Serdica
Bulgariacae mathematicae publicationes

Сердика
Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
20 (1994) 364-369

ON GENERALIZED ABSOLUTE ALMOST SUMMABILITY

E. SAVAŞ

Abstract. The purpose of this paper is to introduce and discuss the spaces of generalized absolute almost summable sequences which are defined by using a matrix information.

1. Introduction. Let l_∞ be the set of all real or complex sequences $x = (x_n)$ with the norm $\|x\| = \sup|x_n| < \infty$. A linear functional L on l_∞ is said to be a Banach limit if it has the properties:

(i) $L(x) \geq 0$ if $x \geq 0$ (i. e. $x_n \geq 0$ for all n)

(ii) $L(e) = 1$, where $e = (1, 1, ...)$,

(iii) $L(Sx) = L(x)$,

where the shift operator S is defined by $(Sx) = x_{n+1}$.

Let β be the set of all Banach limits on l_∞. A sequence x is said to be almost convergent to a number s if $L(x) = s$ for all $L \in \beta$. Lorentz (1948) has shown that x is almost convergent to s if and only if

$$t_{km}(x) = \frac{x_{m+1} + \cdots + x_{m+k}}{k+1} \longrightarrow s \tag{1.1}$$

as $k \longrightarrow \infty$ uniformly in m. Let us denote by f the set of all almost convergent sequences. We write $f - \lim x = s$ if x is almost convergent to s.

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. We write $A_x = (A_n(x))$ if $A_n(x) = \sum_k a_{nk}x_k$ converges for each n.

Let X and Y be any two nonempty subsets of the set of all sequences with real or complex terms. If $x = (x_k) \in X$ implies that $A_x = (A_n(x)) \in Y$, we say that A defines a matrix transformation from X into Y and we denote it by $A : X \longrightarrow Y$. By (X, Y) we mean the class of matrices A such that $A : X \longrightarrow Y$. Quite recently, some new sequence spaces which arose naturally from the concept of almost convergence have been introduced by Savaş (1990). If $p = (p_k)$ is a given bounded sequence of positive real numbers, then we define (see,[4]),

$$(\widehat{A}, p) = \{x : \sum_k |t_{km}(Ax)|^{p_k} \quad \text{converges uniformly in} \quad m\}$$

$$(\widehat{\widehat{A}}, p) = \{x : \sup_m \sum_k |t_{km}(Ax)|^{p_k} < \infty\},$$

where

$$t_{km}(x) = \sum_n a(m, n, k) x_n$$

and $a(m, n, k) = \dfrac{1}{k+1} \displaystyle\sum_{i=0}^{k} a_{m+i,n}$.

It may be recalled that $|\widehat{A}, p|$ and $|\widehat{\widehat{A}}, p|$ spaces have been introduced and studied by Nanda (1985) and are defined as

$$|\widehat{A}, p| = \{x : \sum_k |t_{km}(Ax) - t_{k-1,m}(Ax)|^{p_k} \quad \text{converges uniformly in} \quad m\},$$

$$|\widehat{\widehat{A}}, p| = \{x : \sup_m \sum_k |t_{km}(Ax) - t_{k-1,m}(Ax)|_k^p < \infty\}.$$

2. Generalized Absolute almost Summability. The object of this section is to introduce the following sequence spaces.

We define

$$|\widehat{w}_A, p| = \{x : \sum_{k=0}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} \quad \text{converges uniformly in} \quad m\},$$

$$|\widehat{\widehat{w}}_A, p| = \{x : \sup_m \sum_k |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} < \infty\},$$

where we set

$$d_{nm}(Ax) = \frac{1}{n+1} \sum_{k=0}^{n} t_{km}(Ax).$$

If $p(p_k)$ is a constant sequence we write $|\widehat{w}_A|_p$ and $|\widehat{\widehat{w}}_A|_p$ instead of $|\widehat{w}_A, p|$ and $|\widehat{\widehat{w}}_A, p|$ respectivelly.

The following inclusion relation holds.

Theorem 1. $|\widehat{w}_A, p| \subset |\widehat{\widehat{w}}_A, p|$.

Proof. Let $x \in |\widehat{w}_A, p|$. Then there is an integer M such that

$$\sum_{k>M} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} \leq 1 \text{ for each } m. \tag{2.1}$$

Hence it is enough to show that for a fixed k and for each m

$$|d_{km}(Ax) - d_{k-1,m}(Ax)| \leq K.$$

But it follows from (2.1.) that

$$|d_{km}(Ax) - d_{k-1,m}(Ax)| \leq 1, \tag{2.2}$$

for every fixed $k > M$ and each m. Since

$$n(n+1)(d_{nm}(Ax) - d_{n-1,m}(Ax)) = \sum_{k=0}^{n} k\left(t_{km}(Ax) - t_{k-1,m}(Ax)\right), \tag{2.3}$$

we have

$$n(n+1)(d_{nm}(Ax) - d_{n-1,m}(Ax)) - (n-1)(d_{n-1,m}(Ax) - d_{n-2,m}(Ax))$$

$$= t_{nm}(Ax) - t_{n-1,m}(Ax) \tag{2.4}$$

Hence it follows from (2. 2.) and (2. 4.) that

$$|t_{nm}(Ax) - t_{n-l,m}(Ax| \leq K(n), \tag{2.5}$$

for every $n > M$ and for all m, where $K(n)$ is a constant depending upon n. Again from the definition of $t_{nm}(Ax)$, we have

$$t_{nm}(Ax) - t_{m-l,m}(Ax) = \sum_{k=0}^{\infty} \frac{1}{n(n+1)} \sum_{v=1}^{n} v a_{m+v,k} x_k, \tag{2.6}$$

so that

$$\begin{aligned} \sum_{k=0}^{\infty} a_{m+n,k} x_k &= (n+1)(t_{nm}(Ax) - t_{n-1,m}(Ax)) \\ &\quad -(n-1)(t_{n-1,m}(Ax) - t_{n-2,m}(Ax)). \end{aligned} \tag{2.7}$$

Hence it follows from (2. 5.) that for each fixed $n > M$

$$\left|\sum_{k=0}^{\infty} a_{m+n,k} x_k\right| \leq K(n) \text{ for each } m. \tag{2.8}$$

Now choose $n = p + 1$. Let

$$K = \max\{K(p+1), \left|\sum_{k=0}^{\infty} a_{1k}x_k\right|, \left|\sum_{k=0}^{\infty} a_{2k}x_k\right|, ..., \left|\sum_{k=0}^{\infty} a_{p+1}x_k\right|\}$$

Hence it follows from (2. 8.) that

$$\left|\sum_{k=0}^{\infty} a_{vk}x_k\right| \leq K \text{ for all } v \tag{2.9}$$

and K is independent of v. It now follows from (2. 6.) that

$$|t_{nm}(Ax) - t_{n-1,m}(Ax)| \leq K \text{ for all } n \text{ and } m \tag{2.10}$$

And from (2. 3.) and (2. 10.) that

$$|d_{km}(Ax) - d_{k-1,m}(Ax)| \leq K$$

for all n and m.

This completes the proof.

Theorem 2. *$|\widehat{w}_A, p|$ is a linear topological space paranomed by the function*

$$g(x) = \sup_m(\sum_{k=0}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k})^{1/M} \tag{2.11}$$

where $M = \max(1, \sup p_k)$. The space $|\widehat{\widehat{w}}_A, p|$ is paranormed by (2.11).

Proof. The proof is a routine verification by using standard techniques and therefore we omit it.

Theorem 3. *Let $p = (p_m), q = (q_m)$ and $0 < p_m \leq q_m$. Then*

$$|\widehat{w}_A, p| \subset |\widehat{w}_A, q|, \tag{i}$$

$$|\widehat{\widehat{w}}_A, p| \subset |\widehat{\widehat{w}}_A, q|. \tag{ii}$$

Proof. (i) Suppose that, $x \in |\widehat{w_A}, p|$. Then there exists an integer M such that

$$\sum_{k=M}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} < 1. \tag{2.12}$$

Hence

$$|d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k}) < 1$$

for $k \geq M$ and for all m. This implies that

$$|d_{km}(Ax) - d_{k-1,m}(Ax)|^{q_k} \leq |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k}. \tag{2.13}$$

The uniform convergence of $\sum_{k=0}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{q_k}$ now follows from (2.13) by the uniform convergence of

$$\sum_{k=0}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k}.$$

(ii) The proof of (ii) differs from the proof of (i), as we cannot assert (2.12). Suppose that $x \in |\widehat{\widehat{w}}_A, p|$. Then there is a constant $K > 1$ such that

$$\sum_{k=0}^{\infty} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} \leq K,$$

and hence

$$|d_{km}(Ax) - d_{k-1,m}(Ax)| \leq K^{1/p_k} \leq K^{1/\delta}$$

where $p_k \geq \delta > 0$.

Hence,

$$|d_{km}(Ax) - d_{k-1,m}(Ax)|^{q_k} =$$

$$|d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k - q_k}$$

$$\leq K^{(p_k - q_k)/\delta} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k}$$

$$\leq K^{\sup p_k/\delta} |d_{km}(Ax) - d_{k-1,m}(Ax)|^{p_k}. \tag{2.14}$$

The result now follows from $(2,14)$ by taking the sum with respect to k and then taking the supremum with respect to m.

Theorem 4.

(i) $$|\widehat{A}|_p \subset |\widehat{w}_A|_p,$$

(ii) $$|\widehat{\widehat{A}}|_p \subset |\widehat{\widehat{w}}_A|_p$$

P r o o f. (i) Let $x \in |\widehat{A}|_p$. By Hölder's inequality for $p > 1$ and trivially for $p = 1$ it can be easily verified that

$$|d_{nm}(Ax) - d_{n-1,m}(Ax)|^p \leq \frac{1}{n(n+1)^p} \sum_{k=1}^{n} k^p \, |t_{km}(Ax) - t_{k-1,m}(Ax)|^p \, .$$

Hence,

$$\sum_{k=0}^{\infty} |d_{nm}(Ax) - d_{d-1,m}(Ax)|^p \leq \sum_{k=1}^{\infty} k^p \, |t_{km}(Ax) - t_{k,1,m}(Ax)|^p \sum_{n=k}^{\infty} \frac{1}{n(n+1)^p}$$

$$\leq \sum_{k=l}^{\infty} |t_{km}(Ax) - t_{k-1,m}(Ax)|^p. \tag{2.14}$$

Now the uniform convergence of $\sum_n |d_{nm}(Ax) - d_{n-1,m}(Ax)|^p$ follows from the uniform convergence of $\sum_k |t_{km}(Ax) - t_{k-1,m}(Ax)|^p$ and this completes the proof.

(ii) The proof of (ii) follows from the inequality (2.14) by taking the supremum with respect to m.

REFERENCES

[1] G. Das and S. K. Sahoo. On Some Sequence Spaces. *J. Math. Analysis and Appl.*, **164** (1992), 381.

[2] G. G. Lorentz. Contribution to the theory of divergent sequences. *Acta Math.*, **80** (1948), 167.

[3] S. Nanda. Absolute almost convergence and absolute almost summability. *Rendiconti di Math.*, Serie VII, **5**, No. 1-2 (1985), 41.

[4] E.Savaş. Almost convergence and Almost Summability. *Tamkang J. Math.*, **21**, 4 (1990), 327.

[5] S.Simons. The sequance spaces $l(p_v)$ and $M(p_v)$. *Proc. London Math. Soc.* **3**, 15 (1969), 422.

Department Of Mathematics
Yüzüncü Yil University
Van
TURKEY

Received 25.03.1994