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PROBABILISTIC APPROACHES TO THE ROUNDING PROBLEM

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Abstract. Very often sums of proportions in reported sets of tables do not add to unity. It occurs so frequently, that if the proportions were to add to exactly 1, one begins to suspect the reporter of forcing the situation. In relation to this problem, fundamental work was done by Mosteller, Youtz and Zahn (1967) and Diaconis and Freedman (1979) who assessed the probability that a table of conventionally rounded proportions adds to 1, as well as by Balinski and Rachev (1992) who introduced some rules of rounding that can improve the conventional rule. Investigating and developing further the so-called K-stationary divisor rules of rounding, we compute, for several of these rules, the limiting probability that the rounded percentages add to 100%.

Introduction. In this paper, we deal with the problem of developing and comparing various rules—mainly probabilistic—of rounding percentages reported in statistical tables. Surprisingly enough, the rounded percentages rarely add to 100%.

The importance and frequency of this problem has led to significant interest and research within academia. Fundamental work was done, in 1967, by Mosteller, Youtz and Zahn who investigated how frequently the rounded percentages fail to add up correctly and what the distributions of sums of rounded percentages are for (1) an empirical set of data, (2) the multinomial distribution in small samples, (3) spacings between points dropped on an interval—the broken stick model—and (4) simulation for several categories. They found that the probability that the sum of rounded percentages adds to exactly 100% is certain for two categories, about three-fourths for three categories, about two-thirds for four categories, and about $\sqrt{6/n\pi}$ for a larger number n of categories.

In 1979, Diaconis and Freedman assessed the probability that a table of rounded percentages adds to 100%. Extending the work of Mosteller, Youtz and Zahn, they gave a mathematical treatment of this phenomenon when the table is drawn from a multinomial distribution or from a mixture of multinomial distributions. Their principal result concerned the Mosteller, Youtz and Zahn broken-stick model.

Balinski and Rachev (1992) continued the work of Diaconis and Freedman by introducing the so-called stationary rules and considering vectors and matrices under varying assumptions concerning the probabilistic structure of the data to be rounded.

In what follows we investigate a class of rounding rules, called divisor rules of rounding, and in particular, we study the K-stationary divisor rules of rounding.

We first describe the vector problem $(\vec{p}, h)$ and a variety of possible rounding rules, with which we treat a particular case of the vector problem where $\vec{p}$ is uniformly distributed on the simplex S_n.

We compute the limiting probability that the sum of the rounded percentages equals the rounding of the sum of the percentages for different rules of rounding and for various probabilistic models generating the data.

1. The vector problem of rounding: K-stationary divisor rules. We start by investigating the so-called K-stationary divisor rules of rounding percentages. Given a vector problem $(\vec{p} = (p_1, \ldots, p_n), h)$ and a K-stationary rule $\vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})$, we first try to evaluate the chance that $x_N^{(K)} := x_1^{(K)} + \ldots + x_n^{(K)} = h$ and then we find the particular rule which maximizes this chance.

This extends the works of Mosteller, Youtz and Zahn (1967), Diaconis and Freedman (1979), and Balinski and Rachev (1992), who assessed the probability that a table of rounded percentages add to 100%.

In Section 1.1 we define the vector problem of rounding and the K-stationary divisor rules of rounding.

In Section 1.2, we show that the maximum of $\lim_{t\to\infty} P[1 - \frac{\Delta}{t} \le x_N^{(K)} \le 1 + \frac{\Delta}{t}]$ for every $\Delta = 0, 1, 2, \ldots$ does not change if, instead of rounding the p_i's, $i = 1, \ldots, n$ with the best of 0-stationary divisor rule, we round them with the best of any other K-stationary divisor rule ($K \ge 1$).

In Section 1.3, we display several computer simulations to support our theoretical results.

1.1 Notation and Preliminaries on the Vector Problem of Rounding. A *vector problem* is a pair, $(\vec{p}, h)$, where $\vec{p} = (p_j)$, $j \in N = \{1, \ldots, n\}$ is a vector of real numbers and h is a real number such that $p_N := p_1 + \ldots + p_n = h$. Unless otherwise specified, we assume $p_j \ge 0$, $j \in N$ and set $p_N = 1$, which is not a restriction for h.

Given any positive real number t, a *rule* ρ_t of $1/t$*-rounding* assigns to each vector $\vec{p}$ a set $\{\vec{x} : \vec{x} = \rho_t(\vec{p})\} \subseteq \{\vec{x} = (x_j) : x_j = \frac{k_j}{t}, k_j \text{ integer}, j \in N\}$. For example, if $\vec{p} = \{0.32, 0.17, 0.25, 0.26\}$ and $t = 10$, then $\vec{x}$ may be $\{1, 0, 0, 0\}$ or $\{0.3, 0.2, 0.2, 0.3\}$ or $\{0.3, 0.2, 0.3, 0.3\}$ or $\{0.2, 0.3, 0.1, 0.4\}$, etc. In our present work, $k_j \in \{[tp_j] - 1, [tp_j], [tp_j] + 1\}$, $j \in N$, where $[tp_j]$ denotes the largest integer contained in tp_j. Since the rule of $\frac{1}{t}$-rounding does not depend on p_N, it is important, first to evaluate the change that the sum $x_N := x_1 + \ldots + x_n$ is exactly p_N, and then, find the rule that maximizes this chance.

A *divisor rule* $\rho_{t,d}$ of $\frac{1}{t}$*-rounding* assigns to each vector $\vec{p}$ a set of vectors $\{\vec{x}_d : \vec{x}_d = \rho_{t,d}(\vec{p})\} \subseteq \{\vec{x} : \vec{x} = \rho_t(\vec{p})\}$ defined by:

$$(1)\ (\vec{x}_d)_j := [p_j]_{t,d} := \begin{cases} (k+1)/t & \text{if } \{k+1/2 < tp_j \le k+1\} \text{ or } \{k+1/2 = tp_j,\ k \text{ odd}\}, \\ k/t & \text{if } \{k \le tp_j < k+1/2\} \quad \text{ or } \{k+1/2 = tp_j,\ k \text{ even}\}, \end{cases}$$

where for $k \in Z$, $d(k) = k + C \in [k, k+1]$ is said to be the divisor criterion.

In the literature of apportionment problems (see Balinski and Young (1982)), we find the following divisor criteria: for $k \in Z$, $Z = \{0, \pm 1, \pm 2, \ldots\}$

$$\begin{array}{lll} \text{Adams} & : & d(k) = k, \\ \text{Dean} & : & d(k) = k(k+1)/(k+1/2), \\ \text{Hill} & : & d(k) = \sqrt{k(k+1)}, \\ \text{Webster} & : & d(k) = k + 1/2, \\ \text{Jefferson} & : & d(k) = k+1. \end{array}$$

Mosteller, Youtz and Zahn (1967) were the first to discuss the conventional rule (for short, MYZ-rule) of $1/t$-rounding $\vec{x} = \rho_t(\vec{p})$ for the problem $(\vec{p}, 1)$. The conventional rule rounds p_j, $j \in N$, to the nearest k/t and, therefore, is the divisor rule with $d(k) = k + 1/2$, that is,

$$(2) \qquad x_j \equiv (\vec{x}_d)_j := [p_j]_{t,k+1/2} := \begin{cases} (k+1)/t & \text{if } \{k+1/2 < tp_j \le k+1\} \\ & \text{or } \{k+1/2 = tp_j, k \text{ odd}\}, \\ k/t & \text{if } \{k \le tp_j < k+1/2\} \\ & \text{or } \{k+1/2 = tp_j, k \text{ even}\}, \end{cases}$$

Mosteller, Youtz and Zahn computed the probability that $x_N := x_1 + \ldots + x_n = 1$ for several probability models generating $\vec{p}$ and found that the probability that $x_N = 1$ is 1 for $n = 2$, about 3/4 for $n = 3$, about 2/3 for $n = 4$ and about $\sqrt{6/\pi n}$ for $n > 4$. Their argument is persuasive and backed by extensive empirical evidence (rounding behavior of 565 tables in the National Halothane Study).

Diaconis and Freedman (1979) assessed the limit probability of $x_N = 1$. They showed that if $\vec{p}$ has an absolutely continuous distribution on the simplex S_n, n large, and $\vec{x}$ is obtained by (2), then, as $t \to \infty$, $P\{x_N = 1\}$ converges to the probability that $-1/2 \le V_1 + V_2 + \ldots + V_{n-1} \le 1/2$, where the V_j's are independent and uniformly distributed on $[-1/2, 1/2]$. In particular, as $t \to \infty$, $P\{x_N = 1\} \to \sqrt{\frac{6}{\pi(n-1)}} + O\left(\frac{1}{\sqrt{n^3}}\right)$.

Balinski and Rachev (1992) slightly extended the above theorem: They stated that if $\vec{p}$ has an absolutely continuous distribution on the simplex S_n, n large, and $\vec{x}'$ is obtained by (1) with $d(k) = k + C$, $k \in Z$, $C \in [0, 1]$, then as $t \to \infty$, $P\{x'_N = 1\}$ converges to the probability that $C - 1 \le V_1 + \ldots + V_{n-1} \le C$, where the V_j's are independent and uniformly distributed on $[-C, 1-C]$. Taking $C = 1/2$, the limit is maximized and, in this case, as $t \to \infty$, $P\{x'_N = 1\} \to \sqrt{\frac{6}{\pi(n-1)}} + O\left(\frac{1}{\sqrt{n^3}}\right)$.

A *K-stationary divisor rule* $\rho_t^{(K)}$ *of 1/t-rounding* assigns to each vector $\vec{p}$ a set $\{\vec{x}^{(K)} : \vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})\} \subseteq \{\vec{x} : \vec{x} = \rho_t(\vec{p})\}$ defined by (1) where, for $k \in Z$,

$$d(k) = \begin{cases} k + C_k, C_k \in [0,1] & \text{if } \ k < K, \\ k + C \quad C \in [0,1] & \text{if } \ k \geq K \end{cases} \tag{3}$$

Notice that $\{\vec{x}_d : \vec{x}_d = \rho_{t,d}(\vec{p})\} \subseteq \{\vec{x}^{(K)} : \vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})\}$ and, in fact $\{\vec{x}_d\} \equiv \{\vec{x}^{(0)}\}$.

Remark: The Mosteller, Youtz and Zahn divisor rule maximizes the lim $P\{x_N = 1\}$ as $t \to \infty$, and therefore it is to be the best among all 0-stationary divisor rules in that it maximizes the limit of $P\{x_N = 1\}$.

We, further on, study the K-stationary divisor rules with $K \geq 1$. Our primary objective is to enquire if the K-stationary divisor rules can or cannot lead to a better limiting probability of $x_N = 1$.

1.2. K-stationary divisor rules (K $\geq$ 1) for the Vector Problem of Rounding. We define the K-stationary divisor rule $\vec{x}^{(K)} := \rho_t^{(K)}(\vec{p})$ of $1/t$-rounding of a vector $\vec{p}$ by:

$$x_j^{(K)} := [p_j]_{t,d}^K := \begin{cases} k/t & \text{if } \ \{k \leq p_j t < d(k)\} \ \text{ or } \ \{p_j t = d(k), k \ \text{ even}\}, \\ (k+1)/t & \text{if } \ \{d(k) < p_j t \leq k+1\} \ \text{ or } \ \{p_j t = d(k), k \ \text{ odd}\}, \end{cases} \tag{4}$$

where, for $0 \leq k \leq K-1$, $d(k) = k + C_k$, $C_k \in [0,1]$ and, for $k \geq K$, $d(k) = k + C$, $C \in [0,1]$.

Theorem 1.1. *Suppose $\vec{p}$ is uniformly distributed on the simplex $S_n (n > 1)$ and $\vec{x}^{(K)}$ is obtained by a K-stationary divisor rule, $\vec{x}^{(K)} = \rho_t^K(\vec{p})$ (see (4)). Then*

$$\max \left\{ \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N^{(K)} \leq 1 + \frac{\Delta}{t}\right) : \vec{x}^{(K)} = \rho_t^{(K)}(\vec{p}) \right\}, \ \Delta = 0, 1, 2, \ldots$$

is attained for any K-stationary rule ($K \geq 0$) when $C = 1/2$ and C_k is any point in $[0,1]$ for every $0 \leq k \leq K-1$. Moreover, if $\vec{x} = \rho_t(\vec{p})$ (see (2)) then, $\forall \Delta = 0, 1, \ldots$

$$\max \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N^{(K)} \leq 1 + \frac{\Delta}{t}\right) = \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N \leq 1 + \frac{\Delta}{t}\right).$$

We will later sketch the proof of the above theorem for $K > 1$. Next, we look at the case $K = 1$ and prove theorem 1.2.

According to the definition of K-stationary divisor rule of $1/t$-rounding, a *1-stationary divisor rule* $\vec{x}^{(1)} = \rho_t^{(1)}(\vec{p})$ *of 1/t-rounding* of a vector $\vec{p}$ is defined by

$$x_j^{(1)} := [p_j]_t^1 := \begin{cases} k + 1/t, & \text{if } \ k \neq 0 \ \text{ and } \ k + C < p_j t \leq k+1, \\ k/t & \text{if } \ k \neq 0 \ \text{ and } \ k \leq p_j t < k + C, \\ 1/t & \text{if } \ C_0 < p_j t \leq 1 + C, \\ 0 & \text{if } \ 0 \leq p_j t \leq C_0, \end{cases} \tag{5}$$

where $C_0,\ C \in [0,1]$.

Theorem 1.2. *Suppose $\vec{p}$ is uniformly distributed on the simplex $S_n (n > 1)$. There is no 1-stationary rule $\vec{x}^{(1)} = \rho_t^{(1)}(\vec{p})$ (see (5)) of $1/t$-rounding that is "better" than the Mosteller, Youtz and Zahn rule $\vec{x} = \rho_t(\vec{p})$ (see (2)) in the sense that $\vec{x}^{(1)}$ cannot improve the limiting probability $P(x_N = 1)$ as $t \to \infty$. In fact,*

$$\max\left\{\lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \le x_N^{(1)} \le 1 + \frac{\Delta}{t}\right) : \vec{x}^{(1)} = \rho_t^{(1)}(\vec{p})\right\}$$

$$= \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \le x_N \le 1 + \frac{\Delta}{t}\right), \quad \Delta = 0, 1, 2, \dots, \ .$$

For proof of Theorem 1.2 we need the following two lemmas:

Lemma 1.3. *Let $m_1, m_2, \dots, m_{n-1}$ be positive integers whose sum is at most $t - n + 1$ for t and n fixed, and t large enough. Denote by $A_t(m_1, \dots, m_{n-1})$ the set*

$$A_t(m_1, \dots, m_{n-1}) = \left\{(p_1, \dots, p_{n-1}) : \frac{m_i}{t} \le p_i < \frac{m_i + 1}{t},\ i = 1, \dots, n-1\right\}$$

and let A_t be the union of these $A_t(m_1, \dots, m_{n-1})$ over all choices of $m_1, \dots, m_{n-1}$. Then:

(i) *The probability of A_t tends to 1, as $t \to \infty$.*

(ii) *Given $A_t(m_1, \dots, m_{n-1})$, the random variables $\tilde{V}_i := t(x_i^{(1)} - p_i)$ (rounding errors of a 1-stationary divisor rule), $i = 1, \dots, n-1$ are conditionally independent and uniformly distributed over the $(n-1)$-fold Cartesian product $\otimes_{n-1}[-C, 1-C]$.*

Proof.

(i) From the definition of A_t, we obtain as $t \to \infty$

$$P(A_t) = \frac{t}{t} \cdot \frac{t-1}{t} \cdot \frac{t-2}{t} \cdots \frac{t-n+2}{t} = \frac{t(t-1)\cdots(t-n+2)}{t^{n-1}} \longrightarrow 1.$$

(ii) The distribution of $(p_1, \dots, p_{n-1})$ is uniform over the region

$$\left\{x_i^{(1)} \ge 0,\ 1 \le i \le n-1,\ \sum_{i=1}^{n-1} x_i^{(1)} \le 1\right\}.$$

Also, the hypercube defining $A_t(m_1, \dots, m_{n-1})$ is wholly contained in this region. So given $A_t(m_1, \dots, m_{n-1})$, the first $n-1$ of p_i's are independent, each being uniformly distributed over $[m_i/t,\ (m_i + 1)/t]$ (over its edge of the hypercube). Next, we show that if p_i is uniformly distributed over $[m_i/t,\ (m_i+1)/t]$ then $\tilde{V}_i := t(x_i^{(1)} - p_i)$ is uniformly distributed over $[-C,\ 1-C]$. By the definition of 1-stationary divisor rule

$$x_i^{(1)} := [p_i]_t^1 = \begin{cases} (m_i+1)/t & \text{if } \ (m_i+C)/t < p_i \le (m_i+1)/t \\ & \text{or } \ \Big\{ p_i = (m_i+C)/t \ \text{ and } \ m_i \ \text{ is odd} \Big\}, \\ m_i/t & \text{if } \ m_i/t \le p_i < (m_i+C)/t \\ & \text{or } \ \Big\{ p_i = (m_i+C)/t \ \text{ and } \ m_i \ \text{ is even} \Big\}. \end{cases}$$

Therefore, for $0 < \tau \le 1-C$,

$$P(0 \le \tilde{V}_i < \tau) = P(0 \le t(x_i^{(1)} - p_i) < \tau) = P\left(0 \le t\left(\frac{m_i+1}{t} - p_i\right) < \tau\right)$$

$$= P\left(0 \le \frac{m_i+1}{t} - p_i < \frac{\tau}{t}\right) = P\left(\frac{m_i+1}{t} - \frac{\tau}{t} < p_i \le \frac{m_i+1}{t}\right) = \frac{\tau}{t} \cdot t = \tau.$$

Similarly, for $-C \le \tau < 0$,

$$P\left(\tau < \tilde{V}_i \le 0\right) = P\left(\tau < t\left(x_i^{(1)} - p_i\right) \le 0\right) = P\left(\frac{\tau}{t} < x_i^{(1)} - p_i \le 0\right)$$

$$= P\left(\frac{\tau}{t} < \frac{m_i}{t} - p_i \le 0\right) = P\left(\frac{m_i}{t} \le p_i < \frac{m_i}{t} - \frac{\tau}{t}\right) = -\frac{\tau}{t} \cdot t = -\tau.$$

Therefore, for any $\tau \ge 0$,

$$P\left(-C < \tilde{V}_i < \tau\right) = P\left(-C < \tilde{V}_i < 0\right) + P\left(0 \le \tilde{V}_i < \tau\right) = C + \tau$$

and, for any $\tau < 0$

$$P\left(-C < \tilde{V}_i < \tau\right) = P\left(-C < \tilde{V}_i < 0\right) - P\left(\tau < \tilde{V}_i \le 0\right) = C - (-\tau) = C + \tau.$$

Thus $\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_{n-1}$ are uniformly distributed over $\otimes_{n-1}[-C,\ 1-C]$ and, given $A_t(m_1, \dots, m_{n-1})$, are conditionally independent. □Further on, we denote by I the indicator function, i.e.,

$$I(S) = \begin{cases} 1 & \text{if } \ S \ \text{ is true}, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 1.4. (i) *For every real σ_2, the 1-stationary rule of $1/t$-rounding $[\bullet]_t^1$ gives $\left[\frac{\sigma_2}{t}\right]_t^1 = \frac{1}{t}[\sigma_2]_1^1$.*

(ii) *For every integer $\sigma_1 \ne 0$ and every real σ_2 the 1-stationary rule of $1/t$-rounding $[\bullet]_t^1$ gives the following:*

(a) *For* $C \leq C_0$*, assuming that* $\sigma_2 \neq -\sigma_1$, 1

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^1 = \frac{\sigma_1}{t} + \frac{1}{t}[\sigma_2]_1^1 - \frac{1}{t}I\{\sigma_1+\sigma_2 \in (C, C_0]\} + \frac{1}{t}I\{\sigma_2 \in (C, C_0)\}.$$

(b) *For* $C > C_0$*, assuming that* $\sigma_2 \neq -\sigma_1$, 1,

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^1 = \frac{\sigma_1}{t} + \frac{1}{t}[\sigma_2]_1^1 + \frac{1}{t}I\{\sigma_1+\sigma_2 \in (C_0, C]\} - \frac{1}{t}I\{\sigma_2 \in (C, C_0)\}.$$

Proof.
(i) From the definition of 1-stationary divisor rule of 1/t-rounding $[\bullet]_1^1$, we obtain

$$\left[\frac{\sigma_2}{t}\right]_t^1 = \begin{cases} (k+1)/t & \text{if } \ k \neq 0, \ k+C < \sigma_2 \leq k+1, \\ k/t & \text{if } \ k \neq 0, \ k \leq \sigma_2 \leq k+C, \\ 1/t & \text{if } \ C_0 < \sigma_2 \leq 1, \\ 0 & \text{if } \ 0 \leq \sigma_2 \leq C_0, \end{cases}$$

$$= \frac{1}{t}\begin{cases} k+1 & \text{if } \ k \neq 0, \ k+C < \sigma_2 \leq k+1, \\ k & \quad k \neq 0, \ k \leq \sigma_2 \leq k+C, \\ 1 & \text{if } \ C_0 < \sigma_2 \leq 1, \\ 0 & \text{if } \ 0 \leq \sigma_2 \leq C_0, \end{cases}$$

$$= \frac{1}{t}\,[\sigma_2]_1^1\,.$$

(ii) Let $\sigma_1 \in Z$ and $\sigma_1 \neq 0$. Then by the definition of 1-stationary divisor rule of 1/t-rounding $[\bullet]_t^1$, we obtain

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^1 = \begin{cases} (k+1)/t & \text{if } \ k \neq 0, \ k+C < \sigma_1+\sigma_2 \leq k+1, \\ k/t & \text{if } \ k \neq 0, \ k \leq \sigma_1+\sigma_2 \leq k+C, \\ 1/t & \text{if } \ C_0 \leq \sigma_1+\sigma_2 \leq 1, \\ 0 & \text{if } \ 0 \leq \sigma_1+\sigma_2 \leq C_0, \end{cases}$$

$$= \frac{\sigma_1}{t} + \frac{1}{t}\cdot\begin{cases} k+1-\sigma_1 & \text{if } \ k \neq 0,1, k-\sigma_1+C < \sigma_2 \leq k+1-\sigma_1, \\ k-\sigma_1 & \text{if } \ k \neq 0,1, \ k-\sigma_1 \leq \sigma_2 \leq k-\sigma_1+C, \\ 1-\sigma_1 & \text{if } \ C_0-\sigma_1 < \sigma_2 \leq 1-\sigma_1, \\ -\sigma_1 & \text{if } \ -\sigma_1 \leq \sigma_2 \leq C_0-\sigma_1, \end{cases}$$

$$= \frac{\sigma_1}{t} + \frac{1}{t}\cdot I\left\{\sigma_2 \notin [-\sigma_1, 1-\sigma_1]\right\}\cdot\begin{cases} k+1-\sigma_1 & \text{if } \ k-\sigma_1+C < \sigma_2 \leq k+1-\sigma_1, \\ k-\sigma_1 & \text{if } \ k-\sigma_1 \leq \sigma_2 \leq k-\sigma_1+C, \end{cases}$$

$$+\frac{1}{t}(1-\sigma_1)\cdot I\left\{\sigma_2 \in (C_0-\sigma_1, 1-\sigma_1]\right\}$$

$$+\frac{1}{t}(-\sigma_1)\cdot I\left\{\sigma_2\in[-\sigma_1,C_0-\sigma_1]\right\}$$
$$=\frac{\sigma_1}{t}+\frac{1}{t}\cdot A+\frac{1}{t}\cdot B+\frac{1}{t}\cdot C,$$

where

$$A:=I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\cdot\begin{cases}k-1-\sigma_1 & \text{if } \ k-\sigma_1+C<\sigma_2\le k+1-\sigma_1,\\ k-\sigma_1 & \text{if } \ k-\sigma_1\le\sigma_2\le k-\sigma_1+C,\end{cases}$$

$$B:=(1-\sigma_1)I\left\{\sigma_2\in(C_0-\sigma_2,1-\sigma_1]\right\}\quad\text{and}\quad C:=(-\sigma_1)I\left\{\sigma_2\in[-\sigma_1,C_0-\sigma_1]\right\}.$$

To evaluate A, recall that $\sigma_1\neq 0$, $\sigma_1\in Z$. Then we have:

$$A=I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\cdot\begin{cases}k+1-\sigma_1\text{if } k\neq\sigma_1 \text{ and } k-\sigma_1+C<\sigma_2\le k+1-\sigma_1,\\ k-\sigma_1 \qquad \text{if } k\neq\sigma_1 \text{ and } k-\sigma_1\le\sigma_2\le k-\sigma_1+C,\end{cases}$$

$$+I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\cdot\begin{cases}k+1-\sigma_1\text{if } k=\sigma_1 \text{ and } k-\sigma_1+C<\sigma_2\le k+1-\sigma_1,\\ k-\sigma_1 \qquad \text{if } k=\sigma_1 \text{ and } k-\sigma_1\le\sigma_2\le k-\sigma_1+C,\end{cases}$$

$$=A_1+A_2.$$

Now A_1 can be evaluated as follows:

$$A_1=I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\begin{cases}k+1-\sigma_1\text{if } k\neq\sigma_1 \text{ and } k-\sigma_1+C<\sigma_2\le k+1-\sigma_1,\\ k-\sigma_1 \qquad \text{if } k\neq\sigma_1 \text{ and } k-\sigma_1\le\sigma_2\le k-\sigma_1+C,\end{cases}$$

$$=[\sigma_2]_1^1\, I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\} I\left\{\sigma_2\notin(0,1)\right\}$$
$$=[\sigma_2]_1^1-[\sigma_2]_1^1\, I\left\{\sigma_2\notin(0,1)\cup[-\sigma_1,1-\sigma_1]\right\}$$
$$=[\sigma_2]_1^1-I\left\{\sigma_2\in(C_0,1)\right\}-(1-\sigma_1)I\left\{\sigma_2\in(-\sigma_1+C,1-\sigma_1]\right\}$$
$$-(-\sigma_1)I\left\{\sigma_2\in[-\sigma_1,\sigma_1+C]\right\}.$$

As for A_2, since $\sigma_1\neq 0$, we obtain

$$A_2=I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\cdot\begin{cases}k+1-\sigma_1\text{if } k=\sigma_1 \text{ and } k-\sigma_1+C<\sigma_2\le k+1-\sigma_1,\\ k-\sigma_1 \qquad \text{if } k=\sigma_1 \text{ and } k-\sigma_1\le\sigma_2\le k-\sigma_1+C,\end{cases}$$

$$=I\left\{\sigma_2\notin[-\sigma_1,1-\sigma_1]\right\}\cdot\begin{cases}1 & \text{if } \ C<\sigma_2\le 1,\\ 0 & \text{if } \ 0\le\sigma_2\le C,\end{cases}$$

$$=I\left\{\sigma_2\in(C,1]\right\},\quad\text{since}\quad\sigma_1\neq 0.$$

Summing the expressions for A_1 and A_2, we obtain the following for $A=A_1+A_2$:

$$A=[\sigma_2]_1^1-I\left\{\sigma_2\in(C_0,1)\right\}-(1-\sigma_1)I\left\{\sigma_2\in(-\sigma_1+C,1-\sigma_1]\right\}$$
$$-(-\sigma_1)I\left\{\sigma_2\in[-\sigma_1,-\sigma_1+C]\right\}+I\left\{\sigma_2\in(C,1]\right\}.$$

Therefore, taking into account the values of A, B and C we obtain:

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_1^1 = \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^1 - \frac{1}{t}(1-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C, 1-\sigma_1]\right\}$$

$$+\frac{1}{t}(1-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C_0, 1-\sigma_1]\right\} - \frac{1}{t}(-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1, -\sigma_1+C]\right\}$$

$$+\frac{1}{t}(-\sigma_1)I\left\{\sigma_2 \in [-\sigma_1, -\sigma_1+C_0]\right\} - \frac{1}{t}I\left\{\sigma_2 \in (C_0, 1)\right\} + \frac{1}{t}I\left\{\sigma_2 \in [C, 1]\right\}.$$

Therefore:

(a) For $C \leq C_0$, since $\sigma_2 \neq -\sigma_1, 1$

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^1 = \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^1 - \frac{1}{t}(1-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C, -\sigma_1+C_0]\right\}$$

$$+\frac{1}{t}(-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C, -\sigma_1+C_0]\right\} + \frac{1}{t}I\left\{\sigma_2 \in (C, C_0]\right\}$$

$$= \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^1 - \frac{1}{t}I\left\{\sigma_1+\sigma_2 \in (C, C_0]\right\} + \frac{1}{t}I\left\{\sigma_2 \in (C, C_0]\right\}.$$

(b) Similarly for $C > C_0$, we obtain

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^1 = \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^1 + \frac{1}{t}(1-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C_0, -\sigma_1+C]\right\}$$

$$-\frac{1}{t}(-\sigma_1)I\left\{\sigma_2 \in (-\sigma_1+C_0, -\sigma_1+C]\right\} - \frac{1}{t}I\left\{\sigma_2 \in (C_0, C]\right\}$$

$$= \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^1 + \frac{1}{t}I\left\{\sigma_1+\sigma_2 \in (C_0, C]\right\} - \frac{1}{t}I\left\{\sigma_2 \in (C_0, C]\right\}.$$

P r o o f of Theorem 1.2. By the definition of the 1-stationary divisor rule of $1/t$-rounding $[\bullet]_t^1$ and the corresponding "rounding errors" $\tilde{V}_i$, we obtain

$$x_i^{(1)} = [p_i]_t^1 = p_i + \frac{1}{t}\tilde{V}_i,\ i = 1, \ldots, n-1 \ \text{ and, therefore,}$$

$$p_n = 1 - \sum_{i=1}^{n-1} p_i = 1 - \sum_{i=1}^{n-1} x_i^{(1)} + \frac{1}{t}\sum_{i=1}^{n-1}\tilde{V}_i = \frac{1}{t}\left[\left(t - t\sum_{i=1}^{n-1} x_i^{(1)}\right) + \sum_{i=1}^{n-1}\tilde{V}_i\right].$$

Using Lemma 1.4, with $\sigma_1 = t - t\sum_{i=1}^{n-1} x_i^{(1)} \neq 0$ and $\sigma_2 = \sum_{i=1}^{n-1}\tilde{V}_i \neq 1, -\sigma_1$, we obtain

$$x_n^{(1)} = [p_n]_t^1 = 1 - \sum_{i=1}^{n-1} x_i^{(1)} + \frac{1}{t}\left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 + \frac{1}{t}R$$

$$= 1 - \sum_{i=1}^{n-1}\left(p_i + \frac{1}{t}\tilde{V}_i\right) + \frac{1}{t}\left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 + \frac{1}{t}R$$

$$= p_n + \frac{1}{t}\left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 + \frac{1}{t}\sum_{i=1}^{n-1}\tilde{V}_i + \frac{1}{t}R.$$

According to Lemma 1.4, if $C_0 \geq C$, the remainder $R = R_t$ in the above expression equals

$$R_t = I\{\sigma_1 + \sigma_2 \in (C, C_0]\} + I\{\sigma_2 \in (C, C_0]\}.$$

By Lemma 1.3, $\sigma_2 := \sum_{i=1}^{n-1}\tilde{V}_i$ is, conditionally on A_t, a continuous random variable, so without loss of generality we can assume that $\sigma_2 \notin \{-\sigma_1, 1\}$. Consequently, as $t \to \infty$, with probability 1,

$$R_t = -I\left\{t - t\left(\sum_{i=1}^{n-1}x_i^{(1)} - \frac{1}{t}\sum_{i=1}^{n-1}\tilde{V}_1\right) \in (C, C_0]\right\} + I\left\{\sum_{i=1}^{n-1}\tilde{V}_i \in (C, C_0]\right\}$$

$$= -I\left\{tp_n \in (C, C_0]\right\} + I\left\{\sum_{i=1}^{n-1}\tilde{V}_i \in (C, C_0]\right\} \longrightarrow I\left\{\sum_{i=1}^{n-1}V_i \in (C, C_0]\right\}$$

where, by Lemma 1.3, V_i's are i.i.d. uniforms on $[-C, 1-C]$.

Since $\frac{1}{t}\tilde{V}_i := x_i^{(1)} - p_i$ and, thus, $\frac{1}{t}\sum_{i=1}^{n-1}\tilde{V}_i = \sum_{i=1}^{n-1}x_i^{(1)} - \sum_{i=1}^{n-1}p_i$, we obtain $x_n^{(1)} = p_n + \frac{1}{t}\left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 - \sum_{i=1}^{n-1}x_i^{(1)} + \sum_{i=1}^{n-1}p_i + \frac{1}{t}R_t$. Since $x_N^{(1)} := \sum_{i=1}^{n}x_i^{(1)}$ and $\sum_{i=1}^{n}p_i = 1$, we, finally, conclude that $x_N^{(1)} = 1 + \frac{1}{t}\left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 + \frac{1}{t}R_t$, or else, $t\left\{x_N^{(1)} - 1\right\} = \left[\sum_{i=1}^{n-1}\tilde{V}_i\right]_1^1 + R_t$.

By virtue of Lemma 1.3 and since $\sigma_1 \neq 0$ means $\sum_{i=1}^{n-1}x_i^{(1)} \neq 1$, we conclude that

$$t\left\{x_N^{(1)} - 1\right\} I\left\{\sum_{i=1}^{n-1}x_i^{(1)} \neq 1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1}V_i\right]_1^1 + I\left\{\sum_{i=1}^{n-1}V_i \in (C, C_0]\right\}$$

and

$$t\left\{x_N^{(1)} - 1\right\} I\left\{\sum_{i=1}^{n-1}x_i^{(1)} = 1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1}V_i\right]_1^1$$

where V_i's are i.i.d. uniforms over $[-C, 1-C]$.

In particular, as $t \to \infty$,

$$\lim P\left(x_N^{(1)} = 1\right) = \lim P\left(x_N^{(1)} = 1, \sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right) + \lim P\left(x_N^{(1)} = 1, x_n^{(1)} = 0\right)$$

$$= \lim P\left(x_N^{(1)} = 1, \sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right).$$

Therefore,

$$P\left(x_N^{(1)} = 1\right) \longrightarrow P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 0, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right),$$

$$+P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = -1, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right),$$

where, obviously, $P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = -1, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right) = 0$

Hence,

$$P\left(x_N^{(1)} = 1\right) \underset{t\to\infty}{\longrightarrow} P\left(-1 + C < \sum_{i=1}^{n-1} V_i < C_0, \sum_{i=1}^{n-1} V_i \notin (C, C_0)\right)$$

$$= P\left(-1 + C < \sum_{i=1}^{n-1} V_i < C\right)$$

and, therefore, the limiting probability does not depend on C_0.

Similarly, if $C_0 < C$, with $\sigma_1 = t - t\sum_{i=1}^{n-1} x_i^{(1)} \neq 0$ and $\sigma_2 = \sum_{i=1}^{n-1} \tilde{V}_i$, we obtain

$$R_t = I\{\sigma_1 + \sigma_2 \in (C_0, C]\} - I\{\sigma_2 \in (C_0, C]\}$$

$$= I\{tp_n \in (C_0, C]\} - I\left\{\sum_{i=1}^{n-1} \tilde{V}_i \in (C_0, C]\right\} \underset{t\to\infty}{\longrightarrow} -I\left\{\sum_{i=1}^{n-1} V_i \in (C_0, C]\right\}.$$

Therefore,

$$t\left\{x_N^{(1)} - 1\right\} I\left\{\sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1} V_i\right]_1^1 - I\left\{\sum_{i=1}^{n-1} V_i \in (C_0, C]\right\}$$

and

$$t\left\{x_N^{(1)}-1\right\} I\left\{\sum_{i=1}^{n-1} x_i^{(1)}=1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1} V_i\right]_1^1,$$

where V_i's are i.i.d. uniforms on $[-C, 1-C]$.

In particular, as $t \to \infty$,

$$\lim P\left(x_N^{(1)}=1\right)=\lim P\left(x_N^{(1)}=1, \sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right)+\lim P\left(x_N^{(1)}=1, x_n^{(1)}=0\right)$$

$$=\lim P\left(x_N^{(1)}=1, \sum_{i=1}^{n-1} x_n^{(1)} \neq 1\right).$$

Hence, as $t \to \infty$,

$$\begin{aligned} Pr\left\{x_N^{(1)}=1\right\} \longrightarrow P & \left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1=1, \sum_{i=1}^{n-1} V_i \in (C_0, C]\right) \\ & +P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1=0, \sum_{i=1}^{n-1} V_i \notin (C_0, C]\right) \\ & =P\left(C_0<\sum_{i=1}^{n-1} V_i<C\right)+P\left(-1+C<\sum_{i=1}^{n-1} V_i<C_0\right) \\ & =P\left(-1+C<\sum_{i=1}^{n-1} V_i<C\right) \end{aligned}$$

and therefore, once again the limiting probability does not depend on C_0.

Next, we wish to find the optimal C that maximizes

$$\left\{P\left(-1+C<\sum_{i=1}^{n-1} V_i<C\right): 0 \leq C \leq 1,\ V_i' \text{s are i.i.d. uniforms on } [-C, 1-C]\right\}.$$

Define $U_i := V_i + C - \frac{1}{2}$. Then the above maximum becomes

$$\max_{C\in[0,1]}\left\{P\left[-1+C<\sum_{i=1}^{n-1}\left(U_i-C+\frac{1}{2}\right) \leq C\right],\right.$$

$$\left. U_i' \text{s are i.i.d. uniforms on } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\}$$

$$=\max_{C\in[0,1]}\left\{P\left[-1+C+(n-1)C-\frac{n-1}{2}<\sum_{i=1}^{n-1} U_i \leq C+(n-1)C-\frac{n-1}{2}\right],\right.$$

U_i's are i.i.d. uniforms on $\left.\left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$.

Since $\sum_{i=1}^{n-1} U_i$ has a symmetric distribution around zero, the optimal C is determined by the equation $-\left(-1 + C + (n-1)C - \frac{n-1}{2}\right) = C + (n-1)C - \frac{n-1}{2}$, which results in $C = \frac{1}{2}$.

Therefore, the limiting probability of $\left\{x_N^{(1)} = 1\right\}$, for a 1-stationary rule, $\vec{x}^{(1)} = \rho_t^{(1)}(\vec{p})$ attains its maximum for the rule with divisor points C_0 and C, where C_0 is any point on $[0,1]$ while $C = \frac{1}{2}$.

Thus, we have proven Theorem 1.2 for $\Delta = 0$. Next, we let $\Delta \in \{1, 2, \ldots\}$ be fixed and we consider the limit of $P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}\right)$, as $t \to \infty$. Assuming $C \leq C_0$, we have seen that

$$t\left\{x_N^{(1)} - 1\right\} I\left\{\sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1} V_i\right]_1^1 + I\left\{\sum_{i=1}^{n-1} V_i \in (C, C_0]\right\}$$

and

$$t\left\{x_N^{(1)} - 1\right\} I\left\{\sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right\} \xrightarrow{w} \left[\sum_{i=1}^{n-1} V_i\right]_1^1,$$

where V_i's are i.i.d. uniforms on $[-C, 1-C]$.

In particular, as $t \to \infty$,

$$\lim P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}\right) =$$

$$\lim P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}, \sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right) +$$

$$\lim P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}, \sum_{i=1}^{n-1} x_i^{(1)} = 1\right) =$$

$$\lim P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}, \sum_{i=1}^{n-1} x_i^{(1)} \neq 1\right).$$

Hence,

$$P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}\right) \underset{t\to\infty}{\longrightarrow} P(-\Delta \leq L_n \leq \Delta)$$

where

$$L_n := \left[\sum_{i=1}^{n-1} V_i\right]_1^1 + I\left\{\sum_{i=1}^{n-1} V_i \in (C, C_0]\right\}.$$

If $\Delta = 1$, then

$$P(-\Delta \le L_n \le \Delta) = Pr(-1 \le L_n \le 1) =$$

$$= P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = -1, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right) + P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 0, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right)$$
$$+P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = -1, \sum_{i=1}^{n-1} V_i \in (C, C_0]\right) + P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 1, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right)$$
$$+P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 0, \sum_{i=1}^{n-1} V_i \in (C, C_0]\right)$$
$$= P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = -1\right) + P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 0, \sum_{i=1}^{n-1} V_i \notin (C, C_0]\right)$$
$$+P\left(\left[\sum_{i=1}^{n-1} V_i\right]_1^1 = 1\right) + P\left(\sum_{i=1}^{n-1} V_i = 0, \sum_{i=1}^{n-1} V_i \in (C, C_0]\right)$$
$$= P\left(-2 + C < \sum_{i=1}^{n-1} V_i < -1 + C\right) + P\left(-1 + C < \sum_{i=1}^{n-1} V_i < C\right)$$
$$+P\left(C_0 < \sum_{i=1}^{n-1} V_i < 1 + C\right) + P\left(-1 + C < \sum_{i=1}^{n-1} V_i < C_0\right)$$
$$= P\left(-2 + C < \sum_{i=1}^{n-1} V_i < 1 + C\right).$$

Similarly, for $\Delta \ge 2$, we obtain

$$P\left(-1 + \frac{\Delta}{t} \le x_N^{(1)} \le 1 + \frac{\Delta}{t}\right) \underset{t\to\infty}{\longrightarrow} Pr(-\Delta \le L_n \le \Delta)$$

$$= P\left(-\Delta - 1 + C \le \sum_{i=1}^{n-1} V_i \le \Delta + C\right).$$

Therefore, $\forall \Delta \ge 1$, the limiting probability of $\left(1 - \frac{\Delta}{t} \le x_N^{(1)} \le 1 + \frac{\Delta}{t}\right)$, as $t \to \infty$, does not depend on C_0. Next, we wish to find the C which maximizes the limiting probability.

Recall the definition $U_i := V_i + C - \frac{1}{2}$. Then

$$\max\Big\{P\left(-\Delta - 1 + C \leq \sum_{i=1}^{n-1} V_i \leq \Delta + C\right) : C \in [0,1],$$

$$V_i^{\prime}\text{s are i.i.d. uniforms on } [-C, 1-C]\Big\}$$

$$= \max\Big\{P\left(-\Delta - 1 + C \leq \sum_{i=1}^{n-1}\left(U_i - C + \frac{1}{2}\right) \leq \Delta + C\right) : C \in [0,1],$$

$$U_i\text{'s are i.i.d. uniforms on } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\Big\}$$

$$= \max_{C\in[0,1]}\Big\{P\left(-\Delta - 1 + C + (n-1)(C - \frac{1}{2})\right) \leq \sum_{i=1}^{n-1} U_i \leq \Delta + C + (n-1)(C - \frac{1}{2}) :$$

U_i's are i.i.d. uniforms on $\left[-\frac{1}{2}, \frac{1}{2}\right]\Big\}$. Since $\sum_{i=1}^{n-1} U_i$ has a symmetric distribution, the optimal C in the above maximum is determined by the equation

$$-\Delta - 1 + C + (n-1)(C - \frac{1}{2}) = -\left[\Delta + C + (n-1)(C - \frac{1}{2})\right],$$

that is, $C = \frac{1}{2}$.

Therefore,

$$\max \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}\right)$$

$$= P\left(-\Delta - 1 + \frac{1}{2} \leq \sum_{i=1}^{n-1} V_i \leq \Delta + \frac{1}{2}\right) = P\left(-\Delta - \frac{1}{2} \leq \sum_{i=1}^{n-1} V_i \leq \Delta + \frac{1}{2}\right),$$

where V_i's are i.i.d. uniforms on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Hence, if $\vec{x}$ is obtained by the Mosteller-Youtz-Zahn divisor rule $\vec{x} = \rho_t(\vec{p})$,

$$\max \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N^{(1)} \leq 1 + \frac{\Delta}{t}\right) \equiv \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \leq x_N \leq 1 + \frac{\Delta}{t}\right),$$

$\forall \Delta = 0, 1, 2, \ldots$

Remark: For the Mosteller-Youtz-Zahn divisor rule of rounding and, thus, for the best of the 1-stationary divisor rules (with $C = 1/2$), as $t \to \infty$, we obtain

$$\lim P\left(1 - \frac{1}{t} \le x_N \le 1 + \frac{1}{t}\right) = P\left(-1 \le \left[\sum_{i=1}^{n-1} V_i\right]_1 \le 1\right)$$

$$= P\left(-2 + \frac{1}{2} \le \sum_{i=1}^{n-1} V_i \le 1 + \frac{1}{2}\right) \quad \left(V_i\text{'s are i.i.d. uniforms on } \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$$

$$= P\left(\frac{-\frac{3}{2}}{\sqrt{(n-1)\frac{1}{12}}} \le \frac{\sum_{i=1}^{n-1} V_i}{\sqrt{(n-1)\frac{1}{12}}} \le \frac{\frac{3}{2}}{\sqrt{(n-1)\frac{1}{12}}}\right)$$

$$= P\left(\frac{-\sqrt{27}}{\sqrt{n-1}} \le Z \le \frac{\sqrt{27}}{\sqrt{n-1}}\right) \quad (Z \text{ is standard normal})$$

$$= \int_{\frac{-\sqrt{27}}{\sqrt{n-1}}}^{\frac{\sqrt{27}}{\sqrt{n-1}}} \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx \simeq \sqrt{\frac{54}{\pi(n-1)}} + O\left(n^{-\frac{3}{2}}\right). \quad \square$$

Next, we will sketch the proof of Theorem 1.1. First, we need the following two lemmas.

Lemma 1.5. *Let $\vec{m} = (m_1, \ldots, m_{n-1})$ be a vector of integers $m_i \ge K$ whose sum is at most $t - n + 1$ for t and n fixed, $n > 1$ and t large enough. Let*

$$A_t(\vec{m}) = \left\{(p_1, \ldots, p_{n-1}) : \frac{m_i}{t} \le p_i < \frac{m_i + 1}{t}, i = 1, \ldots, n-1\right\}$$

and $A_t = \bigcup_{\vec{m}} A_t(\vec{m})$. Then $Pr(A_t) \to 1$, as $t \to \infty$. Moreover, given $\vec{p} \in A_t$, the random variable

$$\tilde{V}_i = V_i^{\vec{m},t} := t(x_i^{(K)} - p_i), i = 1, \ldots, n-1$$

(rounding errors of a K-stationary rule of rounding $\vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})$), are independent and uniformly distributed on $[-C, 1 - C]$.

The proof parallels that of Lemma 1.3. Note that m_i's were assumed to be greater than 0, while here $m_i \ge K$. Clearly, as $t \to \infty$, the probability for a proportion p_i to be in the interval $[0, K/t]$ is negligible.

Lemma 1.6. *For every integer $\sigma_1 \notin \{0, -1, \ldots, -K+1\}$ and every continuous random variable, the K-stationary rule of $1/t$-rounding $\vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})$ (4), gives the following:*

$$\left[\frac{\sigma_1 + \sigma_2}{t}\right]_t^K = \frac{\sigma_1}{t} + \frac{1}{t}\left[\sigma_2\right]_1^K + \frac{1}{t}\sum_{j=0}^{K-1} \left(I\left\{\sigma_2 \in [j, j + C_j]\right\} - I\left\{\sigma_2 \in [j, j + C]\right\}\right)$$

$$+\frac{1}{t}\sum_{j=0}^{K-1}\left(I\left\{\sigma_1+\sigma_2\in[j,j+C]\right\}-I\left\{\sigma_1+\sigma_2\in[j,j+C_j]\right\}\right).$$

In particular, if $C \le C_j$, $j=0,\dots,K-1$, *then*

$$\left[\frac{\sigma_1+\sigma_2}{t}\right]_t^K=\frac{\sigma_1}{t}+\frac{1}{t}\left[\sigma_2\right]_1^K+\frac{1}{t}\sum_{j=0}^{K-1}I\left\{\sigma_2\in(j+C,j+C_j]\right\}$$

$$+\frac{1}{t}\sum_{j=0}^{K-1}I\left\{\sigma_1+\sigma_2\in(j+C,j+C_j]\right\}.$$

The proof parallels the proof of Lemma 1.4.

Sketch of proof of Theorem 1.1. Applying Lemma 1.6, and using the expression $x_i^{(K)}=[p_i]_t^K=p_i+\frac{1}{t}\tilde{V}_i$, $i=1,\dots,n-1$ we obtain the following expression for $x_n^{(K)}$:

$$x_n^{(K)}=[p_n]_t^K=\left[1-\sum_{i=1}^{n-1}x_i^{(K)}+\frac{1}{t}\sum_{i=1}^{n-1}\tilde{V}_i\right]_t^K=\left[\frac{\left(t-t\sum_{i=1}^{n-1}x_i^{(K)}\right)+\sum_{i=1}^{n-1}\tilde{V}_i}{t}\right]_t^K$$

$$=1-\sum_{i=1}^{n-1}x_i^{(K)}+\frac{1}{t}\left[\sum_{i=1}^{n-1}V_i\right]_1^K+\frac{1}{t}R_t,$$

where for R_t we have the following:

If $C\le C_j$, $j=0,1,\dots,K-1$ (the general case can be handled in the same way) and $\sigma_1\notin\{0,-1,\dots,-K+1\}$,

$$R_t=\sum_{j=0}^{K-1}I\left\{\sigma_2\in(j+C,j+C_j]\right\}-\sum_{j=0}^{K-1}I\left\{\sigma_1+\sigma_2\in(j+C,j+C_j]\right\},$$

where $\sigma_1=t-t\sum_{i=1}^{n-1}x_i^{(K)}$ and $\sigma_2=\sum_{i=1}^{n-1}\tilde{V}_i$. Consequently, $\sigma_1+\sigma_2=tp_n$ and, as $t\to\infty$, $\sum_{j=0}^{K-1}I\left\{\sigma_1+\sigma_2\in(j+C,j+C_j]\right\}\to 0$, with probability 1. On the other hand, by Lemma 1.5, as $t\to\infty$

$$\sum_{j=0}^{K-1}I\left\{\sigma_2\in(j+C,j+C_j]\right\}\longrightarrow\sum_{j=0}^{K-1}I\left\{\sum_{j=0}^{K-1}V_i\in(j+C,j+C_j]\right\},$$

where V_i's are i.i.d. uniforms on $[-C,1-C]$. Summing up all expressions of x_i, $i=1,\dots,n$, we obtain

$$x_N^{(K)}=\sum_{i=1}^{n-1}x_i^{(K)}=1+\frac{1}{t}\left[\sum_{i=1}^{n-1}V_i\right]_1^K+\frac{1}{t}R_t$$

and consequently,

$$t\left\{x_N^{(K)}-1\right\}=\left[\sum_{i=1}^{n-1}V_i\right]_1^K+R_t.$$

Recall that, to apply Claim 1.5, we must assume $\sigma_1=t-t\sum_{i=1}^{n-1}x_i^{(K)}\notin\{0,-1,\ldots,-K+1\}$. Then, as $t\to\infty$,

$$\lim P(x_N^{(K)}=1)=\lim P\left(x_N^{(K)}=1,\sum_{i=1}^{n-1}x_i^{(K)}\neq 1-\frac{\Delta}{t},\ \forall\Delta\in\{0,1,\ldots,K-1\}\right)$$

$$+\lim P\left(x_N^{(K)}=1,\sum_{i=1}^{n-1}x_i^{(K)}=1-\frac{\Delta}{t},\quad\text{for some}\Delta\in\{0,1,\ldots,K-1\}\right).$$

The probability of the second term on the right-hand side is, in fact, equal to the $P(x_N^{(K)}=1,x_n^{(K)}=\frac{\Delta}{t}$ for some $\Delta\in\{0,1,\ldots,K-1\})$ and, as $t\to\infty$, it converges to 0, since $\vec{p}$ is uniformly distributed on the simplex S_n. Therefore,

$$\lim_{t\to\infty}P\left(x_N^{(K)}=1\right)=\lim_{t\to\infty}P\left(t\left\{x_N^{(K)}-1\right\}=0,\sigma_1\notin\{0,-1,\ldots,-K+1\}\right)$$

$$=P\left(\left[\sum_{i=1}^{n-1}V_i\right]_1^K+\sum_{j=0}^{K-1}I\left\{\sum_{i=1}^{n-1}V_i\in(j+C,j+C_j]\right\}=0\right).$$

The latter probability can be expressed as a sum of K terms, say $T_0,\ldots,T_{K-1}$ where

$$T_0=P\left(\left[\sum_{i=1}^{n-1}V_i\right]_1^K=0\ \text{ and }\ I\left\{\sum_{i=1}^{n-1}V_i\in(j+C,j+C_j]\right\}=0\ \ \forall j\in\{0,\ldots,K-1\}\right)$$

$$T_1=P\left(\left[\sum_{i=1}^{n-1}V_i\right]_1^K=-1,I\left\{\sum_{i=1}^{n-1}V_i\in(j_0+C,j_0+C_{j_0}]\right\}=1,\right.$$

$$\left.\text{for some } j_0\in\{0,\ldots,K-1\}\right)$$

$$\left.\text{and }\ I\left\{\sum_{i=1}^{n-1}V_i\in(j+C,j+C_j]\right\}=0\ \ \forall j\neq j_0,j\in\{0,\ldots,K-1\}\right)$$

and so on,

$$T_{K-1}=P\left(\left[\sum_{i=1}^{n-1}V_i\right]_1^K=-K\ \text{ and }\ I\left\{\sum_{i=1}^{n-1}V_i\in(j+C,j+C_j]\right\}=1\right.$$

$$\forall j \in \{0, \ldots, K-1\}\Big)$$

Note that $T_1 = T_2 = \ldots = T_{k-1} = 0$. Consequently,

$$\lim P\left(x_N^{(K)} = 1\right) = P\Bigg(-1 + C < \sum_{i=1}^{n-1} V_i < C_0 \text{ and } \sum_{i=1}^{n-1} V_i \notin (j + C, j + C_j)\,,$$

$$\forall j \in \{0, 1, \ldots, K-1\}\Bigg)$$

$$= P\left(-1 + C < \sum_{i=1}^{n-1} V_i < C\right).$$

Hence the limiting probability does not depend on $C_0, \ldots, C_{K-1}$.

The rest of the proof parallels that of Theorem 1.2 and leads us to the following conclusion:

The limiting probability of $\left\{x_N^{(K)} = 1\right\}$ for a K-stationary divisor rule $\vec{x}^{(K)} = \rho_t^{(K)}(\vec{p})$ attains its maximum for the rule with divisor points $C_0, C_1, \ldots, C_{k-1}, C$ where C_j, $0 \le j \le K-1$ may be any point on $[0,1]$ while $C = \frac{1}{2}$. Moreover, the maximum of the $\lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \le x_N^{(K)} \le 1 + \frac{\Delta}{t}\right)$, $\forall \Delta = 0, 1, 2, \ldots$ is attained by the K-stationary rule with divisor point $C_0, C_1, \ldots, C_{K-1}, C$ as described above.

In addition, if $\vec{x}$ is obtained by the Mosteller-Youtz-Zahn divisor rule $\vec{x} = \rho_t(\vec{p})$, then

$$\max \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \le x_N^{(K)} \le 1 + \frac{\Delta}{t}\right) = \lim_{t\to\infty} P\left(1 - \frac{\Delta}{t} \le x_N \le 1 + \frac{\Delta}{t}\right). \quad \square$$

1.3. Simulation Studies. Simulation studies have been conducted to support our theoretical results: Suppose $\tilde{p} = (p_1, \ldots, p_n)$ is uniformly distributed over the simplex S_n and $\vec{x}^{(MYZ)} = (x_1, \ldots, x_n)$, $\vec{x}^{(1)} = \left(x_1^{(1)}, \ldots, x_n^{(1)}\right)$ and $\vec{x}^{(2)} = \left(x_1^{(2)}, \ldots, x_n^{(2)}\right)$ are the $1/t$-roundings of $\vec{p}$ obtained by the Mosteller-Youtz-Zahn rule, 1-stationary and 2-stationary rules respectively. Then, if $x_N^{(\bullet)} = x_1^{(\bullet)} + \ldots + x_n^{(\bullet)}$,

$$\begin{aligned}\lim_{t\to\infty} P\left(x_N^{(MYZ)} = 1\right) &= \lim_{t\to\infty} P\left(x_N^{(1)} = 1\right) \\ &= \lim_{t\to\infty} P\left(x_N^{(2)} = 1\right) = \sqrt{\frac{6}{\pi(n-1)}} + O\left(\frac{1}{\sqrt{n^3}}\right)\end{aligned} \tag{6}$$

where the second term of the sum is equal to $-\sqrt{\dfrac{3}{2\pi(n-1)^3}} + O\left(\dfrac{1}{\sqrt{n^5}}\right)$.

Our simulations (see Tables 1.1–1.5) show that for $n \geq 100$, the numerical results approach the theoretical results of (6) when we round in the sixth or seventh decimal point, that is, for $t = 10^6$ or $t = 10^7$. If we wish to obtain precision up to the second term on the right-hand side of (6), we need, first, to consider for rounding at least 10^6 vectors $\vec{p}$ and second, to round at least to the 10th decimal point.

In our simulations, the C_0 of the 1-stationary rule and the C_0, C_1 of the 2-stationary rule have been assigned values taken from the interval $[0.35, 0.65]$. The further from 0.5 these values are, the larger the rounding number t should be, in order to obtain the first and second equality in (6). In order to get the desired results in the cases where C_0 and C_1 take values outside the interval $[0.35, 0.65]$, we need, once more, to round at least to the 10th decimal.

The expected results in (6) change subject to changes on the number n of components that each vector $\vec{p}$ consists of. The following table displays the values of $\sqrt{\frac{6}{\pi(n-1)}}$ and $\sqrt{\frac{3}{2\pi(n-1)^3}}$ for several values of n:

n	$\sqrt{\frac{6}{\pi(n-1)}}$	$\sqrt{\frac{3}{2\pi(n-1)^3}}$
100	0.1389	0.000701662
500	0.0619	0.000062005
1000	0.0437	0.000021889
1500	0.0357	0.000011909
2000	0.0309	0.000007733

In rounding five thousand vectors $\vec{p} = (p_1, \ldots, p_n)$ for each $n \in \{100, 500, 1000, 1500, 2000\}$ we obtain the following Tables:

TABLE 1.1: $n = 100$

t	$P(x_N^{(MYZ)} = 1)$	$P(x_N^{(1)} = 1)$	$P(x_N^{(2)} = 1)$
10^3	0.1340	0.1332	0.1240
10^4	0.1362	0.1346	0.1322
10^5	0.1378	0.1366	0.1360
10^6	0.1380	0.1380	0.1376
10^7	0.1382	0.1382	0.1380

TABLE 1.2: $n = 500$

t	$P(x_N^{(MYZ)} = 1)$	$P(x_N^{(1)} = 1)$	$P(x_N^{(2)} = 1)$
10^3	0.0200	0.0102	0.0224
10^4	0.0536	0.0528	0.0506
10^5	0.0582	0.0590	0.0594
10^6	0.0616	0.0614	0.0618
10^7	0.0618	0.0614	0.0616

TABLE 1.3: $n = 1000$

t	$P(x_N^{(MYZ)} = 1)$	$P(x_N^{(1)} = 1)$	$P(x_N^{(2)} = 1)$
10^3	0.0000	0.0000	0.0034
10^4	0.0405	0.0314	0.0342
10^5	0.0416	0.0414	0.0393
10^6	0.0425	0.0420	0.0418
10^7	0.0436	0.0435	0.0436

TABLE 1.4: $n = 1500$

t	$P(x_N^{(MYZ)} = 1)$	$P(x_N^{(1)} = 1)$	$P(x_N^{(2)} = 1)$
10^3	0.0000	0.0000	0.0000
10^4	0.0225	0.0225	0.0128
10^5	0.0325	0.0325	0.0315
10^6	0.0345	0.0340	0.0340
10^7	0.0355	0.0355	0.0355

TABLE 1.5: $n = 2000$

t	$P(x_N^{(MYZ)} = 1)$	$P(x_N^{(1)} = 1)$	$P(x_N^{(2)} = 1)$
10^3	0.0000	0.0000	0.0000
10^4	0.0165	0.0165	0.0018
10^5	0.0260	0.0260	0.0268
10^6	0.0290	0.029?	0.0280
10^7	0.0300	0.030?	0.0290

REFERENCES

[1] B. Athanasopoulos. Rounding proportions with applications to fair representation. Proceedings of the VI Latin Ibero American Conference on Operations Research, 1992, to appear.

[2] M. L. Balinski and G. Demange. Algorithms for Proportional Matrices in Reals and Integers. *Math Programming*, North Holland, **45** (1989), 193-210.

[3] M. L. Balinski and S. T. Rachev. Rounding Proportions: Rules of Rounding. *Technical Report*, **384** Laboratoire d'Econometrie, École Polytechnique, (1992).

[4] P. Billingsley. Probability and Measure, 2nd ed., Wiley, 1986.

[5] P. Diaconis and D. Freedman. On Rounding Percentages. *Journal of the American Statistical Association*, **74** (1979), 359-364.

[6] W. Feller. An Introduction to Probability Theory and Its Applications. John Wiley & Sons, New York, Vol. II, 2nd ed., 1970, 504-515.

[7] M. Maejima and S. T. Rachev. An ideal metric and the rate of convergence to a self-similar process. *Annals of Probability*, **15** (1987), 702-727.

[8] F. Mosteller, C. Youtz and D. Zahn. The Distribution of Sums of Rounded Percentages. *Demography*, **4** (1967), 850-858.

[9] R. Pyke. Spacings. *The Journal of the Royal Statistical Society* Series B, **27**, *3* (1965), 395-449.

[10] S. T. Rachev. Probability Metrics and the Stability of Stochastic Models. Wiley, New York, 1991.

[11] A. M. Turing. Rounding-Off Errors in Matrix Processes. *Quart. J. Mech.*, **1**, (1948), 287-308.

[12] J. H. Wilkinson. Rounding Errors in Algebraic Processes. Prentice-Hall, Englewood Cliffs, NJ, 1963.

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