\{\alpha, -\frac{1}{\mu}\left(\nu + \frac{p-q}{2} + \gamma(r) - \frac{1}{2}\right), \frac{2\rho}{\mu}\right\} \text{ and}$$

$$\beta_1 = \min\left\{\beta, -\frac{2}{\mu}\left(\nu + \frac{p-q}{2} + \rho\right)\right\}$$

with $\rho > \max\left\{\frac{\mu\beta}{2}, \frac{1}{2}\left(\gamma(r) - \frac{1}{2} - \nu - \frac{p-q}{2}\right), -\frac{\mu}{2}\alpha - \nu - \frac{p-q}{2}\right\}$. Then $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ can be extended to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ for every $s \geq r$ being $s' > \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$. If either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \leq 2$, then T is one to one on $\mathcal{L}_{\gamma,r}$. Also if γ does not belong to the exceptional set of $\mathfrak{H}$ the equality

$$(19) \quad T\left(\mathcal{L}_{\gamma,r}\right) = \left(N_{\frac{2}{\mu}}\ M_{\frac{1}{2}\left(\nu+\frac{p-q}{2}+1\right)}\ h_{2,\nu+\frac{p-q}{2}+2\rho-1}\right)\left(\mathcal{L}_{1-\frac{\mu(1-\gamma)}{2}-\frac{1}{2}\left(\nu+\frac{p-q}{2}+1\right),\, r}\right)$$

while if γ is in the exceptional set of $\mathfrak{H}$, then $T\left(\mathcal{L}_{\gamma,r}\right)$ is a subset of the set on the right hand side of (19). *Equality* (12) *holds provided that $f, g \in \mathcal{L}_{\gamma,r}$, with $1 < r \leq 2$ and $r > \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$ then* (13) *(respectively* (14)*) holds with $\lambda > -\gamma$ (respectively $\lambda < -\gamma$) for every $f \in \mathcal{L}_{\gamma,r}$.*

P r o o f. Firstly note that $\nu + (1-\gamma)\mu + \frac{p-q}{2} \leq 0$ because $\gamma(r) \geq \frac{1}{2}$. Hence by Proposition 3 $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ and for every $f \in \mathcal{L}_{\gamma,2}$

$$\left(\mathcal{M}Tf\right)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma.$$

Define the function

$$L(s) = \frac{\Gamma\left(\rho - \frac{\mu}{2}s\right)}{\Gamma\left(\nu + \frac{p-q}{2} + \rho + \frac{\mu}{2}s\right)}\eta^s \mathfrak{H}(s)$$

being $a = \left|\frac{\mu}{2}\right|^{\mu}\ \eta$ and $\rho > \max\left\{\frac{\mu\beta}{2}, \frac{1}{2}\left(\gamma(r) - \frac{1}{2} - \nu - \frac{p-q}{2}\right), -\frac{\mu}{2}\alpha - \nu - \frac{p-q}{2}\right\}$. L is holomorphic on $\alpha_2 = max\left\{\alpha, \frac{2\rho}{\mu}\right\} < \Re(s) < \beta$. Moreover $L \in \mathcal{A}$. Indeed, by using (4) and (6) we obtain

$$|L(\sigma+it)| \sim (2\pi)^{\delta} \prod_{j=1}^{p} \alpha_j^{-a_j+\frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j-\frac{1}{2}} \left|\frac{\mu}{2}\right|^{-\nu-\frac{p-q}{2}}, \quad \text{as} \quad |t| \to \infty,$$

uniformly in σ when σ is in a bounded subset of R, and from (5) and (7) one infers

$$\frac{d}{dt}L(\sigma+it) = iL(\sigma+it)\left\{-\frac{\mu}{2}\left(\Psi\left(\rho - \frac{\mu}{2}(\sigma+it)\right)\right) + \Psi\left(\nu + \frac{p-q}{2} + \rho + \frac{\mu}{2}(\sigma+it)\right)\right.$$

$$\left. + \log a + \mu\log|t| - \log\eta + \frac{\nu + \mu\sigma + \frac{p-q}{2}}{it} + O\left(t^{-2}\right)\right\} = O\left(t^{-2}\right), \quad \text{as} \quad |t| \to \infty$$

for every $\alpha_2 < \sigma < \beta$.

Since $L \in \mathcal{A}$ with $\alpha(L) = \alpha_2$ and $\beta(L) = \beta$, Theorem 2.1 [16] allows to find for every $1 < r < \infty$ and $\alpha_2 < \epsilon < \beta$ a $\mathfrak{L} \in [\mathcal{L}_{\epsilon,r}]$ such that if $1 < r \leq 2$ $\mathfrak{L}$ is one to one and

$$(20) \qquad (\mathcal{M}\mathfrak{L}f)(s) = L(s)(\mathcal{M}f)(s), \quad \Re(s) = \epsilon$$

provided that $f \in \mathcal{L}_{\epsilon,r}$.

We now consider the function

$$\mathfrak{G}(s) = \mathfrak{G}^{1,0}_{0,2}\left(\begin{array}{cc} 0 & 0 \\ \nu + \frac{p-q}{2} + \rho & 1-\rho \end{array}\middle| s\right) = \frac{\Gamma\left(\nu + \frac{p-q}{2} + \rho + s\right)}{\Gamma(\rho - s)}$$

In notations used by P.G. Rooney [16] §3 the parameters associated with this $\mathfrak{G}$–function are $\delta_{\mathfrak{G}} = n_{\mathfrak{G}} = p_{\mathfrak{G}} = 0$, $q_{\mathfrak{G}} = 2$, $m_{\mathfrak{G}} = k_{\mathfrak{G}} = 1$, $l_{\mathfrak{G}} = -1$, $\nu_{\mathfrak{G}} = \nu + \frac{p-q}{2} + 1$, $\alpha_{\mathfrak{G}} = -\nu - \frac{p-q}{2} - \rho$ and $\beta_{\mathfrak{G}} = +\infty$. Note that if $1 - \gamma < \min\left\{\beta, -\frac{2}{\mu}\left(\nu + \frac{p-q}{2} + \rho\right)\right\}$, then $\alpha_{\mathfrak{G}} = -\nu - \frac{p-q}{2} - \rho < \frac{\mu(1-\gamma)}{2} < \beta_{\mathfrak{G}}$. Also $2\left(\frac{1}{2} - \left(1 - \frac{\mu(1-\gamma)}{2}\right)\right) + \nu_{\mathfrak{G}} \leq \frac{1}{2} - \gamma(r)$ because $1 - \gamma > \alpha_1 \geq -\frac{1}{\mu}\left(\nu + \frac{p-q}{2} + \gamma(r) - \frac{1}{2}\right)$. Hence according to Theorem 6.3 [16] there exists $T_{\mathfrak{G}} \in \left[\mathcal{L}_{1-\frac{\mu}{2}(1-\gamma),r}, \mathcal{L}_{\frac{\mu}{2}(1-\gamma),s}\right]$ for every $s \geq r$ such that $s' \geq \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$. Moreover if either $1 < r \leq 2$ or γ is not in the exceptional set of $\mathfrak{H}$ (that implies $1 - \frac{\mu}{2}(1-\gamma)$ is not in the exceptional set of $\mathfrak{G}$, then $T_{\mathfrak{G}}$ is one to one and for each $1 < r \leq 2$ and $f \in \mathcal{L}_{1-\frac{\mu}{2}(1-\gamma),r}$

$$(21) \qquad (\mathcal{M}T_{\mathfrak{G}}f)(s) = \mathfrak{G}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = \frac{\mu}{2}(1-\gamma).$$

Furthermore $2(\frac{1}{2}- \alpha_{\mathfrak{G}})- \nu_{\mathfrak{G}} - 1 = \nu + \frac{p-q}{2} +2\rho - 1 > -1$ because $\rho > -\frac{1}{2}(\nu + \frac{p-q}{2} + \frac{1}{2} - \gamma(r)) \geq -\frac{1}{2}\left(\nu + \frac{p-q}{2}\right)$. Therefore if γ does not belong to the exceptional set of $\mathfrak{H}$, then

(22) $$T_{\mathfrak{G}}\left(\mathcal{L}_{1-\frac{\mu(1-\gamma)}{2},r}\right) = \left(M_{\frac{1}{2}\left(\nu+\frac{p-q}{2}+1\right)}h_{2,\nu+\frac{p-q}{2}+2\rho-1}\right)\left(\mathcal{L}_{1-\frac{\mu(1-\gamma)}{2}-\frac{1}{2}\left(\nu+\frac{p-q}{2}+1\right),r}\right)$$

we introduce the operator

$$\mathfrak{L}_1 = N_{\frac{2}{\mu}}\, T_{\mathfrak{G}}\, R\, N_{\frac{\mu}{2}}\, D_{|\frac{\mu}{2}|^{\mu}\eta}\, \mathfrak{L}\, R.$$

By taking into account the behavior of the operators N_a, D_a, R (Definition 2.2 [16]) $T_{\mathfrak{G}}$ and $\mathfrak{L}$ we conclude that $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$. Moreover from (20) and (21) we deduce that if $f \in \mathcal{L}_{\gamma,2}$

$$\begin{aligned}(\mathcal{M}\mathfrak{L}_1 f)(s) = {} & \mathcal{M}\left[T_{\mathfrak{G}}\, R\, N_{\frac{\mu}{2}}\, D_{|\frac{\mu}{2}|^{\mu}\eta}\, \mathfrak{L}\, R\, f\right](s\tfrac{\mu}{2})\,|\tfrac{\mu}{2}| = \\ = {} & \mathfrak{G}\left(s\tfrac{\mu}{2}\right)\ \mathcal{M}\ \left[N_{\frac{\mu}{2}}\, D_{|\frac{\mu}{2}|^{\mu}\eta}\, \mathfrak{L}\, R\, f\right](s\tfrac{\mu}{2})\,|\tfrac{\mu}{2}| = \\ = {} & \mathfrak{G}\left(s\tfrac{\mu}{2}\right)\ \mathcal{M}\ \left[D_{|\frac{\mu}{2}|^{\mu}\eta}\, \mathfrak{L}\, R\, f\right](s) = \\ = {} & \mathfrak{G}\left(s\tfrac{\mu}{2}\right)|\tfrac{\mu}{2}|^{-\mu s}\,\eta^{-s}\ \mathcal{M}\left[\mathfrak{L}\, R\, f\right](s) = \\ = {} & \mathfrak{H}(s)\mathcal{M}[f](1-s), \qquad \Re(s) = 1-\gamma.\end{aligned}$$

Then $(\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}Tf)(s)$, for $\Re(s) = 1-\gamma$ and $f \in \mathcal{L}_{\gamma,2}$. Hence $\mathfrak{L}_1 = T$ on $\mathcal{L}_{\gamma,2}$ and $\mathfrak{L}_1$ is the unique extension of T to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$. In the sequel we will denote such an extension $\mathfrak{L}_1$ also by T.

On the other hand if either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \leq 2$, then T is one to one because the operators $T_{\mathfrak{G}}$ and $\mathfrak{L}$ are one to one. Also if γ is not in the exceptional set of $\mathfrak{H}$, then (19) holds because (22) is true and $\mathfrak{L}(\mathcal{L}_{1-\gamma,r}) = \mathcal{L}_{1-\gamma,r}$ (this last one can be seen as the corresponding property in Proposition 4. When γ is in the exceptional set of $\mathfrak{H}$, $\mathfrak{L}(\mathcal{L}_{1-\gamma,r})$ is a subset of $\mathcal{L}_{1-\gamma,r}$ and hence $T(\mathcal{L}_{\gamma,r})$ is a subset of the set in the right hand side of (19).

The equality (12) and the representation formulas (13) and (14) can be proved as the corresponding ones in Proposition 3. □

Proposition 7. *Assume that $\xi = 0$, $\mu > 0$, $1 < r < \infty$ and $\alpha_1 < 1-\gamma < \beta_1$ where*

$$\alpha_1 = \max\left\{\alpha, -1 - \frac{2}{\mu}\left(\nu + \frac{p-q}{2} + \rho\right)\right\} \textit{ and}$$

$$\beta_1 = \min\left\{\beta, -\frac{1}{\mu}\left(\nu + \frac{p-q}{2} + \gamma(r) - \frac{1}{2}\right), \frac{1+2\rho}{\mu}\right\} \textit{with}$$

$$\rho > \max\left\{\frac{-\mu(1-\alpha)}{2}, \frac{1}{2}\left(\gamma(r) - \frac{1}{2} - \nu - \frac{p-q}{2}\right), \frac{\mu}{2}(1-\beta) - \nu - \mu - \frac{p-q}{2}\right\}$$

The transformation $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ *can be extended to* $\mathcal{L}_{\gamma,r}$ *as a member of* $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ *provided that* $s \geq r$ *being* $s' > \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$. *If either* γ *is not in the exceptional set of* $\mathfrak{H}$ *or* $1 < r \leq 2$, T *is one to one on* $\mathcal{L}_{\gamma,r}$. *Also if* γ *does not belong to the exceptional set of* $\mathfrak{H}$ *the equality*

$$T\left(\mathcal{L}_{\gamma,r}\right) = \left(M_{-1-\frac{2}{\mu}} N_{\frac{-2}{\mu}} M_{\frac{-1}{2}\left(\nu+\frac{p-q}{2}+\mu+1\right)} h_{-2,\nu+\frac{p-q}{2}+\mu+2\rho-1}\right)\left(\mathcal{L}_{\frac{1}{2}\left(\nu+\frac{p-q}{2}+\mu(1-\gamma)+1\right),r}\right) \tag{23}$$

while if γ *is in the exceptional set of* $\mathfrak{H}$ *then* $T\left(\mathcal{L}_{\gamma,r}\right)$ *is a subset of the set in the right hand side of* (23). *Equality* (12) *is satisfied when* $f, g \in \mathcal{L}_{\gamma,r}$, *with* $1 < r \leq 2$ *and* $r > \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$. *Moreover, if* $1 < r \leq 2$ *and* $r > \left[\frac{1}{2} - \mu(1-\gamma) - \nu - \frac{p-q}{2}\right]^{-1}$, *then* (13) *(respectively* (14)*) holds with* $\lambda > -\gamma$ *(respectively* $\lambda < -\gamma$*) for every* $f \in \mathcal{L}_{\gamma,r}$.

Proof. Since $1-\gamma < -\frac{1}{\mu}\left(\nu + \frac{p-q}{2} + \gamma(r) - \frac{1}{2}\right)$ and $\gamma(r) \geq \frac{1}{2}$, $\mu(1-\gamma) + \nu + \frac{p-q}{2} < 0$ and according to Proposition 3 $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ and for every $f \in \mathcal{L}_{\gamma,2}$

$$\left(\mathcal{M}Tf\right)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \Re(s) = 1-\gamma.$$

We define $L(s) = \mathfrak{H}(1-s)$, for $1-\beta < \Re(s) < 1-\alpha$. Note that L is also a function of (3) type. The parameters associated with L, which we will denote by prime, are related to the corresponding numbers for $\mathfrak{H}$ through: $\xi' = \xi$, $\mu' = -\mu$, $\eta' = \eta^{-1}$, $\nu' = \nu + p - q + \mu$, $p' = q$, $q' = p$, $m' = n$, $n' = m$, $\beta'_j = \alpha_j$, $b'_j = 1 - a_j - \alpha_j$ $(j = 1, \ldots, p)$, $\alpha'_j = \beta_j$, $a'_j = 1 - b_j - \beta_j$, $(j = 1, \ldots, q)$, $\alpha' = 1-\alpha$ and $\beta' = 1-\beta$. Also ϵ is in the exceptional set of L if, and only if, $1-\epsilon$ is in the exceptional set of $\mathfrak{H}$. Hence according to Proposition 6 for every $1 < r < \infty$ $\alpha'_1 < 1-\epsilon < \beta'_1$, where

$$\alpha'_1 = \max\left\{\alpha', -\frac{1}{\mu'}\left(\nu' + \frac{p'-q'}{2} + \gamma(r) - \frac{1}{2}\right), \frac{2\rho}{\mu'}\right\} \text{ and}$$

$$\beta'_1 = \min\left\{\beta', -\frac{2}{\mu'}\left(\nu' + \frac{p'-q'}{2} + \rho\right)\right\} \text{ with}$$

$$\rho > \max\left\{\frac{\mu'\beta'}{2}, \frac{1}{2}\left(\gamma(r) - \frac{1}{2} - \nu' - \frac{p'-q'}{2}\right), -\frac{\mu'\alpha'}{2} - \nu' - \frac{p'-q'}{2}\right\};$$ and $s \geq r$ being $s' > \left[\frac{1}{2} - \mu'(1-\epsilon) - \nu' - \frac{p'-q'}{2}\right]^{-1}$, there exists $\mathfrak{L} \in [\mathcal{L}_{\epsilon,r}, \mathcal{L}_{1-\epsilon,r}]$ such that if $f \in \mathcal{L}_{\epsilon,r}$ and $1 < r \leq 2$ then

$$\left(\mathcal{M}\mathfrak{L}f\right)(s) = L(s)(\mathcal{M}f)(1-s) \quad \Re(s) = 1-\epsilon. \tag{24}$$

Note that the conditions which we have just listed agree with our hypotheses when ϵ is replaced by $1-\gamma$.

Introduce now the operator $\mathfrak{L}_1 = R\ \mathfrak{L}\ R$. According to our hypotheses $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$. Moreover by (24) if $f \in \mathcal{L}_{\gamma,r}$ and $1 < r \leq 2$, then

$$(\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}\mathfrak{L}Rf)(1-s) = L(1-s)(\mathcal{M}Rf)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s),\ \Re(s) = 1-\gamma.$$

In particular $(\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}Tf)(s)$, for $f \in \mathcal{L}_{\gamma,2}$, and $\Re(s) = 1-\gamma$. Hence $\mathfrak{L}_1 f = Tf$, $f \in \mathcal{L}_{\gamma,2}$, and $\mathfrak{L}_1$ is the unique extension of T to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$. T also will denote in the sequel to such extension. The one – to – one property follows without difficulty from Proposition 6. Also by taking into account that γ is in the exceptional set of $\mathfrak{H}$ if, and only if, $1-\gamma$ is in the exceptional set of L and the relations $R\ N_a = M_{a-1}\ N_a\ R$, $R\ M_a = M_{-a}\ R$, $R\ h_{a,b} = h_{-a,b}\ R$, (19) leads to

$$\begin{aligned}
&T\left(\mathcal{L}_{\gamma,r}\right)\\
&= \left(RN_{\frac{2}{\mu'}} M_{\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right)} h_{2,\nu'+\frac{p'-q'}{2}+2\rho-1}\right)\left(\mathcal{L}_{1-\frac{\mu'\gamma}{2}-\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right),r}\right)\\
(25)\quad &= \left(M_{\frac{2}{\mu'}-1} N_{\frac{2}{\mu'}} M_{-\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right)} h_{-2,\nu'+\frac{p'-q'}{2}+2\rho-1}\right)\left(\mathcal{L}_{\frac{\mu'\gamma}{2}+\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right),r}\right)\\
&= \left(M_{-1-\frac{2}{\mu}} N_{-\frac{2}{\mu}} M_{-\frac{1}{2}\left(\nu+\frac{p-q}{2}+\mu+1\right)} h_{-2,\nu+\frac{p-q}{2}+\mu+2\rho-1}\right)\left(\mathcal{L}_{\frac{1}{2}\left(\nu+\frac{p-q}{2}+\mu(1-\gamma)+1\right),r}\right)
\end{aligned}$$

provided that γ is not in the exceptional set of $\mathfrak{H}$. Moreover if γ is in the exceptional set of $\mathfrak{H}$, then $T\left(\mathcal{L}_{\gamma,r}\right)$ is a subset of the last set in (25) because

$$\mathfrak{L}\left(\mathcal{L}_{1-\gamma,r}\right) \subseteq \left(N_{\frac{2}{\mu'}} M_{\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right)} h_{2,\nu'+\frac{p'-q'}{2}+2\rho-1}\right)\left(\mathcal{L}_{1-\frac{\mu'\gamma}{2}-\frac{1}{2}\left(\nu'+\frac{p'-q'}{2}+1\right),r}\right).$$

Equality (12) and representation formulas (13) and (14) can be established in the usual way. □

5. Boundedness and range of the $\mathcal{H}$–transformation on $\mathcal{L}_{\gamma,r}$ for $\xi > 0$. We established in Proposition 3 and Corollary 2 that if $\xi > 0$ and $\alpha < 1-\gamma < \beta$, then the transformation T defined by

$$(Tf)(x) = \int_0^\infty \mathcal{H}_{p,q}^{m,n}\left(\left.\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{smallmatrix}\right| xt\right) f(t)\,dt, \quad \text{for } f \in \mathcal{L}_{\gamma,2}$$

is in $[\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ and for each $f \in \mathcal{L}_{\gamma,2}$

$$(26)\qquad (\mathcal{M}Tf)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma.$$

In this section we prove that for every $1 < r < \infty$ and $\alpha < 1-\gamma < \beta$ this transformation T can be extended to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ when $\xi > 0$ and s is suitable. We also analyze the range $T(\mathcal{L}_{\gamma,r})$ of T on $\mathcal{L}_{\gamma,r}$. $T(\mathcal{L}_{\gamma,r})$ is described through the operators defined at the beginning of Section 4 and a modified Laplace transformation defined by

$$(L_{a,b}f)(x) = \int_0^\infty (xt)^{-|b|} e^{-|a|(xt)^{\frac{1}{a}}} f(t)dt, \quad a \neq 0 \text{ and } \; b \in R.$$

The behavior of $L_{a,b}$ on $\mathcal{L}_{\gamma,r}$ is investigated in Theorem 5.1,(d)[16].

Proposition 8. *Let $\xi > 0$, $\alpha < 1-\gamma < \beta$ and $1 \le r \le s \le \infty$. Then the transformation $T \in [\mathcal{L}_{\gamma,2}, \mathcal{L}_{1-\gamma,2}]$ can be extended to $\mathcal{L}_{\gamma,r}$ as a member of $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$. T is one to one provided that $1 \le r \le 2$. Moreover equality (12) holds for every $f \in \mathcal{L}_{\gamma,r}$ and $g \in \mathcal{L}_{\gamma,s'}$.*

Proof. Let $\alpha < 1-\gamma < \beta$. We choose ϵ_i, $i = 1, 2$, such that $\alpha < \epsilon_1 < 1-\gamma < \epsilon_2 < \beta$. According to (10) for every $1 \le l < \infty$ one has

$$\int_0^\infty \left| x^{1-\gamma}\mathcal{H}(x)\right|^l \frac{dx}{x} \le C\left\{\int_0^1 x^{(1-\gamma-\epsilon_1)l-1}dx \;+\; \int_1^\infty x^{(1-\gamma-\epsilon_2)l-1}dx\right\} < \infty$$

here C denotes a suitable positive constant. Hence $\mathcal{H} \in \mathcal{L}_{1-\gamma,l}$ and by virtue of Lemma 5.1(b)[16] the transformation T defined by (2) is in $[\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ when $1 \le r \le s \le \infty$. Moreover if $f \in \mathcal{L}_{\gamma,r}$ with $1 \le r \le 2$, then

$$(\mathcal{M}Tf)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \quad \Re(s) = 1-\gamma\ .$$

Since the zeros of $\mathfrak{H}(s)$ are isolated $Tf = 0$ implies that $(\mathcal{M}f)(s) = 0$ except, at most, when s is an isolated set. By invoking well-known properties of the Mellin transformation we obtain that $f = 0$ if and only if, $Tf = 0$. Then T is one to one provided that $1 \le r \le 2$.

To prove (12) for $f \in \mathcal{L}_{\gamma,r}$ and $g \in \mathcal{L}_{\gamma,s'}$ it is sufficient to apply Hölder's inequality and to take into account that (12) holds for each $f, g \in C_0$. $\square$

We now investigate the range $T(\mathcal{L}_{\gamma,r})$ of T on $\mathcal{L}_{\gamma,r}$ when $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. Our study is divided in five cases.

Proposition 9. *Assume that $\xi > 0$, $\mu_1 < 0$, $\mu_2 > 0$, $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. If either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \le 2$, then T is one to one. If $\omega = 1 + \mu_2\alpha + \mu_1\beta + \nu - \frac{1}{2}(q-p)$ and γ is not in the exceptional set, then*

$$T(\mathcal{L}_{\gamma,r}) = \left(L_{\mu_2,\alpha}\, L_{-\mu_1,1-\beta+\frac{\omega}{\mu_1}}\right)(\mathcal{L}_{1-\gamma,r}), \quad \textit{when } \; \omega \ge 0 \tag{27}$$

and

$$T(\mathcal{L}_{\gamma,r}) = \left(\mathcal{J}_{\frac{1}{\mu_2},-\omega,-\mu_2\alpha} L_{\mu_2,\alpha}\, L_{-\mu_1,1-\beta}\right)(\mathcal{L}_{1-\gamma,r}), \quad \textit{when } \; \omega < 0 \tag{28}$$

Moreover if γ is in the exceptional set of $\mathfrak{H}$ the range $T(\mathcal{L}_{\gamma,r})$ of T on $\mathcal{L}_{\gamma,r}$ is a subset of the set on the right hand side of (27) *and* (28).

Proof. Note that $m > 0$ because $\mu_2 > 0$ and $n > 0$ because $\mu_1 < 0$. Hence $\alpha > -\infty$ and $\beta < +\infty$.

Assume firstly $\omega = 1 + \mu_2\alpha + \mu_1\beta + \nu - \frac{1}{2}(q-p) \geq 0$. We introduce the function

$$L(s) = \mu_2^{\mu_2(s-\alpha)-1}(-\mu_1)^{\mu_1(s-\beta)+\omega-1}\eta^s \frac{\mathfrak{H}(s)}{\Gamma(\mu_2(s-\alpha))\,\Gamma(\mu_1(s-\beta)+\omega)}, \quad \alpha < \Re(s) < \beta.$$

By taking into account that $\xi = \mu_2 - \mu_1$ and $\mu = \mu_2 + \mu_1$, from Proposition 1 and the properties (6) and (7) for the Γ–function we deduce

$$(29) \qquad |L(\sigma + it)| \sim (2\pi)^{\delta-1}(-\mu_1\mu_2)^{-\frac{1}{2}} \prod_{j=1}^{p} \alpha_j^{-a_j+\frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j-\frac{1}{2}}, \quad \text{as} \quad |t| \to \infty$$

uniformly in σ when σ is in a bounded subset of R, and

$$\begin{aligned} &\tfrac{d}{dt}L(\sigma+it) = iL(\sigma+it)\Big\{\mu_2\log\mu_2 + \mu_1\log(-\mu_1) + \log\eta - \\ (30) \qquad &-\mu_2\Psi(\mu_2(\sigma+it-\alpha)) - -\mu_1\Psi(\mu_1(\sigma+it-\beta)+\omega) + \mu\log|t| - \log\eta + \\ &+i\tfrac{\pi}{2}\,\xi\,\operatorname{sgn} t + \frac{\nu + \mu\sigma + \frac{p-q}{2}}{it} + O\left(t^{-2}\right)\Big\} = O\left(t^{-2}\right), \quad \text{as} \quad |t| \to \infty. \end{aligned}$$

Therefore $L \in \mathcal{A}$ being $\alpha(L) = \alpha$ and $\beta(L) = \beta$. Then by virtue of Theorem 2.1 [16] for every $1 < r < \infty$ and $\alpha < \epsilon < \beta$ there exists a transformation $\mathfrak{L} \in [\mathcal{L}_{\epsilon,r}]$ such that for every $f \in \mathfrak{L}_{\epsilon,r}$, with $1 < r \leq 2$ and $\alpha < \epsilon < \beta$

$$(\mathcal{M}\mathfrak{L}f)(s) = L(s)(\mathcal{M}f)(s), \quad \Re(s) = \epsilon \ .$$

Moreover $\mathfrak{L}$ is one to one on $\mathcal{L}_{\epsilon,r}$ provided that $1 < r \leq 2$ and $\alpha < \epsilon < \beta$ because the zeroes of $L(s)$ are isolated. Also if $\dfrac{1}{L} \in \mathcal{A}$, then $\mathfrak{L}$ is one to one and onto $\mathfrak{L}_{\epsilon,r}$ with $1 < r < \infty$ and $\max\left(\alpha, \alpha\left(\dfrac{1}{L}\right)\right) < \epsilon < \min\left(\beta, \beta\left(\dfrac{1}{L}\right)\right)$. $L(s)$ has at most a finite numbers of zeros on the strip $\alpha < \Re(s) < \beta$. By $\{\sigma_i\}_{i=1}^{n}$ we denote the family of zeros of L on the strip $\alpha < \Re(s) < \beta$, where $\sigma_i < \sigma_{i+1}$, $i = 1, \ldots, n-1$. We understand $n = 0$ when L has no zeros on $\alpha < \Re(s) < \beta$. If γ is in the exceptional set of $\mathfrak{H}$, then $\sigma_i < 1 - \gamma < \sigma_{i+1}$, for some $i = 0, \ldots, n$, being $\sigma_0 = \alpha$ and $\sigma_{n+1} = \beta$. Moreover from (29) and (30) we infer that $\dfrac{1}{L} \in \mathcal{A}$ with $\alpha\left(\dfrac{1}{L}\right) = \sigma_i$ and $\beta\left(\dfrac{1}{L}\right) = \sigma_{i+1}$, for each

$i = 0, \ldots, n$. Hence T is one to one from $\mathfrak{L}_{1-\gamma,r}$ onto $\mathfrak{L}_{1-\gamma,r}$ provided that γ is not in the exceptional set of $\mathfrak{H}$.

Define now $\mathfrak{L}_1 = L_{\mu_2,\alpha} L_{-\mu_1,1-\beta+\frac{\omega}{\mu_1}} D_\eta \mathfrak{L} R$. Since $\mu_1 \left(1-\gamma-\beta+\dfrac{\omega}{\mu_1}\right) > 0$ and $\mu_2\,(1-\gamma-\alpha) > 0$, $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$ when $1 < r \le s < \infty$. Also $\mathfrak{L}_1$ is one to one. Moreover by virtue of Theorem 5.1 (d) [16] for every $f \in \mathcal{L}_{\gamma,2}$ one has

$$\begin{aligned}
(\mathcal{M}\mathfrak{L}_1 f)(s) &= \frac{\Gamma(\mu_2(s-\alpha))}{\mu_2^{\mu_2(s-\alpha)-1}} \left(\mathcal{M} L_{-\mu_1,1-\beta+\frac{\omega}{\mu_1}} D_\eta \mathfrak{L} R f\right)(1-s) \\
&= \frac{\Gamma(\mu_2(s-\alpha))}{\mu_2^{\mu_2(s-\alpha)-1}} \frac{\Gamma(\mu_1(s-\beta)+\omega)}{(-\mu_1)^{\mu_1(s-\beta)+\omega-1}} \mathcal{M}\left[D_\eta \mathfrak{L} R f\right](s) \qquad (31) \\
&= \frac{\Gamma(\mu_2(s-\alpha))}{\mu_2^{\mu_2(s-\alpha)-1}} \frac{\Gamma(\mu_1(s-\beta)+\omega)}{(-\mu_1)^{\mu_1(s-\beta)+\omega-1}} \eta^{-s} \mathcal{M}\left[\mathfrak{L} R f\right](s) \\
&= \mathfrak{H}(s)\mathcal{M}[f](1-s), \quad \Re(s) = 1-\gamma\,.
\end{aligned}$$

Hence from (26) and (32) we deduce that $\mathfrak{L}_1 f = Tf$, for every $f \in C_0$. Then since C_0 is a dense subset of $\mathcal{L}_{\gamma,r}$ and $\mathfrak{L}_1$ and $T \in [\mathcal{L}_{\gamma,r}, \mathcal{L}_{1-\gamma,s}]$, $\mathfrak{L}_1 = T$.

The operators R, D_η and $L_{a,b}$ are one to one. Therefore T is one to one when, and only when, $\mathfrak{L}$ is one to one. This holds either for $1 < r \le 2$ or for γ not belonging to the exceptional set of $\mathfrak{H}$. Moreover if γ is not in the exceptional set of $\mathfrak{H}$, then $\mathfrak{L}(\mathcal{L}_{1-\gamma,r}) = \mathcal{L}_{1-\gamma,r}$, for $1 < r < \infty$. Hence if γ does not belong to the exceptional set of $\mathfrak{H}$ (27) holds because $R(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{1-\gamma,r}$ and $D_\eta(\mathcal{L}_{\gamma,r}) = \mathcal{L}_{\gamma,r}$ for every $1 < r < \infty$. On the other hand if γ is in the exceptional set of $\mathfrak{H}$ and $1 < r < \infty$ $T(\mathcal{L}_{\gamma,r})$ is a subset of the set on the right hand side of (27) because $\mathfrak{L}(\mathcal{L}_{1-\gamma,r}) \subseteq \mathcal{L}_{1-\gamma,r}$.

Now let $\omega < 0$. To see (28) we consider the function

$$\Omega(s) = \mu_2^{\mu_2(s-\alpha)-1}(-\mu_1)^{\mu_1(s-\beta)-1}\eta^s \frac{\Gamma(\mu_2(s-\alpha)-\omega)\,\mathfrak{H}(s)}{\left[\Gamma(\mu_2(s-\alpha))\right]^2 \Gamma(\mu_1(s-\beta))}, \quad \alpha < \Re(s) < \beta,$$

and we proceed as in the previous case. Ω is in $\mathcal{A}$ with $\alpha(\Omega) = \alpha$ and $\beta(\Omega) = \beta$. Hence according to Theorem 2.1 [16] for every $1 < r < \infty$ and $\alpha < \epsilon < \beta$ there exists a transformation $\mathcal{W} \in [\mathcal{L}_{\epsilon,r}]$ such that for each $1 < r \le 2$ and $\alpha < \epsilon < \beta$

$$(\mathcal{M}\mathcal{W}f)(s) = \Omega(s)\,(\mathcal{M}f)(1-s), \quad \Re(s) = \epsilon. \qquad (32)$$

By virtue of Theorem 5.1 (b) and (d) [16] and by (32) the operator $\mathcal{W}_1$ defined by

$$\mathcal{W}_1 = \mathcal{J}_{\frac{1}{\mu_2}\,,\,-\omega\,,\,-\mu_2\alpha}\, L_{\mu_2,\alpha}\, R\, L_{\mu_1\ \beta}\, R\, D_\eta\, \mathcal{W}\, R \qquad (33)$$

coincides with T on $\mathcal{L}_{\gamma,r}$ provided that $1 < r < \infty$ and $\alpha < 1-\gamma < \beta$. The range $T(\mathcal{L}_{\gamma,r})$ of T on $\mathcal{L}_{\gamma,r}$ can be now analyzed by means of (33). □

Proposition 10. *Let $\xi > 0$, $\mu_1 = 0$, $\mu_2 > 0$, $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. Then T is one to one on $\mathcal{L}_{\gamma,r}$ provided that either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \le 2$. Moreover if $\omega = \mu_2\alpha + \frac{1}{2} + \nu - \frac{1}{2}(q-p)$ and γ is not in the exceptional set of $\mathfrak{H}$ then*

$$T(\mathcal{L}_{\gamma,r}) = L_{\mu_2,\alpha-\frac{\omega}{\mu_2}}(\mathcal{L}_{\gamma,r}), \quad \text{when } \omega \ge 0 \tag{34}$$

and

$$T(\mathcal{L}_{\gamma,r}) = \left(\mathcal{J}_{\frac{1}{\mu_2},-\omega,-\mu_2\alpha} L_{\mu_2,\alpha}\right)(\mathcal{L}_{\gamma,r}), \quad \text{when } \omega < 0 \tag{35}$$

while if γ is in the exceptional set of $\mathfrak{H}$ the range $T(\mathcal{L}_{\gamma,r})$ of T on $\mathcal{L}_{\gamma,r}$ is a subset of the set on the right hand side of (34) and (35).

Proof. Note that $\alpha < -\infty$ because $\mu_2 > 0$. Suppose firstly that $\omega \ge 0$ and define the function

$$L(s) = \mu_2^{\mu_2(s-\alpha)+\omega-1}\eta^s \frac{\mathfrak{H}(s)}{\Gamma(\mu_2(s-\alpha)+\omega)}, \quad \alpha < \Re(s) < \beta.$$

Proposition 1 and the properties (6) and (7) allow to prove that $L \in \mathcal{A}$ being $\alpha(L) = \alpha$ and $\beta(L) = \beta$. Hence, by virtue of Theorem 2.1 [16] for every $1 < r < \infty$ and $\alpha < \epsilon < \beta$ we can find a transformation $\mathfrak{L} \in [\mathcal{L}_{\epsilon,r}]$ such that for each $f \in \mathcal{L}_{\epsilon,r}$ with $1 < r \le 2$ and $\alpha < \epsilon < \beta$

$$(\mathcal{M}\mathfrak{L}f)(s) = \mathfrak{H}(s)(\mathcal{M}f)(s), \quad \Re(s) = \epsilon. \tag{36}$$

From (36) we infer that $\mathfrak{L}$ is one to one on $\mathcal{L}_{\epsilon,r}$ provided that $1 < r \le 2$ and $\alpha < \epsilon < \beta$ because the zeros of $\mathfrak{H}(s)$ on $\alpha < \Re(s) < \beta$ are isolated. Moreover if γ is not in the exceptional set of $\mathfrak{H}$, then $\sigma_1 < 1-\gamma < \sigma_2$ where σ_1 and σ_2 are two consecutive zeros of $L(s)$. Hence according to Theorem 2.1 [16] $\mathfrak{L}$ is one to one from $\mathcal{L}_{1-\gamma,r}$ onto itself when $1 < r < \infty$ and γ is not in the exceptional set of $\mathfrak{H}$.

We now introduce the operator $\mathfrak{L}_1 = L_{\mu_2,\alpha-\frac{\omega}{\mu_2}} R D_\eta \mathfrak{L} R$. By using Theorem 5.1 (d) [16] we conclude that $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}\ \mathcal{L}_{1-\gamma,s}]$, for $1 < r \le s < \infty$. Also for every $f \in \mathcal{L}_{\gamma,2}$ one has

$$\begin{aligned}(\mathcal{M}\mathfrak{L}_1 f)(s) &= \frac{\Gamma(\mu_2(s-\alpha)+\omega)}{\mu_2^{\mu_2(s-\alpha)+\omega-1}}(\mathcal{M}D_\eta\mathfrak{L}Rf)(1-s) \\ &= \frac{\Gamma(\mu_2(s-\alpha)+\omega)}{\mu_2^{\mu_2(s-\alpha)+\omega-1}}\eta^{-s}\mathcal{M}(\mathfrak{L}Rf)(s) = \mathfrak{H}(s)(\mathcal{M}f)(1-s), \qquad (37)\end{aligned}$$

$$\Re(s) = 1-\gamma\,.$$

From (26) and (37) it follows that $\mathfrak{L}_1 f = Tf$ for $f \in C_0$. Then since $\mathfrak{L}_1$ and $T \in [\mathcal{L}_{\gamma,r}\ \ \mathcal{L}_{\gamma,s}]$, for $1 < r \le s < \infty$, $\mathfrak{L}_1 = T$ on $\mathcal{L}_{\gamma,r}$.

Therefore if either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \le 2$, T is one to one because then T is a composition of one to one operators. Moreover

$$T\left(\mathcal{L}_{\gamma,r}\right) = \left(L_{\mu_2\,,\,\alpha-\frac{\omega}{\mu_2}}\ \ R\ \ D_\eta\ \ \mathfrak{L}\right)\left(\mathcal{L}_{1-\gamma,r}\right) \subseteq L_{\mu_2\,,\,\alpha-\frac{\omega}{\mu_2}}\left(\mathcal{L}_{\gamma,r}\right)$$

and the equality holds provided that γ does not belong to the exceptional set of $\mathfrak{H}$.

To prove (35) for $\omega < 0$ we must consider the function

$$\Omega(s) = \mu_2^{\mu_2(s-\alpha)-1}\eta^s \frac{\Gamma\left(\mu_2(s-\alpha)-\omega\right)\mathfrak{H}(s)}{\left[\Gamma\left(\mu_2(s-\alpha)\right)\right]^2}$$

instead of $L(s)$ and to proceed as in the proof of (34). □

Proposition 11. *Assume that $\xi > 0$, $\mu_1 < 0$, $\mu_2 = 0$, $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. Then T is one to one on $\mathcal{L}_{\gamma,r}$ when either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \le 2$. Moreover if $\omega = \mu_1\beta + \frac{1}{2} + \nu - \frac{1}{2}(q-p)$ and γ is not in the exceptional set of $\mathfrak{H}$, then*

$$T\left(\mathcal{L}_{\gamma,r}\right) = L_{\mu_1,\beta-\frac{\omega}{\mu_1}}\left(\mathcal{L}_{\gamma,r}\right), \quad \textit{when}\ \ \omega \ge 0 \tag{38}$$

and

$$T\left(\mathcal{L}_{\gamma,r}\right) = \left(\mathcal{J}_{-\frac{1}{\mu_1}\,,\,-\omega\,,\,-\mu_1(2-\beta)}\ M_2\ L_{\mu_1,\beta}\right)\left(\mathcal{L}_{\gamma,r}\right), \quad \textit{when}\ \ \omega < 0, \tag{39}$$

while if γ is in the exceptional set of $\mathfrak{H}$ the range $T\left(\mathcal{L}_{\gamma,r}\right)$ of T on $\mathcal{L}_{\gamma,r}$ is a subset of the set on the right hand side of (38) *and* (39).

P r o o f. To establish these assertions we can employ a procedure similar to the one used in the proof of Proposition 7. According to Proposition 10 the function $\mathfrak{H}(1-s)$ has associated a transformation $\mathfrak{L} \in [\mathcal{L}_{1-\gamma,r}\ \ \mathcal{L}_{\gamma,s}]$ provided that $1 < r \le s < \infty$ and $\alpha < 1-\gamma < \beta$. Mellin integral transformation leads to $T = R\ \mathfrak{L}\ R$ and by using Proposition 10 the proof can be completed. □

Proposition 12. *Let $\xi > 0$, $\mu_1 > 0$, $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. Then T is one to one on $\mathcal{L}_{\gamma,r}$ provided that either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \le 2$. If $\beta < \infty$, choose $\omega = \xi c - \frac{1}{2} - \nu + \frac{1}{2}(q-p) \ge \frac{1}{2} + 2\mu_1\beta$ with $c \ge -\alpha$ and then choose $b < -\beta + \frac{\omega}{\mu_1}$ and $b \le \alpha$. When γ is not in the exceptional set of $\mathfrak{H}$*

$$T\left(\mathcal{L}_{\gamma,r}\right) = \left(M_{\frac{1}{2}-\frac{\omega}{2\mu_1}}\ \ h_{2\mu_1\,,\,\omega-1-2\mu_1 b}\ \ L_{-\xi\,,\,c+\frac{1}{2}+\frac{\omega}{2\mu_1}}\right)\left(\mathcal{L}_{\frac{3}{2}-\gamma-\frac{\omega}{2\mu_1}\,,\,r}\right) \tag{40}$$

while if γ is in the exceptional set of $\mathfrak{H}$, $T\left(\mathcal{L}_{\gamma,r}\right)$ is a subset of the set in the right hand side of (40). If $\beta = \infty$, choose $\omega = \xi c - \frac{1}{2} - \nu + \frac{1}{2}(q-p) \ge \frac{1}{2} + 2\mu_1(1-\gamma)$ with $c \ge -\alpha$ and $b \le \alpha$ being $\omega > \mu_1(1-\gamma+b)$. Then (40) *holds provided that γ is not in the* *exceptional set of $\mathfrak{H}$, while if γ belongs to the exceptional set of $\mathfrak{H}$, then $T(\mathcal{L}_{\gamma,r})$ is a subset of the set on the right hand side of* (40).

P r o o f. Suppose firstly that $\beta < \infty$ and define

$$L(s) = \mu_1^{2\mu_1 s - \omega} \xi^{\xi(s+c)-1} \eta^s \frac{\Gamma(\omega - \mu_1(b+s))\, \mathfrak{H}(s)}{\Gamma(\mu_1(s-b))\, \Gamma(\xi(c+s))},$$

where $b < -\beta + \frac{\omega}{\mu_1}$ with $\omega = \xi c - \frac{1}{2} - \nu + \frac{1}{2}(q-p)$ and $c \geq -\alpha$. It is clear that $\Gamma(\omega - \mu_1(b+s))$ is a holomorphic function on $\Re(s) > -b + \frac{\omega}{\mu_1}$. Hence, since $-b + \frac{\omega}{\mu_1} > \beta$ $L(s)$ is holomorphic on the strip $\alpha < \Re(s) < \beta$. Moreover by invoking again Proposition 1 and by (6) and (7) we obtain that

$$|L(\sigma + it)| \sim (2\pi)^{\delta - \frac{1}{2}} \xi^{-\frac{1}{2}} \prod_{j=1}^{p} \alpha_j^{-a_j + \frac{1}{2}} \prod_{j=1}^{q} \beta_j^{b_j - \frac{1}{2}}, \quad \text{as } |t| \to \infty$$

uniformly in σ when σ is in a bounded subset of R; and

$$\frac{d}{dt} L(\sigma + it) = L(\sigma + it) \{\log \eta + 2\mu_1 \log \mu_1 + \xi \log \xi - \mu_1 [\Psi(\omega - \mu_1(\sigma + it + b)) +$$

$$+ \Psi(\mu_1(\sigma + it - b))] - \xi \Psi(\xi(c+s)) + \mu \log |t| - \log \eta + i\tfrac{\pi}{2}\, \xi \, \log |t| +$$

$$\left. + \frac{\nu + \mu\sigma + \frac{p-q}{2}}{it} + O\left(t^{-2}\right) \right\} = O\left(t^{-2}\right), \quad \text{as } |t| \to \infty.$$

for every $\alpha < \sigma < \beta$. Therefore $L \in \mathcal{A}$ with $\alpha(L) = \alpha$ and $\beta(L) = \beta$.

According to Theorem 2.1 [16] there exists a transformation $\mathfrak{L} \in [\mathcal{L}_{\epsilon,r}]$ when $1 < r < \infty$ and $\alpha < \epsilon < \beta$, such that if $1 < r \leq 2$ and $\alpha < \epsilon < \beta$, for every $f \in \mathcal{L}_{\gamma,r}$

$$(\mathcal{M}\mathfrak{L}f)(s) = L(s)(\mathcal{M}f)(s), \quad \Re(s) = \epsilon \tag{41}$$

and $\mathfrak{L}$ is one to one.

Moreover if $L(\epsilon + it) \neq 0$, $t \in R$, then $\mathfrak{L}$ is one to one from $\mathcal{L}_{\epsilon,r}$ onto itself. Note that $L(1 - \gamma + it) \neq 0$, $t \in R$, provided that γ is not in the exceptional set of $\mathfrak{H}$.

We now introduce the operator

$\mathfrak{L}_1 = M_{\frac{1}{2} - \frac{\omega}{2\mu_1}}\; h_{2\mu_1,\, \omega - 1 - 2\mu_1 b}\; L_{-\xi,\, c + \frac{1}{2} + \frac{\omega}{2\mu_1}}\; M_{-\frac{1}{2} + \frac{\omega}{2\mu_1}}\; \mathfrak{L}\; D_\eta\; R$. Since $\alpha < 1 - \gamma < \beta$ and $1 < r < \infty$, $M_{-\frac{1}{2} + \frac{\omega}{2\mu_1}}\; \mathfrak{L}\; D_\eta\; R \in \left[\mathcal{L}_{\gamma,r}\,,\, \mathcal{L}_{\frac{3}{2} - \gamma - \frac{\omega}{2\mu_1},\, r}\right]$. Also by the Theorem 5.1 (d) [16] $L_{-\xi,\, c + \frac{1}{2} + \frac{\omega}{2\mu_1}} \in \left[\mathcal{L}_{\frac{3}{2} - \gamma - \frac{\omega}{2\mu_1},\, r}\, \mathcal{L}_{\gamma - \frac{1}{2} + \frac{\omega}{2\mu_1},\, r}\right]$ for every $1 < r < \infty$ and $\alpha < 1 - \gamma < \beta$ because

$$-\xi\left(\gamma - \frac{1}{2} + \frac{\omega}{2\mu_1} - c - \frac{1}{2} - \frac{\omega}{2\mu_1}\right) = \xi(c + 1 - \gamma) \geq \xi(-\alpha + 1 - \gamma) > 0.$$

Moreover, since $\omega \geq \frac{1}{2} + 2\mu_1\beta$, $1-\gamma < \beta$ and $b \leq \alpha$ the inequalities $\gamma(r) \leq 2\mu_1(\gamma - 1) + \omega - \frac{1}{2} < \omega + \frac{1}{2} - 2\mu_1 b$ hold. Hence from Theorem 5.1 [16] we deduce that

$$h_{2\mu_1\,,\,\omega-1-2\mu_1 b} \in \left[\mathcal{L}_{\gamma-\frac{1}{2}+\frac{\omega}{2\mu_1}\,,\,r}\ ,\ \mathcal{L}_{\frac{3}{2}-\gamma-\frac{\omega}{2\mu_1}\,,\,s}\right]$$

provided that $1 < r \leq s < \infty$ and $s' \geq \left(2\mu_1(\gamma-1)+\omega-\frac{1}{2}\right)^{-1}$. Therefore $\mathfrak{L}_1 \in [\mathcal{L}_{\gamma,r}\ ,\ \mathcal{L}_{1-\gamma,s}]$ under the imposed conditions.

By taking into account the behavior of the Mellin transformation on the operators that appear in the definition of $\mathfrak{L}_1$ (Theorem 5.1 (d) [16]) and by (41) we get for every $f \in \mathcal{L}_{\gamma,r}$ with $1 < r \leq 2$

$$\begin{aligned}
&(\mathcal{M}\mathfrak{L}_1 f)(s) = \left(\mathcal{M}h_{2\mu_1,\omega-1-2\mu_1 b}L_{-\xi,c+\frac{1}{2}+\frac{\omega}{2\mu_1}}M_{-\frac{1}{2}+\frac{\omega}{2\mu_1}}\mathfrak{L}D_\eta R f\right)\left(s+\tfrac{1}{2}-\tfrac{\omega}{2\mu_1}\right) = \\
&= \mu_1^{-2\mu_1 s+\omega}\frac{\Gamma(\mu_1(s-b))}{\Gamma(\omega-\mu_1(b+s))}\left(\mathcal{M}L_{-\xi,c+\frac{1}{2}+\frac{\omega}{2\mu_1}}M_{-\frac{1}{2}+\frac{\omega}{2\mu_1}}\mathfrak{L}D_\eta Rf\right)\left(-s+\frac{1}{2}+\frac{\omega}{2\mu_1}\right) = \\
&= \mu_1^{-2\mu_1 s+\omega}\frac{\Gamma(\mu_1(s-b))\,\Gamma(\xi(s+c))}{\Gamma(\omega-\mu_1(b+s))\,\xi^{-\xi(-s-c)-1}}\left(\mathcal{M}M_{-\frac{1}{2}+\frac{\omega}{2\mu_1}}\mathfrak{L}D_\eta Rf\right)\left(s+\frac{1}{2}-\frac{\omega}{2\mu_1}\right) = \\
&= \mu_1^{-2\mu_1 s+\omega}\xi^{-\xi(s+c)+1}\frac{\Gamma(\mu_1(s-b))\,\Gamma(\xi(s+c))}{\Gamma(\omega-\mu_1(b+s))}(\mathcal{M}\mathfrak{L}D_\eta Rf)(s) = \\
&= \mu_1^{-2\mu_1 s+\omega}\xi^{-\xi(s+c)+1}\frac{\Gamma(\mu_1(s-b))\,\Gamma(\xi(s+c))}{\Gamma(\omega-\mu_1(b+s))}\eta^{-s}L(s)\,(\mathcal{M}f)(1-s) = \\
&= \mathfrak{H}(s)\,(\mathcal{M}f)(1-s)\,, \quad \Re(s) = 1-\gamma.
\end{aligned}$$

Hence in particular for each $f \in C_0$ $\quad (\mathcal{M}\mathfrak{L}_1 f)(s) = (\mathcal{M}Tf)(s)$, $\Re(s) = 1-\gamma$, and then for every $f \in C_0$ $\mathfrak{L}_1 = Tf$. Since $\mathfrak{L}_1$ and $T \in [\mathcal{L}_{\gamma,r}\ \ \mathcal{L}_{1-\gamma,s}]$ and C_0 is dense on $\mathcal{L}_{\gamma,r}$, $T = \mathfrak{L}_1$ on $\mathcal{L}_{\gamma,r}$.

Remainder of the proof for $\beta < +\infty$ follows as in the previous Propositions. When $\beta = +\infty$ the results can be proved in a similar way. □

Proposition 13. *Assume that $\xi > 0$, $\mu_2 < 0$, $\alpha < 1-\gamma < \beta$ and $1 < r < \infty$. Then T is one to one on $\mathcal{L}_{\gamma,r}$ when either γ is not in the exceptional set of $\mathfrak{H}$ or $1 < r \leq 2$. For $\alpha > -\infty$, let $\omega = \xi c - \frac{1}{2} - \nu - \mu - \frac{1}{2}(p-q) \geq \frac{1}{2} - 2\mu_2(1-\alpha)$ with $c \geq \beta - 1$ and let $b < \alpha - 1 - \frac{\omega}{\mu_2}$ and $b \leq 1-\beta$. Then when γ is not in the exceptional set of $\mathfrak{H}$*

$$T(\mathcal{L}_{\gamma,r}) = \left(M_{-\frac{1}{2}-\frac{\omega}{2\mu_2}}\ \ h_{2\mu_2\,,\,\omega-1+2\mu_2 b}\ \ L_{\xi\,,\,-c+\frac{1}{2}+\frac{\omega}{2\mu_2}}\right)\left(\mathcal{L}_{-\frac{1}{2}-\gamma-\frac{\omega}{2\mu_2}\,,\,r}\right) \tag{42}$$

while if γ is in the exceptional set of $\mathfrak{H}$, $T(\mathcal{L}_{\gamma,r})$ is a subset of the set on the right hand side of (42). For $\alpha = -\infty$, $\omega = \xi c - \frac{1}{2} - \nu - \mu - \frac{1}{2}(p-q) \geq \frac{1}{2} - 2\mu_2\gamma$ with $c \geq \beta - 1$ and

$b \leq 1-\beta$ *being* $\omega > -\mu_2(\gamma+b)$, (42) *holds provided that* γ *is not in the exceptional set of* $\mathfrak{H}$*, while if* γ *belongs to the exceptional set of* $\mathfrak{H}$*, then* $T\left(\mathcal{L}_{\gamma,r}\right)$ *is a subset of the set on the right hand side of* (42).

Proof. Our assertions in this proposition can be inferred from Proposition 12 by studying the operator $R\,T\,R$ as in the proof of Proposition 7. □

6. Some special transformations Now we apply the theory developed in previous Sections to study some special integral transformations.

I.- E. Kratzel [7] introduced the integral transformation defined by:

$$\left(\mathbf{L}_\nu^{(n)} f\right)(x) = \int_0^\infty \lambda_\nu^{(n)}(xt) f(t)dt\ ,$$

where

$$\lambda_\nu^{(n)}(z) = \frac{(2\pi)^{\frac{(n-1)}{2}}\sqrt{n}\left(\dfrac{z}{n}\right)^{n\nu}}{\Gamma\left(\nu+1-\frac{1}{n}\right)} \int_1^\infty \left(t^n-1\right)^{\nu-\frac{1}{n}} e^{-zt}dt$$

$$z>0, \qquad n \in N-\{0\} \quad \text{and} \quad \nu > -1+\frac{1}{n}.$$

The $\mathbf{L}_\nu^{(n)}$–transformation reduces to the $\mathcal{K}_\nu-$ transformation([11] [12]) when $n=2$. The Mellin integral transform of $\lambda_\nu^{(n)}$ is (see [7])

$$\mathcal{M}\left[\lambda_\nu^{(n)}(z)\right](s) \ = \ \frac{(2\pi)^{\frac{(n-1)}{2}}}{n^{\frac{1}{2}+n\nu}} \ \frac{\Gamma(s+n\nu)\ \Gamma\left(\frac{s}{n}\right)}{\Gamma\left(\frac{s}{n}+\nu+1-\frac{1}{n}\right)} \ =$$

$$\frac{(2\pi)^{\frac{(n-1)}{2}}}{n^{\frac{1}{2}+n\nu}} \mathfrak{H}_{1\,,\,2}^{2\,,\,0}\left(\left.\begin{matrix}(\nu+1-\frac{1}{n}\,,\,\frac{1}{n})\\(n\nu\,,\,1)\,,\,(0\,,\,\frac{1}{n})\end{matrix}\right| s\right).$$

Hence, by virtue of Propositions 8 and 10 we immediately deduce the following

Corollary 3. *Let* $max\{-n\nu\ ,\ 0\} < 1-\gamma$ *and* $1 \leq r \leq s \leq \infty$*. Then the transformation* $\mathbf{L}_\nu^{(n)} \in [\mathcal{L}_{\gamma\,,\,r}\ ,\ \mathcal{L}_{1-\gamma\,,\,s}]$ *and for every* $f \in \mathcal{L}_{\gamma\,,\,r}$ *and* $g \in \mathcal{L}_{\gamma\,,\,s'}$ *one has*

$$\int_0^\infty \left(\mathbf{L}_\nu^{(n)}\ f\right)(x)g(x)dx = \int_0^\infty f(x)\left(\mathbf{L}_\nu^{(n)}\ g\right)(x)dx.$$

$\mathbf{L}_\nu^{(n)}$ *is one to one on* $\mathcal{L}_{\gamma\,,\,r}$ *when either* $1<r\leq 2$ *or* γ *is not in the exceptional set of* $\mathfrak{H}$*. Moreover if* γ *does not belong to the exceptional set of* $\mathfrak{H}$*, then*

$$\mathbf{L}_\nu^{(n)}\left(\mathcal{L}_{\gamma\,,\,r}\right) = L_{1\,,\,-n\nu+\nu+1-\frac{1}{n}}\ \left(\mathcal{L}_{\gamma\,,\,r}\right)\quad ,\quad for\ \ \nu \geq \frac{1}{n}\ ,$$

$$\mathbf{L}_\nu^{(n)}\left(\mathcal{L}_{\gamma\,,\,r}\right) = \mathcal{J}_{1\,,\,\nu(1-n)+1-\frac{1}{n}\,,\,0}\, L_{1\,,\,0}\left(\mathcal{L}_{\gamma\,,\,r}\right) \quad , \quad for\ \ 0 \le \nu < \frac{1}{n}\ ,$$

$$\mathbf{L}_\nu^{(n)}\left(\mathcal{L}_{\gamma\,,\,r}\right) = \mathcal{J}_{1\,,\,\nu+1-\frac{1}{n}\,,\,n\nu}\, L_{1\,,\,0}\left(\mathcal{L}_{\gamma\,,\,r}\right) \quad , \quad for\ \ 1-\frac{1}{n} < \nu < 0$$

while $\mathbf{L}_\nu^{(n)}\left(\mathcal{L}_{\gamma\,,\,r}\right)$ *is, in each case, a subset of the set on the right hand side of the last equalities when* γ *is in the exceptional set of* h.

II.- The generalized Hankel transformation defined by

$$\left(\mathbf{H}_{\lambda\,,\,\mu} f\right)(x) \;=\; \int_0^\infty \mathbf{k}_{\lambda\,,\,\mu}(xt) f(t)dt \quad , \tag{43}$$

where $\mathbf{k}_{\lambda\,,\,\mu}(z) \;=\; 2^{-\lambda}\; z^{\lambda+\frac{1}{2}}\; \mathbf{J}_\lambda^\mu\left(\frac{z^2}{4}\right)$ and $\mathbf{J}_\lambda^\mu$ denotes a generalized Bessel function usually called Wright function ([18]), was introduced by R. P. Agarwal [1]. Taking $\mu = 1$ in (43) we obtain

$$\left(\mathbf{H}_{\lambda\,,\,1} f\right)(x) \;=\; \left(h_\lambda f\right)(x) \;=\; \int_0^\infty \sqrt{xt}\mathbf{J}_\lambda(xt) f(t)dt$$

i.e. the well-known Hankel transformation. Here $\mathbf{J}_\lambda$ represents, as usual, the Bessel function of the first kind and order λ .The h_λ–transformation was investigated on $\mathcal{L}_{\gamma\,,\,r}$ spaces by P. G. Rooney [15].

The Mellin integral transformation of $\mathbf{k}_{\lambda\,,\,\mu}$ is given by (see [15])

$$\mathcal{M}\left[\mathbf{k}_{\lambda\,,\,\mu}(z)\right](s) \;=\; 2^{s\,-\,\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\left(s+\lambda+\frac{1}{2}\right)\right)}{\Gamma\left(1+\lambda-\frac{\mu}{2}\left(s+\lambda+\frac{1}{2}\right)\right)} \;=$$

$$= \;\mathfrak{H}_{0\,;\,2}^{1\,;\,0}\left(\begin{matrix} - & - \\ \left(\frac{1}{2}\left(\lambda+\frac{1}{2}\right)\ ,\ \frac{1}{2}\right) & \left(\frac{\mu}{2}(\lambda+1)-\lambda\ ,\ \frac{\mu}{2}\right)\end{matrix}\ \middle|\, s\right).$$

Therefore, by taking into account Definition 2 [16], Propositions 8 and 12 lead to.

Corollary 4. *Let* $0 < \mu < 1$ $\lambda > -\frac{1}{2}$, $-\frac{1}{2}\left(\lambda+\frac{1}{2}\right) < 1-\gamma$ *and* $1 < r \le s < \infty$. *Then the transformation* $\mathbf{H}_{\lambda\,,\,\mu} \in \left[\mathcal{L}_{\gamma\,,\,r}\ ,\ \mathcal{L}_{1-\gamma\,,\,s}\right]$ *and for every* $f \in \mathcal{L}_{\gamma\,,\,r}$ *and* $g \in \mathcal{L}_{\gamma\,,\,s'}$

$$\int_0^\infty \left(\mathbf{H}_{\lambda\,,\,\mu}\ f\right)(x) g(x)dx = \int_0^\infty f(x)\left(\mathbf{H}_{\lambda\,,\,\mu}\ g\right)(x)dx.$$

$\mathbf{H}_{\lambda\,,\,\mu}$ *is one to one on* $\mathcal{L}_{\gamma\,,\,r}$ *provided that either* $1 < r \le 2$ *or* γ *is not in the exceptional set of* $\mathfrak{H}$. *Moreover if* γ *does not belong to the exceptional set of* $\mathfrak{H}$ *by choosing* $c \ge \frac{1}{2}\left(\lambda+\frac{1}{2}\right)$,

$$\omega = \frac{1-\mu}{2}c + \frac{1}{2} - \frac{1}{2}\left(\lambda+\frac{1}{2}\right) - \frac{\mu}{2}(\lambda+1) + \lambda \ge \frac{1}{2} + \mu(1-\gamma)$$

and $b \leq -\frac{1}{2}\left(\lambda + \frac{1}{2}\right)$ *being* $\omega > \frac{\mu}{2}(1 - \gamma + b)$, *then*

$$\mathbf{H}_{\lambda\,,\,\mu}(\mathcal{L}_{\gamma\,,\,r}) = M_{\frac{1}{2}-\frac{\omega}{\mu}}\ h_{\mu\,,\,\omega-1-\mu b}\ L_{\frac{\mu-1}{2}\,,\,c+\frac{1}{2}+\frac{\omega}{\mu}}\ \left(\mathcal{L}_{\frac{3}{2}-\gamma-\frac{\omega}{\mu}\,,\,r}\right) \tag{44}$$

while if γ *is in the exceptional set of* $\mathfrak{H}$, $\mathbf{H}_{\lambda\,,\,\mu}(\mathcal{L}_{\gamma\,,\,r})$ *is a subset of the set on the right hand side of* (44).

Results presented in Corollary 4 improve the ones obtained in [2].

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