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ON SUPERCONNECTED SPACES

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Abstract. A topological space is called an S-space (or has the S-topology) if every subset which contains a non-void open subset is open. In this paper we study the class of connected S-spaces', which is a specification of the classical notion of connected spaces. We show that superconnected spaces are exactly the spaces where all non-void open sets form a filter. We prove that a space is superconnected if and only if it is an irreducible α-space or equivalently if and only if it is a connected semi-space. Dense in themselves superconnected spaces are investigated as well as spaces, where the non-void open subsets form an ultrafilter.

Preliminaries. Throughout this paper we consider spaces on which no separation axioms are assumed unless explicitly stated. The word "iff" means "if and only if", a "space" will always mean a topological space and the symbol □ is used to indicate the end or omission of a proof. The topology of a space is denoted by τ and (X, τ) will be replaced by X if there is no chance for confusion.

Next we recall some definitions.

For $A \subset X$, the closure and the interior of A in X are denoted by $\overline{A}$ and Int$A$, respectively. Recall that A is said to be *regular open* (resp. *preopen* [8], *semi-open* [5]) if $A = \mathrm{Int}\overline{A}$ (resp. $A \subset \mathrm{Int}\overline{A}$, $A \subset \overline{\mathrm{Int}A}$). In [10], a topology τ_α has been introduced by defining its open sets to be the α-sets, that is the sets $A \subset X$ with $A \subset \mathrm{Int}\overline{\mathrm{Int}A}$. Such sets are usually called *α-open*. The complement of a regular open set (resp. preopen, semi-open, α-open) is called *regular closed* (resp. *preclosed*, *semi-closed*, *α-closed*).

A space X will be called an *α-space* (resp. *semi-space*) iff every α-open (resp. semi-open) subset of X is open.

We denote the set of all accumulation points of A by $d(A)$. A set with no accumulation points will be called *non-accumulative*. Recall that a set $A \subset X$ is called *dense in itself* (*in sich dicht, dense en soi*) [3] if $A \subset d(A)$ or equivalently if A has no isolated points.

A non-void space X is *irreducible* [1] if every two non-void open subset of X intersect or equivalently if every non-void (semi-)open subset of X is dense. An irreducible space is called sometimes *hyperconnected*. The space X is called *submaximal* [2] if every dense subset of X is open. A space X is *extremally disconnected* (or *extremal*) iff the closure of each open subset of X is open or equivalently iff every semi-open set is preopen. It is called an S-*space* (or has the *S-topology*) [7] if every subset which contains a non-void open subset is open. A space in which every set is either open or closed is called a *door space*.

Given a set X, a non-empty collection $\mathcal{I}$ of subsets of X is called an *ideal* if (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ (heredity), and (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (finite additivity).

Superconnectedness.

Definition 1. *A space X is called* **superconnected** *iff it is a connected S-space.*

Theorem 2.1. *For a space X the following are equivalent:*
(1) *X is superconnected.*
(2) *X is a connected semi-space.*
(3) *X is an irreducible α-space.*
(4) *X is an irreducible S-space.*

Proof. (1) $\Rightarrow$ (2) Let A be a non-void semi-open subset of X. Then for some non-void and open U in X we have $U \subset A \subset \overline{U}$. Since X is an S-space, then A is open. This shows that X is a semi-space. □

(2) $\Rightarrow$ (3) Since every α-open set is semi-open, then X is an α-space. Let A and B be two non-void open subsets of X. We need to show that they intersect. Since X is extremally disconnected, then $\overline{A}$ is open and since X is connected, then $\overline{A}$ is X itself or equivalently A is dense. Thus B meets A.

(3) $\Rightarrow$ (1) Let $U \subset A \subset X$, where U is non-void and open. Since X is irreducible, then $\overline{U} = X$ and thus $U \subset A \subset \overline{U}$. Hence A is semi-open. Since every irreducible space is extremally disconnected, then A is α-open and hence open, since X is an α-space. Thus X is an S-space. On the other hand every irreducible space is connected.

(1) $\Leftrightarrow$ (4) follows from above and from the fact that every irreducible space is connected. □

Theorem 2.2. *For a non-void space X the following are equivalent:*
(1) *X is superconnected.*
(2) *All non-void open subsets of X form a filter on X.*
(3) *For some filter $\mathcal{F}$ on X, $\mathcal{F} \cup \{\emptyset\} = \tau$.*

(4) *The family of all proper closed subsets of X form a topological ideal on X.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{F} = \tau \setminus \{\emptyset\}$. Since X is non-void, then $X \in \mathcal{F}$. If U and $V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$, since X is irreducible by Theorem 2.1. If $U \in \mathcal{F}$ and $U \subset V \subset X$, then $V \in \mathcal{F}$, since X is an S-space again by Theorem 2.1. Thus $\mathcal{F}$ is a filter.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Assume that X is not connected. Let U be a proper non-void clopen set. Then U and $X \setminus U \in \tau \setminus \{\emptyset\} = \mathcal{F}$ and thus $\emptyset = U \cap (X \setminus U) \in \mathcal{F}$. By contradiction X is connected. Now let A be a non-void semi-open subset of X. Then $\text{Int}A \neq \emptyset$ and hence $\text{Int}A \in \tau \setminus \{\emptyset\} = \mathcal{F}$. Thus $A \in \mathcal{F}$. This shows that A is open and hence X is a semi-space. Hence X is superconnected by Theorem 2.1.

(2) $\Leftrightarrow$ (4) is trivial. □

Theorem 2.3. *For a space X with cardinality at least two the following are equivalent:*

(1) *X is superconnected.*

(2) *X is non-discrete S-space.*

Proof. (1) $\Rightarrow$ (2) It is enough to show that X is not discrete. But this is clear since, every discrete space with at least two points is disconnected.

(2) $\Rightarrow$ (1) Assume that X is disconnected. Let U be a non-void proper clopen subset of X. Let $A \subset X$. Since X is an S-space, then $U \cup A$ and $(X \setminus U) \cup A$ are open sets. Then $A = (U \cup A) \cap ((X \setminus U) \cup A)$ is open and hence X is discrete. By contradiction X is connected. Thus X is superconnected. □

Remark 2.4. Note that the notion of superconnectedness is independent from the notion of ultraconnectedness defined in [9] and in [11]. By definition, a space X is *ultraconnected* iff every two non-void closed subsets of X intersect. For example the space $X = \{a, b, c\}$, where the only non-trivial open subset is $\{a\}$ is ultraconnected, but not superconnected. On the other hand the same space $X = \{a, b, c\}$, where the non-trivial open subsets are $\{a\}$, $\{a, b\}$ and $\{a, c\}$ is superconnected, but not ultraconnected. Ultraconnected spaces are studied in [4] and in [6] under the name of strongly connected spaces.

Theorem 2.5. *Let $\mathcal{F}$ be a filter on X and $\tau = \mathcal{F} \cup \{\emptyset\}$ the corresponding superconnected topology. Then the following conditions are equivalent:*

(1) *(X, τ) is a T_0-space.*

(2) *$\cap\{A \mid A \in \mathcal{F}\}$ is either empty or a singleton.*

Also the following conditions are equivalent:

(a) *(X, τ) is a T_1-space.*

(b) *$\cap\{A \mid A \in \mathcal{F}\} = \emptyset$ (i.e. $\mathcal{F}$ is a free filter).*

P r o o f. (1) $\Rightarrow$ (2) Assume that the set $T = \cap\{A \mid A \in \mathcal{F}\}$ contains two different points a and b. If $\emptyset \neq U \in \tau$, then $U \in \mathcal{F}$ and thus $\{a, b\} \subset U$. Hence (X, τ) is not a T_0-space. Thus by contradiction condition (2) is satisfied.

(2) $\Rightarrow$ (1) By (2) for some $p \in X$, we have $T \subset \{p\}$. If a and $b \neq p$, and $a \neq b$, then for some $U \in \mathcal{F}$, we have $a \notin U$. Thus $U \cup \{b\}$ is a neighbourhood of b, which does not contain a and $U \cup \{p\}$ is a neighbourhood of p, which does not contain a. Hence (X, τ) is a T_0-space.

(a) $\Rightarrow$ (b) Let $p \in X$. For every $a \neq p$, there exists by (a) a set $U \in \mathcal{F}$ such that $p \notin U$ and $a \in U$. Then $T \subset U$ and hence $p \notin T$. Thus $T = \emptyset$.

(b) $\Rightarrow$ (a) Let a and b be two different points in X. Then by (b) for some $U \in \mathcal{F}$, we have $a \notin U$. Hence $U \cup \{b\}$ is an open neighbourhood of b, which does not contain a. This shows that (X, τ) is a T_1-space. □

2.6. *Let $\mathcal{F}$ be a filter on X and $\tau = \mathcal{F} \cup \{\emptyset\}$ the corresponding superconnected topology. Then the following conditions are equivalent:*

(1) *(X, τ) is not dense in itself.*

(2) *$\mathcal{F}$ is the point generated ultrafilter $\hat{p} = \{A \subset X \mid p \in A\}$.*

If the condition (1) = (2) holds, then the space (X, τ) has exactly one isolated point and this point is p itself.

P r o o f. (1) $\Rightarrow$ (2) By (1) the space (X, τ) has at least one isolated point. If a and b are isolated points in X, then $\{a\}$ and $\{b\} \in \mathcal{F}$ and hence $\{a\} \cap \{b\} \in \mathcal{F}$. Since this set is non-void, then $a = b$. Hence X has exactly one isolated point p. Thus $\{p\}$ is a non-void open subset of X and it belongs to the filter $\mathcal{F}$. Hence $\hat{p} = \mathcal{F}$.

(2) $\Rightarrow$ (1) Since $\emptyset \neq \{p\} \in \mathcal{F}$, then $\{p\} \in \tau$ and thus p is an isolated point of X. Hence X is not dense in itself. □

Theorem 2.7. *Let $\mathcal{F}$ be a filter on X and let $\tau = \mathcal{F} \cup \{\emptyset\}$. If $A \in \mathcal{F}$, then $X \setminus A$ is a non-accumulative subset of (X, τ), i.e. it is closed and discrete.*

P r o o f. Since A is open, then $X \setminus A$ is closed. If $x \in X \setminus A$, then $(A \cup \{x\}) \cap (X \setminus A) = \{x\}$ and $A \cup \{x\} \in \mathcal{F} \subset \tau$. Thus $X \setminus A$ is also discrete. □

Corollary 2.8. *Let $\mathcal{F}$ be an ultrafilter on X and let $\tau = \mathcal{F} \cup \{\emptyset\}$. Every subset of $X = (X, \tau)$ is either open or non-accumulative. Particularly X is a door space.*

P r o o f. Since $\mathcal{F}$ is an ultrafilter, then either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. In the first case A is open and in the second non-accumulative by Theorem 2.7. Thus every subspace of X is either open or closed or equivalently X is a door space. □

Theorem 2.9. *Let $\mathcal{F}$ be an ultrafilter on X and let $\tau = \mathcal{F} \cup \{\emptyset\}$. Then $\mathcal{F}$ is the collection $\mathcal{D}(X)$ of all dense subsets of (X, τ) and thus X is submaximal. Every non-void dense in itself subset of X belongs to the ultrafilter $\mathcal{F}$. If $\mathcal{F}$ is a free ultrafilter,* *then it is the collection of all non-void dense in themselves subsets of X and in this case X is a maximally dense in itself space.*

Proof. If $A \in \mathcal{F}$, then A is open and non-void and hence dense, since X is irreducible. Assume now that A is a dense subset of X. If A is not open then it is closed by Corollary 2.8. Then $X = \overline{A} = A$ and thus X is not open. By contradiction A is open. Since on the other hand A is non-void, then $A \in \mathcal{F}$. Hence $\mathcal{F}$ is the collection of all dense subsets of X.

Next let A be dense in itself and non-void, i.e. $\emptyset \neq A \subset d(A)$. Then $d(A) \neq \emptyset$ and A is open by Theorem 2.7. Hence $A \in \mathcal{F}$.

Finally let $\mathcal{F}$ be a free ultrafilter. Let $A \in \mathcal{F}$. Since $\mathcal{F}$ is free, then $\mathrm{card}(A) \geq 2$ (A is actually infinite) and X is dense in itself by Theorem 2.6. Since every open subset of a dense in itself space with at least two points is dense in itself, then A is dense in itself. □

Theorem 2.10. *Every irreducible submaximal space X is superconnected.*

Proof. Let A be α-open. Then A is clearly preopen and so $A \subset \mathrm{Int}\overline{A}$. Since A is dense in $\overline{A}$ and $\overline{A}$ is submaximal (every subspace of a submaximal space is submaximal), then A is open $\overline{A}$, hence also in $\mathrm{Int}\overline{A}$. Thus A is open in X, since $\mathrm{Int}\overline{A}$ is open in X. Hence X is an α-space. Thus by Theorem 2.1 X is superconnected. □

Next we give an example of a superconnected space that is not submaximal, not even T_0.

Example 2.11. Let $X = \{a, b, c\}$, where the only non-trivial open subset is $\{a, b\}$. X is not submaximal, since the dense subset $\{b, c\}$ is not open. It is not a T_0-space, since the points a and b have common neighborhoods. But X is clearly superconnected.

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