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ON QUASIDIAGONAL OPERATIONS

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Abstract. For any operator T on a Hilbert Space H, a distance function $Qd(T)$ is introduced and studied. The properties of another distance function $qd(T)$ known as modulus of quasidiagonality are also discussed. It is proved that if T_M is the compression of T to a subspace M of finite co-dimension in H, then

$$qd(T) = qd(T_M) \text{ and } Qd(T) = Qd(T_M).$$

It is also shown that the unitary equvalence in the calkin algebra preserves the values of qd and QD.

An operator T on a Hilbert Space H is said to be quasidiagonal if there exists an increasing sequence $\{P_n\}_n^\infty = 1$ of finite rank orthogonal projections such that $P_n \to I$, the identity operator, strongly and $||TP_n - P_nT|| \to 0$, as $n \to \infty$. The notion of quasidiagonality was introduced by P.R.Halmos [5] in 1970. D.A.Herrero [6] defined the notion of modulus of quasidiagonality $qd(T)$ of any operator T on H as

$$qd(T) = \underline{\operatorname{Lim}}_{\substack{P \in \mathcal{P}(H) \\ P \to I}} ||TP - PT||$$

where $\mathcal{P}(H)$ denotes the directed set of all finite rank (orthogonal) projections on H under the usual ordering. From [5, 902], it follows that T is quasidiagonal if and only if $qd(T) = 0$. the purpose of the present paper is to introduce and study a new distance function Qd and also to discuss the notion of modulus of quasidiagonality.

Troughout the paper H denotes an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(H)$, the set of all bounded linear operators on H. $K(H)$ denotes the ideal of compact operators on H and π is the natural mapping of $\mathcal{B}(H)$ onto quotient algebra $\mathcal{B}(H)/_{K(H)}$. The class of all quasidiagonal operators in $\mathcal{B}(H)$ is denoted by $[QD]$.

For any operator T on H we introduce (see also [4]) the following

$$Qd(T) = \overline{\operatorname{Lim}}_{\substack{P \in \mathcal{P}(H) \\ P \to I}} ||TP - PT||$$

$$= \operatorname{Inf}_{N \in \tau(H)} \operatorname{Sup}_{\substack{M \in \tau(H) \\ M \supset N}} ||TP_M - P_M T||,$$

$$Qd(T) = \operatorname{Lim} \ ||TP - PT|| \ \ P \in P(H) \ \ P \to I$$

$$= \operatorname{Inf} \ \ \operatorname{Sup} \ ||TP_M - P_M T||, N \in \tau(H) M \in \tau(H) M \supset N$$

where P_M denotes the projection on the closed linear subspace M of H and $\tau(H)$ is the collection of all finite - dimensional closed linear subspaces of H. We also define

$$d(T) = \operatorname{Inf}_{S \in [QD]} \|T - S\|.$$

One can easily vertify that the map $T \to QD(T)$ is continuous and QD is a semi-norm. In [2, Corollary 2.2] it is found that T is thin (an operator T is said to be thin if it is of the form $\lambda I + K$, for a scalar λ and for a compact operator K on H) if and only if $Qd(T) = 0$, and $Qd(T) = d(T, [T])$ [2, Theorem2.3], where $d(T, [T])$ denotes the distance of T to the c^*-algebra $[T]$ of all thin operators on H.

D.A.Herrero [6, Theorem 6.13] proved the following

Lemma A. *For any T in $\mathcal{B}(H), qd(T) = d(T)$. We make use of Lemma A to prove the following*

Theorem 1. *If M is a closed linear subspace of H with finite co-dimension, then for any operator T in $B(H)$,*

$$qd(T) = qd(T_M),$$

where T_M denotes the compression of T to M.

Proof. Let $\varepsilon > 0$. Then by definition, there exists an operator S in $[QD]$ such that

$$\|T - S\| < d(T) + \varepsilon.$$

Since dim $M^{\perp} < \infty$, [7, Theorem 4] implies that S_M also belongs to $[QD]$.

Therefore using Lemma A

$$qd(T_M) = d(T_M) \leq \|T_M - S_M\| \leq$$

$$\|T - S\| < d(T) + \varepsilon = qd(T) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get

$$qd(T_M) \leq qd(T).$$

To prove the reverse inequality, we consider any $\varepsilon > 0$. Then by definition, there exists a quasidiagonal operator R in $B(M)$ satisfying

$$\|T_M - R\| < d(T_M) + \varepsilon.$$

Since dim $M^{\perp} < \infty$ it can now be easily seen that $F \oplus 0$ is a quasidiagonal operator on H. Also

$$d(T_M \oplus 0) \leq \|T_M \oplus 0 - F \oplus 0)\|$$
$$= \|T_M - R\| \leq d(T_M) + \varepsilon.$$

Again, since $\dim M^{\perp} < \infty, qd(T_M \oplus 0) = qd(T)$ Hence by Lemma A, we get

$$qd(T) \leq qd(T_M).$$

The desired conclusion follows.

In [3], Douglas proved that if U is a non-unitary isomery in $B(H)$, then $U = S + K$, where S is a unilateral shift of suitable multiplicity and K is a compact operator. The following collorary is a slight extention of this result.

Collorary 2. *If U is a non-unitary isomery in $B(H)$ and M is a subspace of finite co-dimension in H, then $U_M = S + K$ where S is a unilateral shift in $\mathcal{B}(M)$, and K is a compact operator.*

Proof. From Theorem 1. we have

$$1 = qd(U) = qd(U_M) \leq \|U_M\| \leq 1.$$

Since $qd(U_M) = \|U_M\| = 1$, [4, Theorem 2] implies that $U_M = V + L$, where V is a non-unitary isometry and L is a compact operator. The above mentioned result of Douglas states that $V = S + J$, where S is a unilateral shift and J is compact. Since $S + J$ is compact, the proof is completed.

Theorem 3. *For any operator T on H, there exists a sequence $\{M_n\}_{n=1}^{\infty}$ in $\tau(H)$ of increasing subspaces of H such that*

(i) $$\{P_{M_n}\} \to I \quad \textit{strongly},$$

(ii) $$\|TP_{M_n} - P_{M_n}T\| \to Qd(T), \quad \textit{and}$$

(iii) $$Qd(T) = \sup\{\underline{\lim} \quad ||TP_{N_nN} - P_{N_n}T|| : \{N_n\} \subset \tau(H)$$

$$\text{and} \quad \{P_{N_n}\} \to I \quad \text{strongly} \quad \}.$$

P r o o f. Let N be any subspace of H in $\tau(H)$ and $\{N_n\}$ be a sequence of subspaces in $\tau(H)$, such that the corresponding sequence $\{P_{N_n}\}$ of projections converges to the identity operator strongly. Then by [1, Lemma 1.5], there exists a sequence $\{M_m\}$ of subspaces in $\tau(H)$ satisfying

(i) $$M_n \supset N \quad \text{for each } n,$$

(ii) $$\|P_{N_n} - P_{M_n}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Now

$$\sup_{\substack{M \in \tau(H) \\ M \supset N}} \|P_M T - T P_M\| \geq \underline{\text{Lim}}\, \|P_{M_n} T - T P_{M_n}\|$$

$$\geq \underline{\text{Lim}}\, \|P_{N_n} T - T P_{N_n}\|.$$

Consequently

$$Qd(T) = \inf_{N \in \tau(H)} \sup_{\substack{M \in \tau(H) \\ M \supset N}} \|P_M T - T P_M\|$$

$$\geq \underline{\lim} \|P_{N_n} T - T P_{N_n}\|.$$

As the sequence $\{N_n\}$ in $\tau(H)$ satisfying $\{P_{N_n}\} \to I$ strongly is arbitrary, we get

$$Qd(T) \geq \text{Sup} \;\{\underline{\lim} \;\; \|P_{N_n} T - T P_N\| : \{N_n\} \subset \tau(H) \;\; \text{and} \;\; \{P_{N_n}\} \to I \;\; \text{strongly}\}.$$

By the definition of $Qd(T)$, there exists a sequence $\{N_n\}$ of subspaces in $\tau(H)$ such that

$$\lim_{n \to \infty} \left\{ \sup_{\substack{M \in \tau(H) \\ M \sup N_n}} \|P_M T - T P_M\| \right\} = Qd(T).$$

Since $\displaystyle\sup_{\substack{M \in \tau(H) \\ M \supset N_n}} \|P_M t - T P_M\|$ decreases with the increase of the subspaces N_n, we may assume, without loss of generality, that

(i) $$N_n \subset N_{n+1} \quad \text{for each} \quad n,$$

(ii) $$\{P_{N_n}\} \to I \text{ strongly.}$$

Setting $N_1 = M_1$, we can determine by induction a sequence $\{M_n\}$ in $\tau(H)$ of increasing subspaces of H such that

$$M_{n+1} \supset N_{n+1} + M_n$$

and

$$\|P_{M_{n+1}}T - TP_{M_{n+1}}\| > \sup_{\substack{M \in \tau(H) \\ M \supset N_{n+1} + M_n}} ||P_M T - TP_M|| - \tfrac{1}{n}.$$

We also have

$$\|P_{M_{n+1}}T - TP_{M_{n+1}}\| \leq \sup_{\substack{M \in \tau(H) \\ M \supset M_n}} ||P_M T - TP_M||.$$

Therefore

$$\sup_{\substack{M \in \tau(H) \\ M \supset N_{n+1} + M_n}} ||P_M T - TP_M|| - \tfrac{1}{n} \leq \|P_{M_{n+1}}T - TP_{M_{n+1}}\| \leq$$

$$\leq \sup_{\substack{M \in \tau(H) \\ M \supset M_n}} ||P_M T - TP_M||$$

Passing to the limits, we get

$$Qd(T) \geq \lim_{n\to\infty} ||P_{M_{n+1}}T - TP_{Mn+1}|| \geq Qd(T).$$

Hence

$$Qd(T) = \lim_{n\to\infty} ||P_{M_{n+1}}T - TP_{Mn+1}||,$$

which implies

$$Qd(T) \leq \sup\{\underline{\lim}\ \|P_{N_n}T - TP_N\| : \{N_n\} \subset \tau(H) \text{ and } \{P_{N_n}\} \to I \text{ strongly}\}.$$

Corollary 4. *If M is a subspace of H with finite co-dimension, then for any operator T on H,*

$$Qd(T_M) = Qd(T).$$

P r o o f. By Theorem 3 (ii) there exists a sequence $\{P_n\} \in \mathcal{B}(H)$ of finite rank projections converging to the identity operator strongly and satisfying

$$Qd(T_M) \geq \lim_{n\to\infty} \|T_M P_n - P_n T_M\|.$$

With respect to the decomposition $H = M \oplus M^{\perp}$, write $Q_n = P_n \oplus I$; then $\{Q_n\}$ is a sequence of finite rank projections on H converging to the identity operator strongly and satisfying

$$\lim_{n\to\infty} ||(T_M \oplus 0)Q_n - Q_n(T_M \oplus 0)|| = \lim_{n\to\infty} ||(T_M P_n - P_n T_M|| = Qd(T_M).$$

Also

$$Qd(T_M \oplus 0) \geq \lim_{n\to\infty} ||(T_M \oplus 0)Q_n - Q_n(T_M \oplus 0)|| = Qd(T_M).$$

Since $\dim M^{\perp} < \infty$, we have $Qd(T) = Qd(T_M \oplus 0)$ and therefore

$$Qd(T_M) \leq Qd(T).$$

To prove the reverse inequality, let $\{P_n\}$ be a sequence of finite rank projections converging to the identity operator strongly, then by [1, Lemma 1.5], there exists a sequence $\{S_n\}$ of finite fank projections such that

$$\lim_{N\to\infty} ||P_n - S_n|| = 0$$

and for each $n, S_n \geq P$, where P is the projection onto $M^{\perp}$. Let $\{R_n\}$ be any sequence of finite rank projections in $\mathcal{B}(M)$ converging to the indentity operator strongly and $S_n = R_n \oplus I$. Then

$$\begin{aligned} &\|(T_M \oplus 0)P_n - P_n(T_M \oplus 0)\| \\ &= \|(T_M \oplus 0)(P_n - S_n + S_n) - (P_n - S_n + S_n)(T_M \oplus 0)\| \\ &\leq \|(T_M \oplus 0)S_n - S_n(T_M \oplus 0)\| + 2\|P_n - S_n\| \quad \|T_M \oplus 0)\|. \\ &= ||T_M R_n - R_n T_M|| + 2||P_n - S_n|| \quad ||T_M \oplus 0|| \end{aligned}$$

Therefore

$$\underline{\lim}\|(T_M \oplus 0)P_n - P_n(T_M \oplus 0)\| \leq \underline{\lim} \quad \|T_M R_n - R_n T_M\| \leq Qd(T_M),$$

according to Theorem 3. Since the sequence $\{P_n\}$ is arbitrary, Theorem 3 again implies

$$Qd(T_M \oplus 0) \leq Qd(T_M).$$

Since $\dim M^{\perp} < \infty, Qd(T_M \oplus 0) = Qd(T)$. Hence the result follows.

Remark. It is possible to give a shorter proof of the preceding corollary by using the distance formula [2, Theorem 2.3] which states that $Qd(t) = d(T, [T])$ and 2 matrix method.

Theorem 5. *For any two operators T and S. if $\pi(T)$ and $\pi(S)$, are equivalent elements of the algebra $\mathcal{B}(H)/_{K(H)}$, then*

$$qd(T) = qd(S) \quad \textit{and} \quad Qd(T) = Qd(S).$$

P r o o f. By the hypothesis there exists an operator U on H such that $\pi(U)$ is a n unitary element of $\mathcal{B}(H)/_{K(H)}$, satisfying $\pi(U)\pi(T) = \pi(S)\pi(U)$. Since $\pi(U)$ is invertible, it is a Fredholm operator and hence by [3, Theorem 3.1], U is a compact perturbation of a unitary operator , a non-unitary isometry or a non-unitary coisometry, according to whether index $(U) = 0$, index $(U) < 0$ or index $(U) > 0$ respectively.

Assume index $(U) < 0$. Let K be a compact operator on H such that $V = U+K$ is a non-unitary isometry. Let M be the range of V. Since the null space of V is $\{0\}$, therefore $\dim M^{\perp} < \infty$. Let P be the projection onto M, tjen $I - P$ is of finite rank and hence compact. Now $\pi(U)\pi(T) = \pi(S)\pi(U)$ implies $UT - SU = K_1$, where K_1 is compact. Now as V is isometry

$T = V^*VT = V^*(U + K)T = V^*UT + V^*KT = V^*(K_1 + SU) + V^*KT = V^*K_1 + V^*SU + V^*KT$.

On adding and subtracting V^*PSU and V^*PSK this gives

$$T = V^*PS(U + K) + V^*(I - P)SU - V^*PSK + V^*KT + V^*K_1$$

$$= V^*PSV + V^*(I - P)SU - V^*PSK + V^*KT + V^*K_1.$$

As $V^*(I - P)SU - V^*PSK + V^*KT + V^*K_1$ is a compact operator and V^*PSV is unitarily equivalent to S_M, we get

$$qd(T) = qd(V^*PSV) = qd(S_M).$$

Making use of Theorem 1. and Corollary 4, the desired conclusion follows. The case index $U \geq 0$ can be proved similarly by taking adjoints.

The following consequence can easily be obtained.

Corollary 6. *If (T) is a Fredholm operator on H and if $T = UP$ denotes the polar decomposition of $T\perp$ then*

$$qd(UP) = qd(PU) \quad \textit{and} \quad Qd(UP) = Qd(PU).$$

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