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OPERATIONAL CALCULS FOR MODIFIED ERDÉLYI–KOBER OPERATORS

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ABSTRACT. In this paper an operational calculus for the operators $\Delta_{(\delta)} = t^{-\beta\gamma} D_\beta^\delta t^{\beta\gamma}$ and $A_{(\delta)} = t^{-\beta\gamma-\beta\delta} D_\beta^\delta t^{\beta\gamma+\beta\delta}$ is developed, following an algebraic process similar to the one given by Mikusinski and obtaining operational rules to them.

1. Introduction. In 1949, J. Mikusinski [11] used the operator $D = \dfrac{d}{dt}$ as a basis of Operational Calculus. Since then, this theory has been extended to more general operators, but its development has always implied the construction of an algebraic framework in which the considered operator is included. In this sense, and related to the Bessel type operators, we mention, among others, the papers of V. A. Ditkin and A. P. Prudnikov [7], E. L. Koh [9], I. H. Dimovski [4], J. Rodríguez [12], and J. J. Betancor [3]. Recently, V. Kiryakova [8] has applied this method to the modified operator of Erdélyi-Kober:

$$L_{(\delta)} = t^{\beta\delta} I_\beta^{\gamma,\delta} = t^{-\beta\gamma} I_\beta^\delta t^{\beta\gamma}$$

in the space of functions

$$\mathcal{C}_{-\beta(\gamma+1)} = \left\{ f(t) = t^p \widetilde{f}(t) | p > -\beta(\gamma+1) \text{ and } \widetilde{f}(t) \in \mathcal{C}\left([0,\ \infty)\right) \right\},$$

where $I_\beta^{\gamma,\delta} = t^{-\beta\gamma-\beta\delta} I_\beta^\delta t^{\beta\gamma}$ and I_β^δ is the generalized Riemann-Liouville operator of fractional integration [13].

In this work, we study the operational calculus for the operators $\Delta_{(\delta)} = t^{-\beta\gamma} D_\beta^\delta t^{\beta\gamma}$ and $A_{(\delta)} = t^{-\beta\gamma-\beta\delta} D_\beta^\delta t^{\beta\gamma+\beta\delta}$ following an algebraic process similar to J. Mikusinski's.

2. Fractional Integration and Differentiation Operators. The Riemann-Liouville fractional integral operator of order $\delta \geq 0$ is defined [13] by:

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\xi)^{\delta-1} f(\xi) d\xi \quad (\delta > 0) \tag{2.1}$$

$$I^0 f(t) = f(t) \quad (\delta = 0)$$

and its corresponding fractional derivative of order $\delta > 0$ by:

$$D^\delta f(t) = D^n \, I^{n-\delta} f(t) \qquad (n-1 < \delta \le n). \tag{2.2}$$

On the other hand, there exists the generalized integral operator I_β^δ $(\beta > 0, \delta \ge 0)$, defined by

$$I_\beta^\delta f(t) = \frac{\beta}{\Gamma(\delta)} \int_0^t (t^\beta - \xi^\beta)^{\delta-1} \xi^{\beta-1} f(\xi) d\xi \quad (\delta > 0) \tag{2.3}$$

$$I_\beta^0 f = f \qquad (\delta = 0),$$

and its generalized fractional derivative of order $\delta > 0$, like:

$$D_\beta^\delta f(t) = D_\beta^n I_\beta^{n-\delta} f(t) \quad (n-1 < \delta \le n), \ (\beta > 0) \tag{2.4}$$

Going on in this generalization, we have finally the Erdélyi–Kober operator of fractional integration with $\beta > 0, \delta \ge 0$ and $\gamma \in \mathbb{R}$ given by

$$\begin{aligned} I_\beta^{\gamma,\delta} f(t) &= t^{-\beta\gamma-\beta\delta} I_\beta^\delta t^{\beta\gamma} f(t) \\ &= \frac{\beta}{\Gamma(\delta)} t^{-\beta\gamma-\beta\delta} \int_0^t (t^\beta - \xi^\beta)^{\delta-1} \xi^{\beta\gamma+\beta-1} f(\xi) d\xi, \ (\delta > 0) \end{aligned} \tag{2.5}$$

$$I_\beta^{\gamma,0} f(t) = f(t) \qquad (\delta = 0)$$

and the operator used by V. Kiryakova [8]:

$$\Delta_{(\delta)} = t^{-\beta\gamma} D_\beta^\delta t^{\beta\gamma}. \tag{2.6}$$

Moreover, we use another type of operators, the argument power operator given by

$$T^\beta f(t) = f(t^\beta), \qquad (\beta \in \mathbb{R}^+), \ \ (f : [0,\infty) \longrightarrow \mathbb{C}), \tag{2.7}$$

whose main properties are:

$$T^\alpha T^\beta = T^\beta T^\alpha = T^{\alpha\beta} \tag{2.8}$$

$$I^\delta = T^{\frac{1}{\beta}} I_\beta^\delta T^\beta, \quad I_\beta^\delta = T^\beta I^\delta T^{\frac{1}{\beta}} \tag{2.9}$$

$$D^\delta = T^{\frac{1}{\beta}} D_\beta^\delta T^\beta, \quad D_\beta^\delta = T^\beta D^\delta T^{\frac{1}{\beta}}. \tag{2.10}$$

For more details see [1].

3. The extension of $\mathcal{C}_{\beta(\delta-\gamma-1)}$ to the quotient field. Given $\beta > 0,\ \ \delta \geq 1$ and $\gamma \in \mathbb{R}$, we define the function spaces:

$$\mathcal{C}_{\beta(\delta-\gamma-1)} = \Big\{ f(t) = \sum_{k=1}^{\infty} a_k t^{\beta(k\delta-\gamma-1)} \text{ absolutely convergent on compact subsets of} [0,\infty) \Big\} \tag{3.1}$$

and $\mathcal{C}_{\delta-1}$, introduced in [1] and given by

$$\mathcal{C}_{\delta-1} = \Big\{ \widetilde{f}(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1} \text{ absolutely convergent on compact subsets of} [0,\infty) \Big\} \tag{3.2}$$

If we take into account that the linear operators $\Delta_{(\delta)}$ and D^{δ} are linear automorphisms acting on $\mathcal{C}_{\beta(\delta-\gamma-1)}$ and $\mathcal{C}_{\delta-1}$ respectively, that the operator $T^{\frac{1}{\beta}} t^{\beta\gamma}$ is a linear isomorphism from $\mathcal{C}_{\beta(\delta-\gamma-1)}$ to $\mathcal{C}_{\delta-1}$, that by (2.10)

$$T^{\frac{1}{\beta}} t^{\beta\gamma} \Delta_{(\delta)} = D^{\delta} T^{\frac{1}{\beta}} t^{\beta\gamma}$$

and that $*$ is the convolution for the operator D^{δ} defined by

$$(\widetilde{f} * \widetilde{g})(t) = \frac{D^{\delta}}{\Gamma(\delta)} \int_0^t \widetilde{f}(t-\xi) \widetilde{g}(\xi) d\xi \quad \widetilde{f}, \widetilde{g} \in \mathcal{C}_{\delta-1} \tag{3.3}$$

then, according to Meller's similarity theorem [5], we can state the following

Proposition 3.1. *The operation $\circledast$ defined by*

$$\circledast : \mathcal{C}_{\beta(\delta-\gamma-1)} \times \mathcal{C}_{\beta(\delta-\gamma-1)} \longrightarrow \mathcal{C}_{\beta(\delta-\gamma-1)}$$

$$f(t) \circledast g(t) = t^{-\beta\gamma} T^{\beta} \Big[\Big(T^{\frac{1}{\beta}} t^{\beta\gamma} f(t) \Big) * \Big(T^{\frac{1}{\beta}} t^{\beta\gamma} g(t) \Big) \Big]$$

is a convolution for the operator $\Delta_{(\delta)}$.

In particular, for the case of power function of the type (3.1) the convolution turns out to be

$$t^{\beta(k\delta-\gamma-1)} \circledast t^{\beta(m\delta-\gamma-1)} = \frac{\Gamma(k\delta)\Gamma(m\delta)}{\Gamma(\delta)\Gamma[(k+m-1)\delta]} t^{\beta[\delta(k+m-1)-\gamma-1]} \ (k, m \in \mathbb{N})$$

and satisfies the following properties for f, g and $h \in \mathcal{C}_{\beta(\delta-\gamma-1)}$:

$$\begin{array}{rl} i) & f \circledast g = g \circledast f \\ ii) & (f \circledast g) \circledast h = f \circledast (g \circledast h) \\ iii) & f \circledast (g+h) = f \circledast g + f \circledast h \\ iv) & t^{\beta(\delta-\gamma-1)} \circledast f(t) = f(t) \\ v) & f \circledast g = 0 \iff f = 0 \text{ or } g = 0. \end{array}$$

With this, we can state:

Proposition 3.2. $(\mathcal{C}_{\beta(\delta-\gamma-1)}, +, \circledast)$ *is a unitary commutative ring without divisors of zero.*

The above condition allows us to extend $\mathcal{C}_{\beta(\delta-\gamma-1)}$ to its quotient field

$$\mathcal{M}_{\beta(\delta-\gamma-1)} = \mathcal{C}_{\beta(\delta-\gamma-1)} \times \left(\mathcal{C}_{\beta(\delta-\gamma-1)} - \{0\}\right) / \sim$$

where the equivalence relation $\sim$ is defined as usual by

$$(f, g) \sim (h, p) \iff f \circledast p = g \circledast h.$$

According to Mikusinski, we interprete as operators the elements of $\mathcal{M}_{\beta(\delta-\gamma-1)}$ and in what follows, we will denote the pair (f, g) by f/g.

If we define in $\mathcal{M}_{\beta(\delta-\gamma-1)}$ the usual operations of addition, multiplication, and product by scalars by

$$\begin{aligned} \frac{f}{g} + \frac{h}{p} &= \frac{f \circledast p + g \circledast h}{g \circledast p} \\ \frac{f}{g} \cdot \frac{h}{p} &= \frac{f \circledast h}{g \circledast p} \\ \lambda \cdot \frac{f}{g} &= \frac{\lambda f}{g}, \end{aligned}$$

then $\mathcal{M}_{\beta(\delta-\gamma-1)}$ turns out to be an algebra.

The quotient set $\mathcal{M}_{\beta(\delta-\gamma-1)}$ contains a subset $\mathcal{M}'_{\beta(\delta-\gamma-1)}$ which is isomorphic to $\mathcal{C}_{\beta(\delta-\gamma-1)}$, via the mapping:

$$\mathcal{M}'_{\beta(\delta-\gamma-1)} \subset \mathcal{M}_{\beta(\delta-\gamma-1)} \longrightarrow \mathcal{C}_{\beta(\delta-\gamma-1)}$$

$$\frac{t^{\beta(\delta-\gamma-1)} \circledast f(t)}{t^{\beta(\delta-\gamma-1)}} = \frac{f(t)}{t^{\beta(\delta-\gamma-1)}} \longrightarrow f(t).$$

Therefore, the operators of the form $\dfrac{f(t)}{t^{\beta(\delta-\gamma-1)}}$ constitute a subring of $\mathcal{M}_{\beta(\delta-\gamma-1)}$.

4. An Operational Calculus. To prove that the operator $\Delta_{(\delta)}$ defined in (2.6) belongs to $\mathcal{M}_{\beta(\delta-\gamma-1)}$ we will use the modified operator of Erdélyi-Kober [8]:

$$L_{(\delta)} = t^{\beta\delta} I_\beta^{\gamma,\delta} = t^{-\beta\gamma} I_\beta^{\delta} t^{\beta\gamma} \tag{4.1}$$

which is the right inverse operator of $\Delta_{(\delta)}$, i. e,

$$\Delta_{(\delta)} L_{(\delta)} f = f, \tag{4.2}$$

for every $f \in \mathcal{C}_{\beta(\delta-\gamma-1)}$.

But in general, we have:

Proposition 4.1. *For each $f(t) \in \mathcal{C}_{\beta(\delta-\gamma-1)}$, the following equality holds*

$$f(t) = L_{(\delta)}\Delta_{(\delta)} f(t) + \left[t^{-\beta(\delta-\gamma-1)} f(t)\right]_{t=0} t^{\beta(\delta-\gamma-1)}. \tag{4.3}$$

P r o o f.

$$\begin{aligned}
L_{(\delta)}\Delta_{(\delta)} f(t) &= t^{-\beta\gamma} I_\beta^{\delta} t^{\beta\gamma} t^{-\beta\gamma} D_\beta^n I_\beta^{n-\delta} t^{\beta\gamma} f(t) \\
&= t^{-\beta\gamma} T^\beta I^\delta T^{\frac{1}{\beta}} T^\beta D^n T^{\frac{1}{\beta}} T^\beta I_\beta^{n-\delta} T^{\frac{1}{\beta}} t^{\beta\gamma} \Big(\sum_{k=1}^{\infty} a_k t^{\beta(k\delta-\gamma-1)} \Big) \\
&= t^{-\beta\gamma} T^\beta I^\delta D^n I^{n-\delta} \Big(\sum_{k=1}^{\infty} a_k t^{k\delta-1} \Big) = t^{-\beta\gamma} T^\beta I^\delta D^\delta \Big(\sum_{k=1}^{\infty} a_k t^{k\delta-1} \Big) \\
&= t^{-\beta\gamma} T^\beta \Big(\sum_{k=1}^{\infty} a_k t^{k\delta-1} - a_1 t^{\delta-1} \Big) = \sum_{k=1}^{\infty} a_k t^{\beta(k\delta-\gamma-1)} - a_1 t^{\beta(\delta-\gamma-1)} \\
&= f(t) - \left[t^{-\beta(\delta-\gamma-1)} f(t)\right]_{t=0} t^{\beta(\delta-\gamma-1)}.
\end{aligned}$$

To generalize this proposition, we define

$$f_j(0) = \left[t^{-\beta(\delta-\gamma-1)} \Delta_{(\delta)}^{j-1} f(t)\right]_{t=0} \tag{4.4}$$

and then we obtain by induction on m the next assertion

Proposition 4.2. *For $f \in \mathcal{C}_{\beta(\delta-\gamma-1)}$ and $m \in \mathbb{N}$ the following equality holds,*

$$f(t) = L_{(\delta)}^m \Delta_{(\delta)}^m f(t) + \sum_{j=1}^{m} L_{(\delta)}^{j-1} f_j(0) t^{\beta(\delta-\gamma-1)}. \tag{4.5}$$

Proposition 4.3. *For $f \in \mathcal{C}_{\beta(\delta-\gamma-1)}$, we have*

$$\frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{\beta(2\delta-\gamma-1)} \circledast f(t) = L_{(\delta)} f(t). \tag{4.6}$$

P r o o f. From the definition of $\circledast$ we have,

$$\frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{\beta(2\delta-\gamma-1)} \circledast f(t) = t^{-\beta\gamma} T^{\beta} \left[\left(T^{\frac{1}{\beta}} t^{\beta\gamma} \frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{\beta(2\delta-\gamma-1)} \right) * \left(T^{\frac{1}{\beta}} t^{\beta\gamma} f(t) \right) \right]$$

$$= t^{-\beta\gamma} T^{\beta} \left[\frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{2\delta-1} * T^{\frac{1}{\beta}} t^{\beta\gamma} f(t) \right]$$

but, by (2.5) of [1], we obtain,

$$t^{-\beta\gamma} T^{\beta} I^{\delta} T^{\frac{1}{\beta}} t^{\beta\gamma} f(t) = t^{-\beta\gamma} I^{\delta}_{\beta} t^{\beta\gamma} f(t) = L_{(\delta)} f(t).$$

Proposition 4.4. *If $f \in \mathcal{C}_{\beta(\delta-\gamma-1)}$ and $k \in \mathbb{N}$, then*

$$L^{k}_{(\delta)} f(t) = \frac{\Gamma(\delta)}{\Gamma(k\delta+\delta)} t^{\beta(k\delta+\delta-\gamma-1)} \circledast f(t) \tag{4.7.}$$

and therefore, the operators $L^{k}_{(\delta)}$ belong to $\mathcal{M}_{\beta(\delta-\gamma-1)}$.

P r o o f. We can see this by induction on k. For $k = 1$, it was proved in Prop. 4.3. For $k \neq 1$, it is as follows

$$\begin{aligned} L_{(\delta)}(L^{k}_{(\delta)} f(t)) &= \frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{\beta(2\delta-\gamma-1)} \circledast \frac{\Gamma(\delta)}{\Gamma(k\delta+\delta)} t^{\beta(k\delta+\delta-\gamma-1)} \circledast f(t) \\ &= \frac{\Gamma(\delta)}{\Gamma\Big[(k+1)\delta+\delta\Big]} t^{\beta[(k+1)\delta+\delta-\gamma-1]} \circledast f(t) = L^{k+1}_{(\delta)} f(t). \end{aligned}$$

We will call V the inverse operator of $L_{(\delta)}$ for $\circledast$ on $\mathcal{M}_{\beta(\delta-\gamma-1)}$, i. e.

$$V = \frac{\Gamma(2\delta)}{\Gamma(\delta)} \frac{t^{\beta(\delta-\gamma-1)}}{t^{\beta(2\delta-\gamma-1)}} \tag{4.8}$$

and V^{k} the k^{th} iteration of V,

$$V^{k} = \frac{\Gamma(k\delta+\delta)}{\Gamma(\delta)} \frac{t^{\beta(\delta-\gamma-1)}}{t^{\beta(k\delta+\delta-\gamma-1)}}. \tag{4.9}$$

For operator (4.8) we can state the following

Proposition 4.5. *If* $f(t) \in \mathcal{C}_{\beta(\delta-\gamma-1)}$, *then*

$$V f(t) = \Delta_{(\delta)} f(t) + \left[t^{-\beta(\delta-\gamma-1)} f(t)\right]_{t=0} V. \tag{4.10}$$

P r o o f. Applying V to both sides in (4.3) we arrive at (4.10).

Its generalization will be deduced in the next proposition just applying V k times to (4.5).

Proposition 4.6. *If* $k \in \mathbb{N}$ *and* $f(t) \in \mathcal{C}_{\beta(\delta-\gamma-1)}$, *then*

$$V^k f(t) = \Delta^k_{(\delta)} f(t) + \sum_{j=1}^{k} f_j(0) V^{k+1-j}, \tag{4.11}$$

given $f_j(0)$ *by* (4.4).

5. Operational rules. Al-Bassam proved in [2] that $(D^\delta \pm a) f(t) = 0$, $a > 0$, are two differential equations with solutions

$$\widetilde{y}_i(t) = E_\delta\left((-1)^i a,\ t\right) = \sum_{n=1}^{\infty} \frac{\left[(-1)^i a\right]^{n-1} t^{n\delta-1}}{\Gamma(n\delta)} \quad i = 1, 2,$$

the so-called Mittag–Leffler functions.

A similar result for the operator $\Delta_{(\delta)} = t^{-\beta\gamma} D^\delta_\beta t^{\beta\gamma}$ is:

Proposition 5.1. *The differential equations with* $a > 0$

$$(\Delta_{(\delta)} - a) f(t) = 0 \tag{5.1}$$

and

$$(\Delta_{(\delta)} + a) f(t) = 0 \tag{5.2}$$

have the following solutions

$$y_1(t) = F_\delta(a,\ t^\beta) = t^{-\beta\gamma} T^\beta E_\delta(a,\ t) = \sum_{n=1}^{\infty} \frac{a^{n-1} t^{\beta(n\delta-\gamma-1)}}{\Gamma(n\delta)} \tag{5.3}$$

and

$$y_2(t) = F_\delta(-a,\ t^\beta) = t^{-\beta\gamma} T^\beta E_\delta(-a,\ t) = \sum_{n=1}^{\infty} \frac{(-a)^{n-1} t^{\beta(n\delta-\gamma-1)}}{\Gamma(n\delta)}, \tag{5.4}$$

called generalized Mittag–Leffler functions.

Proof. By applying $\Delta_{(\delta)}$ to $F_\delta(a,\ t)$ and using (2.10), we have for the first case

$$\Delta_{(\delta)}F_\delta(a,\ t^\beta) = t^{-\beta\gamma}D^\delta_\beta T^\beta E_\delta(a,\ t) = t^{-\beta\gamma}T^\beta D^\delta E_\delta(a,\ t) = \\ = t^{-\beta\gamma}T^\beta a E_\delta(a,\ t) = aF_\delta(a,\ t^\beta).$$

In an analogous way, we can prove it for the other case.

Since

$$\lim_{t\to 0} f_1(t) = \lim_{t\to 0} t^{\beta(\delta-\gamma-1)} f(t) = \frac{1}{\Gamma(\delta)} \tag{5.5}$$

$$\text{and}\qquad Vf(t) = \Delta_{(\delta)}f(t) + f_1(0)V,$$

we obtain by (5.1), (5.2) and (5.5) that

$$a)\ \frac{V}{V - at^{\beta(\delta-\gamma-1)}} = \Gamma(\delta)t^{-\beta\gamma}E_\delta(a,\ t^\beta) \tag{5.6}$$

$$b)\ \frac{V}{V + at^{\beta(\delta-\gamma-1)}} = \Gamma(\delta)t^{-\beta\gamma}E_\delta(-a,\ t^\beta). \tag{5.7}$$

By a straightforward calculus, we can verify the vality of the following formulas:

$$c)\ \frac{at^{\beta(\delta-\gamma-1)}}{V + at^{\beta(\delta-\gamma-1)}} = t^{\beta(\delta-\gamma-1)} - \frac{V}{V + at^{\beta(\delta-\gamma-1)}},$$

$$d)\ \frac{-at^{\beta(\delta-\gamma-1)}}{V - at^{\beta(\delta-\gamma-1)}} = t^{\beta(\delta-\gamma-1)} - \frac{V}{V - at^{\beta(\delta-\gamma-1)}},$$

$$e)\ \frac{V^2}{V^2 - at^{\beta(\delta-\gamma-1)}} = \frac{\Gamma(\delta)}{2}\left[t^{-\beta\gamma}E_\delta(a,\ t^\beta) + t^{-\beta\gamma}E_\delta(-a,\ t^\beta)\right],$$

$$f)\ \frac{aV}{V^2 - at^{\beta(\delta-\gamma-1)}} = \frac{\Gamma(\delta)}{2}\left[t^{-\beta\gamma}E_\delta(a,\ t^\beta) - t^{-\beta\gamma}E_\delta(-a,\ t^\beta)\right].$$

6. The extension of $\mathcal{C}_{\beta(-\gamma-1)}$ to the quotient field. With $\beta > 0$, $\delta \geq 1$ and $\gamma \in \mathbb{R}$, we consider the set of functions

$$\mathcal{C}_{\beta(-\gamma-1)} = \Big\{ f(t) = \sum_{k=1}^{\infty} a_k t^{\beta[(k-1)\delta-\gamma-1]} \text{ absolutely convergent} \\ \text{on compact subsets of } [0,\infty)\Big\} \tag{6.1}$$

and the operator

$$A_{(\delta)} = t^{-\beta\gamma-\beta\delta}D^\delta_\beta t^{\beta\gamma+\beta\delta}. \tag{6.2}$$

Since $A_{(\delta)}$ and D^δ are linear automorphisms on the spaces $\mathcal{C}_{\beta(-\gamma-1)}$ and $\mathcal{C}_{\delta-1}$ $\big($see (3.2)$\big)$ respectively, and since $T^{\frac{1}{\beta}}t^{\beta\gamma+\beta\delta}$ is a linear isomorphism between them, satisfying by (2.10)

$$T^{\frac{1}{\beta}} t^{\beta\gamma+\beta\delta} A_{(\delta)} = D^{\delta} T^{\frac{1}{\beta}} t^{\beta\gamma+\beta\delta} \tag{6.3}$$

and since $*$ is a convolution for D^{δ} on $\mathcal{C}_{\delta-1}$ $\Big($ see (3.3)$\Big)$, we can apply Meller's theorem [5] to the diagram

$$\begin{array}{ccc} \mathcal{C}_{\beta(-\gamma-1)} & \xrightarrow{\;A_{(\delta)}\;} & \mathcal{C}_{\beta(-\gamma-1)} \\ {\scriptstyle T^{\frac{1}{\beta}} t^{\beta\gamma+\beta\delta}} \Big\downarrow & & \Big\uparrow {\scriptstyle t^{-\beta\gamma-\beta\delta} T^{\beta}} \\ \mathcal{C}_{\delta-1} & \xrightarrow[\;D^{\delta}\;]{} & \mathcal{C}_{\delta-1} \end{array}$$

to establish

Proposition 6.1. *The operation* $\widetilde{\circledast} : \quad \mathcal{C}_{\beta(-\gamma-1)} \times \mathcal{C}_{\beta(-\gamma-1)} \longrightarrow \mathcal{C}_{\beta(-\gamma-1)}$ *defined by:*

$$f(t) \widetilde{\circledast} g(t) = t^{-\beta\gamma-\beta\delta} T^{\beta} \left[\Big(T^{\frac{1}{\beta}} t^{\beta\gamma+\beta\delta} f(t) \Big) * \Big(T^{\frac{1}{\beta}} t^{\beta\gamma+\beta\delta} g(t) \Big) \right] \tag{6.4}$$

is a convolution for the operator $A_{(\delta)}$.

Proceeding in the same way as in Section 3, we can conclude:

Proposition 6.2. *With the operation* $+$ *and* $\widetilde{\circledast}$, $\mathcal{C}_{\beta(-\gamma-1)}$ *is a unitary commutative ring without divisors of zero.*

Therefore, $\mathcal{C}_{\beta(-\gamma-1)}$ can be extended to the fraction field $\mathcal{M}_{\beta(-\gamma-1)}$, which becomes an algebra.

There is a subset $\mathcal{M}'_{\beta(-\gamma-1)}$ of $\mathcal{M}_{\beta(-\gamma-1)}$ isomorphic to $\mathcal{C}_{\beta(-\gamma-1)}$ via the mapping

$$\mathcal{M}'_{\beta(-\gamma-1)} \subset \mathcal{M}_{\beta(-\gamma-1)} \longrightarrow \mathcal{C}_{\beta(-\gamma-1)}$$

$$\frac{f(t)}{t^{\beta(-\gamma-1)}} \longrightarrow f(t)$$

and therefore, the operators of the form $\dfrac{f(t)}{t^{\beta(-\gamma-1)}}$ constitute a subring of $\mathcal{M}_{\beta(-\gamma-1)}$ that can be identified with $f(t)$.

Let $\mathcal{R}_{(\delta)}$ be the operator given by

$$\mathcal{R}_{(\delta)} = I_{\beta}^{\gamma,\delta} t^{\beta\delta} = t^{-\beta\gamma-\beta\delta} I_{\beta}^{\delta} t^{\beta\gamma+\beta\delta}. \tag{6.5}$$

It is easy to see that this operator is the right inverse operator of $A_{(\delta)}$. However, in general it is not its left inverse operator, since it turns out that

$$\begin{aligned} f(t) &= \mathcal{R}_{(\delta)} A_{(\delta)} f(t) + a_1 t^{\beta(-\gamma-1)} = \\ &= \mathcal{R}_{(\delta)} A_{(\delta)} f(t) + \left[t^{\beta(\gamma+1)} f(t)\right]_{t=0} t^{\beta(-\gamma-1)}. \end{aligned} \tag{6.6}$$

Through a similar process to that employed in Prop. 4.3 and 4.4, we can express the operators $\mathcal{R}_{(\delta)}$ and $\mathcal{R}^k_{(\delta)}$ by

$$\mathcal{R}_{(\delta)} = \frac{\Gamma(\delta)}{\Gamma(2\delta)} t^{\beta(\delta-\gamma-1)} \in \mathcal{M}_{\beta(-\gamma-1)} \tag{6.6}$$

$$\mathcal{R}^k_{(\delta)} = \frac{\Gamma(\delta)}{\Gamma(k\delta+\delta)} t^{\beta(k\delta-\gamma-1)} \in \mathcal{M}_{\beta(-\gamma-1)}. \tag{6.7}$$

If we consider the operator

$$\mathcal{V} = \frac{\Gamma(2\delta)}{\Gamma(\delta)} \frac{t^{\beta(-\gamma-1)}}{t^{\beta(\delta-\gamma-1)}} \tag{6.9}$$

which is the inverse one of $\mathcal{R}_{(\delta)}$ in $\mathcal{M}_{\beta(-\gamma-1)}$, we have

Proposition 6.3. *If $f(t) \in \mathcal{C}_{\beta(-\gamma-1)}$, then*

$$\mathcal{V} f(t) = A_{(\delta)} f(t) + \left[t^{\beta(\gamma+1)} f(t)\right]_{t=0} \mathcal{V}. \tag{6.10}$$

Proof. By (6.6) we know that

$$f(t) = \mathcal{R}_{(\delta)} A_{(\delta)} f(t) + \left[t^{\beta(\gamma+1)} f(t)\right]_{t=0} t^{\beta(-\gamma-1)}$$

and applying the operator $\mathcal{V}$ to both sides we get

$$\mathcal{V} f(t) = A_{(\delta)} f(t) + \left[t^{\beta(\gamma+1)} f(t)\right]_{t=0} \mathcal{V}. \tag{6.11}$$

7. Operational rules related to $A_{(\delta)}$. For the differential equations with $a > 0$

$$(A_{(\delta)} - a) f(t) = 0 \tag{7.1}$$

$$(A_{(\delta)} + a) f(t) = 0 \tag{7.2}$$

one can see easily that they have as solutions

$$f(t) = t^{-\beta\gamma-\beta\delta} E_\delta(a,\ t^\beta) \tag{7.3}$$

$$f(t) = t^{-\beta\gamma-\beta\delta} E_\delta(-a,\ t^\beta) \tag{7.4}$$

and since

$$\lim_{t\to 0} t^{\beta(\gamma+1)} f(t) = \frac{1}{\Gamma(\delta)} \tag{7.5}$$

we have by (7.1), (7.2) and (7.5) that

$$a)\ \frac{\mathcal{V}}{\mathcal{V} - at^{\beta(-\gamma-1)}} = \Gamma(\delta) t^{-\beta\gamma-\beta\delta} E_\delta(a,\ t^\beta)$$

$$b)\ \frac{\mathcal{V}}{\mathcal{V} + at^{\beta(-\gamma-1)}} = \Gamma(\delta) t^{-\beta\gamma-\beta\delta} E_\delta(-a,\ t^\beta).$$

Likewise, by a straightforward calculation, one can state

$$c)\ \frac{at^{\beta(-\gamma-1)}}{\mathcal{V} + at^{\beta(-\gamma-1)}} = t^{\beta(-\gamma-1)} - \frac{\mathcal{V}}{\mathcal{V} + at^{\beta(-\gamma-1)}}$$

$$d)\ \frac{-at^{\beta(-\gamma-1)}}{\mathcal{V} - at^{\beta(-\gamma-1)}} = t^{\beta(-\gamma-1)} - \frac{\mathcal{V}}{\mathcal{V} - at^{\beta(-\gamma-1)}}$$

$$e)\ \frac{\mathcal{V}^2}{\mathcal{V}^2 - at^{\beta(-\gamma-1)}} = \frac{\Gamma(\delta)}{2} t^{-\beta\gamma-\beta\delta} \Big[E_\delta(a,\ t^\beta) + E_\delta(-a,\ t^\beta)\Big]$$

$$f)\ \frac{a\mathcal{V}}{\mathcal{V}^2 - at^{\beta(-\gamma-1)}} = \frac{\Gamma(\delta)}{2} t^{-\beta\gamma-\beta\delta} \Big[E_\delta(a,\ t^\beta) - E_\delta(-a,\ t^\beta)\Big].$$

Remark 1. Since the following diagram

$$\begin{array}{ccc}
\mathcal{C}_{\beta(-\gamma-1)} & \xrightarrow{\ A_{(\delta)}\ } & \mathcal{C}_{\beta(-\gamma-1)} \\
\Big\downarrow t^{\beta\delta} & & \Big\uparrow t^{-\beta\delta} \\
\mathcal{C}_{\beta(\delta-\gamma-1)} & \xrightarrow[\ \Delta_{(\delta)}\ ]{} & \mathcal{C}_{\beta(\delta-\gamma-1)}
\end{array}$$

verifies the hypotesis of Meller's theorem, the operation $\odot$ defined as

$$f(t) \odot g(t) = t^{-\beta\gamma}\Big[\Big(t^{\beta\gamma} f(t)\Big) \circledast \Big(t^{\beta\gamma} g(t)\Big)\Big]$$

is a convolution for the operator $A_{(\delta)}$. We could use this convolution instead of that introduced by (6.4) to reach the same results.

Remark 2. Since the Dzrbasjan–Gelfond–Leontiev operator (see [6])

$$l_{\rho,\mu} = t^{\rho(\mu-1)} I_\rho^{\frac{1}{\rho}} t^{\rho(\mu-1)} = t I_\rho^{\mu-1,\frac{1}{\rho}} = L_{(\frac{1}{\rho})},$$

that is to say,

$$l_{\rho,\mu} f(t) = \frac{t}{\Gamma(\frac{1}{\rho})} \int_0^1 (1-\sigma)^{\frac{1}{\rho}-1} \sigma^{\mu-1} f(t\,\sigma^{\frac{1}{\rho}}) d\sigma$$

is an integral operator acting on the functions $f(t)$ of $\mathcal{C}_{\rho(\frac{1}{\rho}-\mu)}$, being due to (4.2) the right inverse operator of (2.6):

$$\Delta_{(\frac{1}{\rho})} = t^{-\rho(\mu-1)} D_\rho^{\frac{1}{\rho}} t^{\rho(\mu-1)},$$

we can establish that the following operation

$$f(t) \circledast g(t) = t^{-\rho(\mu-1)} T^\rho \left[\left(T^{\frac{1}{\rho}} t^{\rho(\mu-1)} f(t) \right) * \left(T^{\frac{1}{\rho}} t^{\rho(\mu-1)} g(t) \right) \right]$$

where $*$ is given by (3.3), is a convolution of the Dzrbasjan–Gelfond–Leontiev operator on the space $\mathcal{C}_{\rho(\frac{1}{\rho}-\mu)}$. Convolutions of operators $l_{\rho,\mu}$ were found by Dimovski and Kiryakova [6], Kiryakova [8] using alternative approaches.

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