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REALCOMPACTIFICATIONS, P–SPACES AND BAIRE ISOMORPHISM OF TOPOLOGICAL SPACES

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Abstract. The concept of realcompactification which is based on the notoin of a complete space of functions is applied to the problem of the Baire homeomorphism of topological spaces and to the study of the Stone–Čech compactifications of P–spaces. It is proved that realcompact P–spaces are homeomorphic if and only if their Stone–Čech compactifications are homeomorphic.

Introduction. In the present paper we investigate the concept of realcompactification of topological spaces. The definition of realcompactification of space X is based on the following properties of Hewitt realcompactification νX of space X:

1. $\nu X \subseteq \beta X$;
2. νX is a realcompact space;
3. $H \cap X \neq \emptyset$ for every non–empty zero–set H of space νX.

The extension rX of space X is a realcompactification of X if is satisfies properties 2 and 3 (see Definition 2.1). This approach is justified by Theorems 3.2, 4.1, 5.5 and Example 5.7.

The notion of realcompactification is applied to the problem of an isomorphism of Baire classes of functions and to the study of the Stone–Čech compactifications of P–spaces. In particular, we shall prove that the realcompact P–spaces X and Y are homeomorphc if and only if their Stone–Čech compactifications βX and βY are homeomorphic.

The notion of complete space of functions enables to apply the Gelfand–Kolmogorov and Stone–Weierstrass theorems to the study of the isomorphisms of the complete algebras of functions.

1. Definitions and notations. We shall consider only Tychonov spaces. We shall use the notations and terminology from [1, 4]. In particular, βX is the Stone–Čech compactification of the space X, νX is the Hewitt realcompactification of space

X, $\omega(X)$ is the weight of space X, the cardinality of a set Y is denoted by $|Y|$, $cl\ H$ or $cl_X H$ denotes the closure of a set H in X, $N = \{1, 2, \dots\}$, R is the field of real numbers, $C(X)$ is the space of all continuous real–valued functions on a space X, $C^*(X)$ presents all bounded functions in $C(X)$.

A space is realcompact if it is homeomorphic to a closed subspace of a product of real lines.

Let S be a set, $B(S)$ be the space of all real–valued functions on S and $B^*(S) = \{f \in B(S) : f \text{ is bounded on } S\}$. The sets $B(S)$ and $B^*(S)$ are lattice–ordered algebras with respect to the pointwise operation. The space $B(S)$ is a Banach algebra with the supremum norm $||f|| = sup\{|f(x)| : x \in S\}$. If $\alpha \in R$, then $\alpha_S(x) = \alpha$ for every $x \in S$. If $f \in B(S)$ and $\alpha \in R$, then we put $f \vee \alpha = f \vee \alpha_S$ and $f \wedge \alpha = f \wedge \alpha_S$.

If $E \subseteq B(S)$, then T_E is the topology on S generated by E and it has a base consisting of all sets of the form $\cap\{f_i^{-1}U_i : i = 1, \dots, n\}$, where $n \in N, f_1, \dots, f_n \in E$ and $U_1, \dots, U_n$ are the open subsets of R. The space E separates the set S if for each pair of distinct points $x, y \in S$ there exists $f \in E$ such that $f(x) \neq f(y)$. The space (S, T_E) is Tychonov if and only if E separates the set S.

Let a subspace E of $B(S)$ separate the set S. Then the mapping $w_E : S \to R^E$, where $w_E(x) = \{f(x) : f \in E\}$, is an embedding of (S, T_E) in R^E. The closure $\nu_E S$ of the set $S = w_E(S)$ in R^E is a realcompact space. The space $\nu_E S$ is compact if and only if $E \subseteq B^*(S)$.

Let $e_1 X$ and $e_2 X$ be the extensions of space X. $e_1 X > e_2 X$ means that there exists a continuous mapping $f : e_1 X \to e_2 X$ such that $f(x) = x$ for every $x \in X$.

A subspace E of $B(S)$ is called a b–complete algebra of functions on S if it is a Banach subalgebra of $B^*(S)$ satisfying the following conditions:

1. E contains all constant functions;
2. E separates S.

A subspace E of $B(S)$ is called a complete space of functions on S if it has the following properties:

3. $mE = E \cap B^*(S)$ is a b–complete algebra of functions on S;
4. If $(f \wedge n) \vee (-n) \in E$ for every $n \in N$, then $f \in E$;
5. If $f \in E$, then $(f \wedge n) \vee (-n) \in E$ for every $n \in N$.

Let E be a b–complete algebra or a complete space of functions on a set S. Let $\beta_E S = \nu_{mE} S$.

1.1. Property. *Let $F \subseteq E \subseteq B(S)$ and F separates the set S. Then $\nu_E S > \nu_F S$.*

Proof. Obvious.

1.2. Property. *Let $E \subseteq B^*(S)$ separate the set S. Then $\nu_E S$ is the smallest compactification of the space (S, T_E) such that all functions of E are continuously extendable over $\nu_E S$.*

P r o o f. Obvious.

1.3. Property. *$\beta X = \beta_{C(X)}X = \nu_{C^*(X)}X$ and $\nu X = \nu_{C(X)}X$ for every space X.*

P r o o f. Obvious.

1.4. Property. *Let E be a b–complete algebra of functions on a set S. Then the operator $u : C(\nu_E S) \to B(S)$, where $u(f) = f|X$, is an isomorphism of $C(\nu_E S)$ onto E.*

P r o o f. Follows from Property 1.2 and the Stone–Weierstrass theorem ([1], p. 191).

1.5. Property. *Let E be a complete space of functions on set S and $mE = E \cap B^*(S)$. Then:*

1. *$\nu_E S > \nu_{mE} S = \beta_E S$ and the natural mapping $\pi : \nu_E S \to \beta_E S$ is an embedding.*

2. *$\nu_E S$ is the maximal subspace of $\beta_E S$ such that all functions of E are continuously extendable over $\nu_E S$.*

3. *$\nu_E S = \beta_E S \setminus \cup\{H \subseteq \beta_E S : H$ is a closed G_δ–subset of $\beta_E S$ and $H \cap S = \emptyset\}$.*

4. *For every $f \in E$ there exist a minimal closed G_δ–subset H of $\beta_E S$ and a continuous function $\beta_E f : \beta_E S \setminus H \to R$ such that $H \cap S = \emptyset$ and $f = \beta_E f|S$.*

5. *E is uniformly closed in $B(S)$.*

6. *If $f \in E$, then $-f \in E$.*

7. *If $m \in N, f \in B(S), f^m \in E$ and $f \geq 0$, then $f \in E$.*

8. *If $f \in E$ and $f(x) \neq 0$ for every $x \in S$, then $g = 1/f \in E$.*

9. *If $E \cap B^*(S) = C^*(S, T_E)$, then E is an algebra of functions and $E = C(S, T_E)$.*

P r o o f. Let $\{f_m : m \in N\} \subseteq E, f \in B(S)$ and $\|f - f_m\| < 2^{-m}$. If $g_n = (f \wedge n) \vee (-n)$ and $g_{mn} = (f_n \wedge n) \vee (-n)$, then $\|g_{mn} - g_n\| < 2^{-m}$ and $g_{mn}, g_n \in E$ for all $m, n \in N$. Hence $f \in E$. Thus Assertion 5 is proved.

Let $f \geq 0$ and $f^m \in E$. Let $f_n = (f \wedge n) \vee (-n)$. Then $f_n^m = (f^m \wedge n^m) \vee (-n^m) \in E$ and $f_n \in E$. Hence $f \in E$. Thus Assertion 7 is proved.

Fix $f \in E$ and suppose that $f(x) \neq 0$ for every $x \in S$. Let $g = 1/f, f_n = (f \wedge n) \vee (-n)$ and $n, m \in N$. Then $f_n \in mE$. Consider functions $f_{mn}(x) = (f_n(x) \wedge (-(m^{-1}))$ if $f_n(x) < 0$ and $f_{mn}(x) = (f_n(x) \vee (m^{-1})$ if $f_n(x) > 0$. By construction, $f_{mn} \in mE$ and $g_{mn}\ 1/f_{mn} \in mE$. Denote $(g \wedge m) \vee (-m) = g_m$. Then $\|g_m - g_{mn}\| \leq n^{-1}$. Hence $g_m \in mE$ for every $m \in N$. This proves Assertion 8.

Fix $f \in E$ and denote $f_n = (f \wedge n) \vee (-n)$. Then for every $n \in N$ there exists a continuous function $g_n : \beta_E S \to R$ such that $f_n = g_n|S$. Let $H_n = g_n^{-1}\{-n, n\}$ and

$H = \cap\{H_n : n \in N\}$. If $U_n = g_n^{-1}(-n, n)$, then $S \subseteq \cup\{U_n : n \in N\} = \beta_E S \setminus H$ and $U_n \subseteq U_{n+1}$ for every $n \in N$. If $x \in U_n$, then we put $\beta_E f(x) = g_n(x)$. Assertion 4 is proved.

Let H be a closed G_δ–subset of $\beta_E S$ and $H \subseteq \beta_E S \setminus S$. Then $g^{-1}(0) = H$ for some $g \in C(\beta_E S)$. Let $f = g|S$. Then $f(x) \neq 0$ for every $x \in S$ and $h = 1/f \in E$. By construction, $\beta_E h(x) = 1/g(x)$ for every $x \in \beta_E S \setminus H$. This fact and Assertion 4 prove Assertions 1, 2 and 3.

Let $f \in E, f_n = (f \wedge n) \vee (-n), h = -f$ and $h_n = (h \wedge n) \vee (-n)$. By construction, $h_n = -f_n$ for every $n \in N$. Therefore $-f = h \in E$. Assertion 6 is proved.

Assertion 9 is obvious. The proof is complete.

1.6. Corollary. *Let E be a b–complete algebra of functions on a set S. Then E is complete if and only if $H \cap S = \phi$ for every non–empty G_δ–subset H of $\nu_E S$.*

1.7. Corollary. *Let E be a complete space of functions on a set S such that (S, T_E) is pseudocompact. Then E is a b–complete algebra of functions on S.*

1.8. Corollary. *Let E be a b–complete algebra or a complete space of functions on a set S. Then E is a sublattice of lattice $B(S)$.*

Fix a space X. Let $B_O(X) = C(X)$ and inductively define the α Baire class $B_\alpha(X)$ for each $\alpha \leq \Omega$ (Ω denotes the first uncountable ordinal) to be the space of pointwise limits of sequences of functions in $\cup\{B_\beta(X) : \beta < \alpha\}$.

For every function $f : X \to R$ we denote $Z(f) = f^{-1}(0)$ and $CZ(f) = X \setminus Z(f)$.

We put $Z_\alpha(X) = \{Z(f) : f \in B_\alpha(X)\} = \{f^{-1}F : f \in B_\alpha(X) \text{and} F \in Z_0(X)\}, CZ_\alpha(X) = \{CZ(f) : f \in B_\alpha(X)\}, Z_\alpha(X) \cap CZ_\alpha(X) = A_\alpha(X)$.

The class $Z_\alpha(X)$ (class $CZ_\alpha(X)$) is a multiplicative (additive) class α of Baire sets of the space X. The sets $A_\alpha(X)$ are called the sets of ambiguous Baire class α. The sets in $Z_0(X)$ are called the zero–sets of space X.

Let PX be the set X with the topology generated by the G_δ–sets in the space X. The topology of PX is called Baire topology of the space X. For every $\alpha \leq \Omega$ the classes $Z_\alpha(X), CZ_{1+\alpha}(X), A_{1+\alpha}(X)$ are open bases of the space PX.

A space X is called a P–space if $X = PX$.

For every $\alpha \leq \Omega$ we denote $\nu_\alpha PX = \nu_{B_\alpha(X)} X$ and $\beta_\alpha PX = \beta_{B_\alpha(X)} X = \nu_{B^*_\alpha(X)} X$, where $B^*_\alpha(X) = B_\alpha(X) \cap B(X)$.

1.9. Property. *For every $\alpha \leq \Omega$ and every space X the set $B_\alpha(X)$ is a complete algebra of functions on a set X.*

P r o o f. Obvious.

1.10. Property. *Let E be a complete space of continuous functions on a Lindelöf space X and T_E be the topology of X. If E is an additive semigroup, then $E = C(X)$.*

P r o o f. Fix the closed and disjoint subsets H and F of X. Denote by H_1 and F_1 the closures of H and F in $bX = \beta_E X$. Then $P_1 = H_1 \cap F_1$ is a compact subset of $bX \setminus X$. Since X is Lindelöf, there exists a subset $P \in Z_0(bX)$ such that $P_1 \subseteq P \subseteq bX \setminus X$. Then there exist continuous functions $f_1 : bX \to [0,1]$ and $g_1 : bX \to [0,2]$ such that $g_1^{-1}(0) = f_1^{-1}(0) = P, g_1(x) = 2^{-1} \cdot f_1(x)$ for every $x \in H_1$ and $g_1(x) = 2 \cdot f_1(x)$ for every $x \in F_1$. Let $f = f_1|X$ and $g = g_1|X$. Then $f(x) \neq 0$ and $g(x) \neq 0$ for every $x \in X$. Hence $1/f \in E$ and $-1/g \in E$. By assumption E is an additive semigroup and $\varphi = (1/f) - (1/g) \in E$. By construction, $H \subseteq \varphi^{-1}(-\infty, -1]$ and $F \subseteq \varphi^{-1}[2^{-1}, \infty)$. Hence $P_1 = \emptyset, bX = \beta X$ and $mE = E \cap B^*(X) = C^*(X)$. From Property 1.5 $E = C(X)$. The proof is complete.

2. Realcompactifications.

2.1. Definition. *A space Y is called a realcompactification of a space X if Y is realcompact, X is a dense subspace of Y and $H \cap X \neq \emptyset$ for every non–empty subset $H \in Z_0(Y)$.*

2.2. Proposition. *Let bX be a realcompact or a compact extension of space X and $rX = bX \setminus \cup\{H \in Z_0(bX) : H \cap X = \emptyset\}$. Then rX is a realcompactification of X generated by the extension bX.*

P r o o f. By virtue of ([1], Theorem 3, 11.10), rX is a realcompact space. By construction, X is dense in rX and $X \cap H \neq \emptyset$ for every non–empty subset $H \in Z_0(rX)$. The proof is complete.

2.3. Proposition. *Let E be a complete space of continuous functions on space X and T_E be the topology of X. Then $\nu_E X$ is a realcompactification of X generated by the compactification $\beta_E X$ of X.*

P r o o f. Follows from Property 1.5.

2.4. Proposition. *Let rX be a realcompactification of space X. Then there exists a maximal complete space E of functions on the set X such that:*

1. *E is an algebra of continuous functions on the space X.*

2. *$rX = \nu_E X$ and $\beta rX = \beta_E X$.*

P r o o f. Let $E = \{f|X : f \in C(rX)\}$ and $mE = \{f|X : f \in C^*(rX)\} = \{f|X : f \in C(\beta rX)\}$. Then, by virtue of ([1], Theorem 3.11.10), $rX = \beta rX \setminus \cup\{H \in Z_0(\beta rX) : H \cap rX = \emptyset\} = \beta rX \setminus \cup\{H \in Z_0(\beta rX) : H \cap X = \emptyset\}$, the realcompactification rX is generated by βrX and mE is a b–complete algebra of functions on X. Let $f \in B(X)$ and $f_n = (f \wedge n) \vee (-n) \in mE$ for every $n \in N$. By construction, $f_n = g_n|X$ for some $g_n \in C(rX)$. We put $g_n^{-1}\{-n, n\} = H_n$ and $H = \cap\{H_n : n \in N\}$. Then $H \in Z_0(rX)$ and $H \cap X = \emptyset$. Hence $H = \emptyset$. The function g, where $g|(rX \setminus H_n) = g_n|(rX \setminus H_n)$ for every $n \in N$, is continuous and $g|X = f$. Therefore $f \in E$ and E is a complete algebra of functions on X. By construction, $rX = \nu_E X$ and $\beta rX = \beta_E X$. It is clear that E is a maximal complete space of functions on a set X for which $rX = \nu_E X$. The proof is complete.

2.5. Proposition. *Let the realcompactification rX of a space X be generated by the compactification bX of X. We put $mE = \{f|X : f \in C(bX)\}$ and $E = \{f \in B(X) : (f \wedge n) \vee (-n) \in mE$ for every $n \in N\}$. Then $rX = \nu_E X$ and $bX = \beta_E X$.*

Proof. By construction, $mE = E \cap B^*(X), E \subseteq C(X)$ and $\nu_E X \subseteq \beta_E X = \nu_{mE} X = bX$. Hence, by virtue of Property 1.5, $rX = \nu_E X$. The proof is complete.

2.6. Corollary. *If eX is an extension of the space X, then the following assertions are equivalent:*

1. *eX is a realcompactification of X.*
2. *$eX = \nu_E X$ for some complete space E of functions on X.*
3. *$eX = \nu_E X$ for some complete algebra E of functions on X.*

2.7. Proposition. *The space X has an unique realcompactification if and only if X is Lindelöf or $|\beta X \setminus X| \leq 1$.*

Proof. Let X be a Lindelöf space and bX be a compactification of X. For every compact subset F of $bX \setminus X$ there exists a compact subset $H \in Z_0(bX)$ such that $F \subseteq H \subseteq bX \setminus X$ (see [1], Problem 3.12.24). Hence X is an unique realcompactification of space X.

Let $|\beta X \setminus X| \leq 1$. Then $\nu X = \beta X$ is the unique realcompactification of X.

Let X be not a Lindelöf space and $|\beta X \setminus X| > 1$. Then there exists a compact subset F of $\beta X \setminus X$ such that $|F| > 1$ and for every compact subset $H \in Z_0(\beta X)$, where $H \subseteq \beta X \setminus X$, we have $F \setminus H \neq \emptyset$. Consider the continuous mapping $h : \beta X \to bX$ onto a compactification bX of X such that $h(F)$ is a singleton set and $h(x) = x$ for every $x \in X$. Then $rX = bX \setminus \cup\{H \in Z_0(bX) : H \cap X = \emptyset\}$ is a realcompactification of $X, h(F) \subseteq rX, rX \neq \nu X$. The proof is complete.

2.8. Example. Let X be a locally compact non–Lindelöf space. Then the Alexandrov one–point compactification αX of X is a realcompactification of X.

2.9. Example. Let X be a locally Lindelöf non–Lindelöf space. Then there exists an extension lX of X such that lX is Lindelöf, every closed Lindelöf subspace of X is closed in lX and $lX \setminus X$ is a singleton set. The space lX is called a one–point lindelöfication of X. The space lX is a realcompactification of X.

2.10. Example. The space X is almost Lindelöf if at least one of any pair of disjoint zero–sets is Lindelöf [5].

Let X be an almost Lindelöf space. Then X is locally Lindelöf. If X is not Lindelöf, then, by virtue of Proposition 5.4 [5], we have $\nu X = lX$.

2.11. Example. Let X be a Lindelöf space and bX be a compactification of X. We put $F = mE = \{f|X : f \in C(bX)\}$ and $E = \{f \in B(X) : (f \wedge n) \vee (-n) \in F$

for every $n \in N\}$. Then E is a complete space of functions, $\nu_E X = X$ and $\beta_E X = bX$. If $bX = \beta X$, then E is a complete algebra of functions and $E = C(X)$.

Let E be a complete algebra of functions on a set X. From Property 1.10 we have $E = C(X)$. Hence E is a complete algebra of functions on X if and only if $E = C(X)$ and $bX = \beta X$. In particular, if $bX \neq \beta X$, then E is a complete space of functions and E is not an algebra of functions.

2.12. Example. Let Q be a space of rational numbers of $[0,1], mE = \{f|Q : f \in C([0,1])\}$ and $E = \{f \in B(Q) : (f \wedge n) \vee (-n) \in mE$ for every $n \in N\}$. Then E is a complete space of functions on the set Q and E is not an algebra of functions on the set Q.

3. Isomorphism of spaces of functions.

3.1. Definition. *Let $E \subseteq B(X)$ and $F \subseteq B(Y)$. The mapping $h : E \to F$ is called a homomorphism if it satisfies the following conditions:*

1. *If $f, g \in E$ and $f + g \in E$, then $h(f + g) = h(f) + h(g)$;*
2. *If $f \in E$ and $-f \in E$, then $h(-f) = -h(f)$;*
3. *If $f, g \in E$ and $f \cdot g \in E$, then $h(f \cdot g) = h(f) \cdot h(g)$.*

The mapping $h : E \to F$ is called isomorphism if h maps E onto F in a one–to–one way and h, h^{-1} are the homomorphisms.

3.2. Theorem. *Let E be a complete space of functions on a set X, F be a complete space of functions on a set Y and $h : E \to F$ be an isomorphism. Then there exists an unique homeomorphism $\psi : \beta_E X \to \beta_E Y$ such that:*

1. *$\psi(Z(\beta_E f)) = Z(\beta_F h(f))$ for every $f \in E$.*
2. *$\psi(\nu_E X) = \nu_F Y$.*

P r o o f. Let $mE = E \cap B^*(X), mF = F \cap B^*(Y), E^+ = \{f \in E : f \geq 0\}$ and $F^+ = \{g \in F : g \geq 0\}$. By Property 1.5, $E^+ = \{f \cdot f : f \in E\}$. Hence $h(E^+) = F^+$. If $f, g \in E$ and $f \leq g$, then $g - f \in E^+$ and $h(f) \leq h(g)$. Therefore h is a lattice isomorphism. It is clear that $h(1_X) = 1_Y$. Hence $h(\alpha_X) = \alpha_Y$ for every $\alpha \in R$, where $\alpha_X(X) = \alpha$. If $f \in mE$, then $-n_X < f < n_X$ for some $n \in N$. Therefore $-n_Y < h(f) < n_Y$ and $h(mE) = mF$.

For every maximal ideal J of the ring mE there exists an unique point $x(J) \in \beta_E X$ such that $J = \{f|X : f \in C(\beta_E X)$ and $f(x(J)) = 0\}$. For every maximal ideal H of the ring mF there exists an unique point $y(H) \in \beta_F Y$ such that $H = \{g|Y : g \in C(\beta_F Y)$ and $g(y(H)) = 0\}$ (see [2], Chapter 4). Then there exists an unique one–to–one mapping $\psi : \beta_E X \to \beta_F Y$ such that $\psi(x(J)) = y(h(J))$ for every maximal ideal J of mE. If $f \in C(\beta_E X)$, then $Z(f) = \{x(J) : J$ is a maximal ideal of mE and $f|X \in J\}$. Hence $Z(\beta_E f) = Z(\beta_E((f \wedge 1) \vee (-1))$ and $Z(\beta_F h(f)) = \psi(Z(\beta_E f))$ for every $f \in E$.

Let $H \in Z_0(\beta_E X)$ and $H \subseteq \beta_E X \setminus X$. Then there exists a continuous function $f \in C(\beta_E X)$ such that $H = Z(f)$. Then $f_1 = f|X \in mE$ and $g = 1/f_1 \in E$. Let $g_1 = \beta_F h(g)$. Then $1 = g_1(y) \cdot h(f)(y)$ for every $y \in Y$. Therefore $g_1(h(x)) = 1/f(x)$ for every $x \in \beta_E X \setminus H$ and $h(H) \subseteq \beta_F Y \setminus Y$. Hence $\psi(\nu_E X) = \nu_F Y$. The proof is complete.

3.3. Example. Let X be a locally compact non–Lindelöf space. By $Y = \alpha X$ we denote the one–point Alexandrov compactification of $X, F = C(Y)$ and $E = \{f|X : f \in F\}$. The mapping $h : F \to E$, where $h(f) = f|X$, is an isomorphism, E is a complete algebra of functions on the set X and F is a complete algebra of functions on the set Y. The spaces X and Y are not homeomorphic. If X is a discrete space of cardinality continuum, then X is realcompact.

4. Realcompactification of P–spaces.

4.1. Theorem. *Let rX be a realcompactification of a P–space X generated by a compactification bY, rY be a realcompactification of a P–space Y generated by a compactification bY and $\psi : bX \to bY$ be a homeomorphism. Then $\psi(rX) = rY$.*

Proof. Let $H \in Z_0(bX)$ and $H \neq \emptyset$. If $H \cap X = \emptyset$, then *int* $H = \emptyset$. If $H \cap X \neq \emptyset$, then *int* $H \neq \emptyset$. Therefore $\psi(rX) = rY$.

4.2. Corollary. *Let rX be a realcompactification of a P–space X, rY be a realcompactification of a P–space Y, E be a maximal complete algebra of functions on X for which $rX = \nu_E X$ and F be a maximal complete algebra of functions on Y for which $rY = \nu_F Y$. Then the following assertions are equivalent:*

1. *E and F are isomorphic.*
2. *rX and rY are homeomorphic.*
3. *βrX and βrY are homeomorphic.*

4.3. Corollary. *Let X and Y be realcompact P–spaces. The compactifications βX and βY are homeomorphic if and only if spaces X and Y are homeomorphic.*

4.4. Corollary. *Let X and Y be P–spaces. The compactifications βX and βY are homeomorphic if and only if the realcompactifications νX and νY are homeomorphic.*

5. Baire isomorphisms. Let X and Y be spaces. The mapping $\varphi : X \to Y$ is called a Baire isomorphism of class (α, β) if $\varphi(Z_\alpha(X)) = Z_\beta(Y)$.

5.1. Lemma. *If X is a realcompact space, then PX is realcompact, too.*

Proof. The space PR is discrete and realcompact (see [1,2]). Hence PX is a closed subspace of a realcompact space $(PR)^{C(X)}$.

5.2. Lemma. *Let X be a dense subspace of a realcompact P–space Y. Then Y is a realcompactification of X.*

P r o o f. . Let $H \in Z_0(Y)$ and $H \neq \emptyset$. Then H is an open subset of Y and $H \cap X \neq \emptyset$. The proof is complete.

5.3. Lemma. *$P\nu X$ is a realcompactification of PX.*

P r o o f. Follows from Lemmas 5.1 and 5.2.

5.4. Lemma. *$\nu_\alpha PX = P\nu X$ for every $\alpha \geq 1$.*

P r o o f. Follows from Lemma 5.1 and P. R. Mayer's theorem ([3], Theorem 7).

5.5. Theorem. *Let X and Y be spaces and $\alpha, \mu \geq 1$. The following assertions are equivalent:*

1. *$B_\alpha(X)$ and $B_\mu(Y)$ are ring isomorphic.*
2. *$B^*_\alpha(X)$ and $B^*_\mu(Y)$ are ring isomorphic.*
3. *$\beta_\alpha PX$ and $\beta_\mu PY$ are homeomorphic.*
4. *There exists a Baire isomorphism $\varphi : \nu X \to \nu Y$ of class (α, μ).*

P r o o f. Implications $1 \to 2 \to 3 \to 2$ and $4 \to 1$ are obvious. Let $\psi : \beta_\alpha PX \to \beta_\mu PY$ be a homeomorphism. From Theorem 4.1 we have $\psi(\nu_\alpha PX) = \nu_\mu PY$. Let $\varphi = \psi|\nu_\alpha PX$. From Lemma 5.4. $\nu_\alpha PX = P\nu X$ and $\nu_\mu PY = P\nu Y$. Let $H \in A_\alpha(P\nu X)$. Then $H = P\nu X \cap H_1$ for some open and closed subset H_1 $\beta_\alpha PX$. The set $\psi(H_1)$ is open and closed in $\beta_\mu PX$ and $\varphi(H) = \psi(H_1) \cap \nu Y \in A_\mu(\nu Y)$. Hence $\varphi : \nu X \to \nu Y$ is a Baire isomorphism of class (α, μ). This proves implication $3 \to 4$. The proof is complete.

5.6. Corollary (J. E. Jayne [4]). *Let X and Y be realcompact spaces and $\alpha, \mu \geq 1$. The following assertions are equivalent:*

1. *$B_\alpha(X)$ and $B_\mu(Y)$ are ring isomorphic.*
2. *$B^*_\alpha(X)$ and $B^*_\mu(Y)$ are ring isomorphic.*
3. *$B_\alpha PX$ and $B_\mu PY$ are homeomorphic.*
4. *There exists a Baire isomorphism $\varphi : X \to Y$ of class (α, μ).*

5.7. Example. Let X be a first countable space and $\nu X \neq X$. Then PX is a discrete realcompact space, $P\nu X$ is a realcompactification of space $PX, PX = \nu PX$ and $\nu PX \neq P\nu X$.

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