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A NEW CLASS OF MULTI-STAGE SCHEMES WITH HIGHER ORDER OF CONVERGENCE

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Abstract. In this note we consider a class of iteration methods for the determination of simple roots of an algebraic equation. Estimates for their order of convergence and efficiency are derived.

1. Introduction. Let the algebraic equation

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1}$$

have only simple roots x_i, $i = 1, \ldots, n$. The following method

$$\begin{aligned} x_i^{k+1} &= x_i^k - \frac{1}{H(x_i^k) - \sum\limits_{\beta_0 \neq i}^{n} \dfrac{1}{x_i^k - x_{\beta_0}^k - \Delta_{\beta_0}^{R,k}}}, \\ H(x_i^k) &= \frac{f'(x_i^k)}{f(x_i^k)}, \\ & i = 1, \ldots, n;\ k = 0, 1, 2, \ldots;\ R = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

where

$$\Delta_s^{R,k} = -\frac{1}{H(x_s^k) - \sum\limits_{l \neq s}^{n} \dfrac{1}{x_s^k - x_l^k - \Delta_l^{R-1,k}}},$$

$$\Delta_j^{0,k} = 0,\ j = 1, \ldots, n;\ k = 0, 1, \ldots$$

for the simultaneous determination of zeroes of the f is given in [1].

Let $0 < q < 1$, $d = \min_{i \neq j} |x_i - x_j|$. Under the condition

$$|x_i^0 - x_i| \leq cq,\ i = 1, 2, \ldots, n$$

and

$$d > 2c(1 + q(2n-1)),$$

(3)
$$c^2 n \left((d-c)(d-2c-2cq)\left(1 - cq\frac{3cq(n-1)}{(d-c)(d-2c-2cq)}\right)\right)^{-1} \le 1$$

the iterative method (2) is convergent with rate of convergence $\tau = 2R+3$.

A class of multi-stage methods based on Weierstrass-Dochev's method

(4)
$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{\beta_0 \neq i}^{n} (x_i^k - x_{\beta_0}^{k-1} - \Delta_{\beta_0}^{R,k-1})},$$

$$i = 1, 2, \ldots, n;\ k = R+1, \ldots$$

is considered in [2]. In the cases when the multiplicity of the roots is not known, it is suitable to modify the calculation procedures and to consider their multipoint analogues, because the multiple roots cause rapid growth of some quantities in a series of one-stage methods.

For other results see [3] and [4].

2. The Method. Define the following computation procedure on the base of the method (2):

(5)
$$x_i^{k+1} = x_i^k - \frac{1}{H(x_i^k) - \sum\limits_{\beta_0 \neq i}^{n} \frac{1}{x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}},$$
$$H(x_i^k) = \frac{f'(x_i^k)}{f(x_i^k)},$$

$$i = 1, 2, \ldots, n;\ k = R+1, \ldots,$$

where

$$\nabla_s^{R,k-1} = -\frac{1}{H(x_s^{k-1}) - \sum\limits_{l \neq s}^{n} \frac{1}{x_s^{k-1} - x_l^{k-2} - \nabla_l^{R-1,k-2}}},$$
$$\nabla_j^{0,t} = 0;\ j, t = 1, 2, \ldots, n.$$

It is shown that these schemes have a superquadratic rate of convergence with respect to the multi-stage degree.

The following theorem gives the rate of convergence of the suggested method

Theorem. *Under the conditions* (3) *and*

(6)
$$|x_i^j - x_i| \le cq^{r^j},\ i = 1, \ldots, n;\ j = 0, 1, \ldots, R+1$$

the following estimate

$$|x_i^k - x_i| \le cq^{r^k},\ i = 1,\dots,n;\ k = R+2,\dots \tag{7}$$

holds, where r is the unique positive root of the equation

$$r^{R+2} - 2r^{R+1} - \cdots - 2r - 1 = 0. \tag{8}$$

P r o o f. We shall prove the theorem by induction on k. We shall consider the $k+1$ approximations

$$x_i^{k+1} - x_i = x_i^k - x_i - \cfrac{1}{\cfrac{1}{x_i^k - x_i} + \sum_{\beta_0 \ne i}^{n} \cfrac{1}{x_i^k - x_{\beta_0}} - \sum_{\beta_0 \ne i}^{n} \cfrac{1}{x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}}$$

$$= x_i^k - x_i - \frac{x_i^k - x_i}{1 + (x_i^k - x_i)\sum_{\beta_0 \ne i}^{n}\left(\cfrac{1}{x_i^k - x_{\beta_0}} - \cfrac{1}{x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}\right)}$$

$$= (x_i^k - x_i)\left(1 - \frac{1}{1 + (x_i^k - x_i)\sum_{\beta_0 \ne i}^{n} \cfrac{x_{\beta_0} - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}{(x_i^k - x_{\beta_0})(x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1})}}\right) \tag{9}$$

$$= \frac{(x_i^k - x_i)^2}{1 + (x_i^k - x_i)\sum_{\beta_0 \ne i}^{n} \cfrac{x_{\beta_0} - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}{(x_i^k - x_{\beta_0})(x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1})}}$$

$$\times \sum_{\beta_0 \ne i}^{n} \frac{x_{\beta_0} - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1}}{(x_i^k - x_{\beta_0})(x_i^k - x_{\beta_0}^{k-1} - \nabla_{\beta_0}^{R,k-1})}.$$

The last sum can be written in the form

$$\sum_{\beta_0\neq i}^{n}\frac{x_{\beta_0}-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})}=$$

$$\sum_{\beta_0\neq i}^{n}\frac{x_{\beta_0}-x_{\beta_0}^{k-1}+\left(\dfrac{f'(x_{\beta_0}^{k-1})}{f(x_{\beta_0}^{k-1})}-\displaystyle\sum_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}\right)^{-1}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})}=$$

$$\sum_{\beta_0\neq i}^{n}\frac{x_{\beta_0}-x_{\beta_0}^{k-1}+\left(\dfrac{1}{x_{\beta_0}^{k-1}-x_{\beta_0}}+\displaystyle\sum_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^{k-1}-x_{\beta_1}}-\sum_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}\right)^{-1}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})}=$$

$$\sum_{\beta_0\neq i}^{n}\frac{(x_{\beta_0}-x_{\beta_0}^{k-1})\left(1-\left(1+(x_{\beta_0}^{k-1}-x_{\beta_0})\displaystyle\sum_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}{(x_{\beta_0}^{k-1}-x_{\beta_1})(x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2})}\right)^{-1}\right)}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})}=$$

$$\sum_{\beta_0\neq i}^{n}\frac{(x_{\beta_0}^{k-1}-x_{\beta_0})^2(x_{\beta_0}-x_i^k)^{-1}(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})^{-1}}{1+(x_{\beta_0}^{k-1}-x_{\beta_0})\displaystyle\sum_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}{(x_{\beta_0}^{k-1}-x_{\beta_1})(x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2})}}$$

$$\times\sum_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}{(x_{\beta_0}^{k-1}-x_{\beta_1})(x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2})}.$$

It should be noted that the sum in the numerator in the last expression is similar to the initial sum with the only difference that R is replaced by $R-1$. Using that dependency recursively we get successively

$$\sum_{\beta_0\neq i}^{n}\frac{x_{\beta_0}-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})}=$$

$$\sum_{\beta_0\neq i}^{n}\frac{(x_{\beta_0}^{k-1}-x_{\beta_0})^2(x_{\beta_0}-x_i^k)^{-1}(x_i^k-x_{\beta_0}^{k-1}-\nabla_{\beta_0}^{R,k-1})^{-1}}{1+(x_{\beta_0}^{k-1}-x_{\beta_0})\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2}}{(x_{\beta_0}^{k-1}-x_{\beta_1})(x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2})}}\times$$

$$\sum_{\beta_1\neq\beta_0}^{n}\frac{(x_{\beta_1}^{k-2}-x_{\beta_1})^2(x_{\beta_1}-x_{\beta_0}^{k-1})^{-1}(x_{\beta_0}^{k-1}-x_{\beta_1}^{k-2}-\nabla_{\beta_1}^{R-1,k-2})^{-1}}{1+(x_{\beta_1}^{k-2}-x_{\beta_1})\sum\limits_{\beta_2\neq\beta_1}^{n}\frac{x_{\beta_2}-x_{\beta_2}^{k-3}-\nabla_{\beta_2}^{R-2,k-3}}{(x_{\beta_1}^{k-2}-x_{\beta_2})(x_{\beta_1}^{k-2}-x_{\beta_2}^{k-3}-\nabla_{\beta_2}^{R-2,k-3})}}\times$$

$$\times\cdots\times \tag{10}$$

$$\sum_{\beta_{R-1}\neq\beta_{R-2}}^{n}\frac{(x_{\beta_{R-1}}^{k-R}-x_{\beta_{R-1}})^2(x_{\beta_{R-1}}-x_{\beta_{R-2}}^{k-(R-1)})^{-1}(x_{\beta_{R-2}}^{k-(R-1)}-x_{\beta_{R-1}}^{k-R}-\nabla_{\beta_{R-1}}^{1,k-R})^{-1}}{1+(x_{\beta_{R-1}}^{k-R}-x_{\beta_{R-1}})\sum\limits_{\beta_R\neq\beta_{R-1}}^{n}\frac{x_{\beta_R}-x_{\beta_R}^{k-(R+1)}}{(x_{\beta_{R-1}}^{k-R}-x_{\beta_R})(x_{\beta_{R-1}}^{k-R}-x_{\beta_R}^{k-(R+1)})}}\times$$

$$\sum_{\beta_R\neq\beta_{R-1}}^{n}\frac{x_{\beta_R}-x_{\beta_R}^{k-(R+1)}}{(x_{\beta_{R-1}}^{k-R}-x_{\beta_R})(x_{\beta_{R-1}}^{k-R}-x_{\beta_R}^{k-(R+1)})}.$$

Let us estimate from above the absolute value of the items in (10). We first show that the estimates,

$$|\nabla_s^{p,k}|\leq 2cq,\ \ s=1,\ldots,n\ \ p=0,1,\ldots \tag{11}$$

hold. The proof can be done by induction on p. When $p=0$, the inequalities (11) are satisfied since $\nabla_s^{0,k}=0$ for $s=1,\ldots,n$. Let us assume that $|\nabla_s^{m-1,k-1}|\leq 2cq,\ \ s=1,\ldots,n$. Then we obtain for $\nabla_s^{m,k},\ \ s=1,\ldots,n$

$$\nabla_s^{m,k}=-\frac{x_s^k-x_s}{\sum\limits_{t=1}^{n}\frac{x_s^k-x_s}{x_s^k-x_t}-(x_s^k-x_s)]\sum\limits_{t\neq s}^{n}\frac{1}{x_s^k-x_t^{k-1}-\nabla_t^{m-1,k-1}}}.$$

It follows from the last equality and (7) that

$$|\nabla_s^{m,k}| \le \frac{cq}{1 - cq\sum_{t\neq s}^{n}\left(\frac{1}{|x_s^k - x_t|} + \frac{1}{|x_s^k - x_t^{k-1}| - |\nabla_t^{m-1,k-1}|}\right)}$$

From

$$\begin{aligned} &|x_s^k - x_t^{k-1}| \ge |x_s - x_t| - |x_s - x_s^k| - |x_t - x_t^{k-1}| \ge d - 2cq, \\ &|x_s^k - x_t| > d - cq \end{aligned} \tag{12}$$

we finally obtain

$$|\nabla_s^{m,k}| \le \frac{cq}{1 - cq(n-1)\left(\frac{1}{d-c} + \frac{1}{d-2c-2cq}\right)} \le \frac{cq}{1 - \dfrac{2cq(n-1)}{d-2c(1+q)}},$$

and for the validity of the inequality $|\nabla_s^{m,k}| \le 2cq$, the following conditions are sufficient:

$$\left(1 - \frac{2cq(n-1)}{d-2c(1+q)}\right)^{-1} \le 2, \;\; 1 - \frac{2cq(n-1)}{d-2c(1+q)} > 0.$$

These two conditions are satisfied if $d > 2c(1 + q(2n-1))$. Now we are able to estimate the sums in (10). In view of (7), (11) and (12) for an arbitrary item in (10), we obtain

$$\frac{1}{|x_\mu - x_\nu^\rho||x_\nu^\rho - x_\mu^{\rho-1} - \nabla_\mu^{s,\rho-1}||1 + (x_\mu^{\rho-1} - x_\mu)\sum\limits_{\lambda\neq\mu}^{n} \dfrac{x_\lambda - x_\lambda^{\rho-2} - \nabla_\lambda^{s-1,\rho-2}}{(x_\mu^{\rho-1} - x_\lambda)(x_\mu^{\rho-1} - x_\lambda^{\rho-2} - \nabla_\lambda^{s-1,\rho-2})}|} \le$$

$$\frac{1}{|x_\mu - x_\nu^\rho|(|x_\nu^\rho - x_\mu^{\rho-1}| - |\nabla_\mu^{s,\rho-1}|)(1 - |x_\mu^{\rho-1} - x_\mu|\sum\limits_{\lambda\neq\mu}^{n} \dfrac{|x_\lambda - x_\lambda^{\rho-2}| + |\nabla_\lambda^{s-1,\rho-2}|}{(d-c)(d-2c-2cq)})} \le$$

$$\frac{1}{(d-c)(d-2c-2cq)(1 - cq(n-1)\dfrac{3cq}{(d-c)(d-2c-2cq)})} = A.$$

It follows from the last inequality, (7),(9),(10) and (3) that

$$|x_i^{k+1} - x_i| \le \left(cq^{r^k}\right)^2 \left(cq^{r^{k-1}}\right)^2 \dots \left(cq^{r^{k-R}}\right)^2 cq^{r^{k-(R+1)}} (nA)^{R+1}$$

$$= cq^{r^{k-R-1}(1+2r+\dots+2r^{R+1})} \left(c^2 nA\right)^{R+1} \le cq^{r^{k-R-1}(1+2r+\dots+2r^{R+1})},$$

since $c^2nA \leq 1$, according to the assumptions. From (8) we finally obtain

$$|x_i^{k+1} - x_i| \leq cq^{r^{k-R-1}r^{R+2}} = cq^{r^{k+1}}.$$

Thus the theorem is proved. □

Let us note that the change of the one-stage method (2), results in a lower speed of convergence and at the same time, the existence of multiple roots ensures that the scheme (5) has a better stability than the one-stage iteration.

The values obtained for $r = r(R)$ are given in the following table

R	r
0	2.4142...
1	2.831...
2	2.9476...
...	...
8	2.99993...

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