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REPRESENTABLE BANACH SPACES AND UNIFORMLY GÂTEAUX-SMOOTH NORMS

Julien Frontisi

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Abstract. It is proved that a representable non-separable Banach space does not admit uniformly Gâteaux-smooth norms. This is true in particular for $C(K)$ spaces where K is a separable non-metrizable Rosenthal compact space.

1. Introduction. The family of spaces $C(K)$ of continuous functions on a compact space K plays a central role in the study of smoothness properties in Banach spaces. First, they form a class which is rich enough to obtain very different results depending on K and that provides many counter-examples. It is known for example that if the Cantor derivative $K^{(\omega_1)}$ of K is empty, then $C(K)$ has a Fréchet-differentiable norm ([2]). On the other hand, Haydon ([8]) constructed trees T such that $C(\hat{T})$ has no Gâteaux differentiable norm. This is also true ([13]) for $C(K)$ when K is the 'two-arrow space' and this result solves the three-space problem for (uniformly) Gâteaux-smooth norms. Also, using theorems of transfer, it is often easy to extend the results from $C(K)$ spaces to larger classes of Banach spaces (see for example [3] VII.4.10). The reader is referred to [3] for further results and references on the topic.

Among compact spaces, the class of Rosenthal compact spaces (see §3) contains many 'natural' examples and shares nice properties with metric spaces like angelicity. Rosenthal separable non-metrizable compact spaces therefore yield a family of $C(K)$

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spaces that may seem close to separable $C(K)$ spaces. However, their smoothness properties can be quite different. Using martingale techniques, it has been proved by Moltó and Troyanski ([10]) that $C(K)$ has no UG norm if K is the space constructed in ([9]) and, more generally, for any scattered separable non-metrizable compact K. We want here to prove this result for a wide class of Banach spaces, namely the class of non-separable representable Banach spaces.

Rosenthal compact sets K have the property that $C(K)$ is analytic in the topology of pointwise convergence σ_D on any countable subset D of K. Universally representable and more generally representable Banach spaces (see definition below) are then a natural extension of that class. It was shown in [6] that the analytic structure of a representable space X allows to construct biorthogonal systems in X as in Stegall's [12], replacing w^*-compactness by analyticity. We use this result along with the proof of Theorem 1 in [7].

We give the notation used here and recall some definitions. The set of positive integers is noted ω, the set of finite (resp. infinite; of length n) $\{0,1\}$-valued sequences is $\mathbf{2}^{<\omega}$ (resp. $\mathbf{2}^{\omega}; \mathbf{2}^{n}$). If $b \in \mathbf{2}^{<\omega}$ or $\mathbf{2}^{\omega}$ and $n \in \omega$, $b\lceil_n$ is the sequence of the first n elements of b and $b(n)$ is the n-th element of b. The set ω^ω is a Polish space and may therefore be considered as a complete metric space. Given a set D, the set $\mathbf{2}^D = \{0,1\}^D$ will be identified to the collection of subsets of D.

A metric space A is called *analytic* if there exists a continuous function $\phi : \omega^\omega \to A$ which is onto. A Banach space X is called *representable* if there exists a countable subset D of X^* which is norming (in the sense that the function $|x|_D = \sup\{f(x); f \in D\}$ defines an equivalent norm on X) such that X is analytic in the topology σ_D.

A norm $\| \ \|$ on a Banach space X is called *uniformly Gâteaux-smooth* (UG) if for all $h \in S_{(X,\| \ \|)}$, $\lim_{t\to 0} \dfrac{\|x+th\| - \|x\|}{t}$ exists and is uniform in $x \in S_{(X,\| \ \|)}$. The norm $\| \ \|_{X^*}$ of X^* is *w^*-uniformly rotund* (W*UR) if for all $f_n, g_n \in S_{(X^*,\| \ \|_{X^*})}$ that satisfy $\lim_{n\to\infty}(2\|f_n\|^2_{X^*} + 2\|g_n\|^2_{X^*} - \|f_n+g_n\|^2_{X^*}) = 0$, then $w^*\text{-}\lim_{n\to\infty}(f_n - g_n)_{X^*} = 0$. The following duality holds: a norm on X is UG if and only if its dual norm on X^* is W*UR (see [3] chapter II).

2. The main result.

Theorem. *Let X be a representable non-separable Banach space. Then no equivalent norm on X is uniformly Gâteaux-smooth.*

Proof. We follow Hájek's proof that any space with a WUR norm is an Asplund space [7]. The following crucial Lemma we are using is in [6]. Its proof is similar to Stegall's in [12]:

Lemma. *Let X be a representable non-separable Banach space, and D a countable norming subset of X^* such that (X, σ_D) is analytic; let $\phi : \omega^\omega \to (X, \sigma_D)$ be* *a continuous mapping onto. Then for all $\varepsilon > 0$, there exists a sequence $A_n \subset \omega^\omega$ that satisfy the following:*

i) For all $n \in \omega$, $A_n = \bigcup_{s\in \mathbf{2}^n} B^n_s$, where B^n_s is a ball of radius less than $\frac{1}{n}$.

ii) For all $n \in \omega$ and $s \in \mathbf{2}^{n+1}$, $B^{n+1}_s \subset B^n_{(s\lceil_n)}$.

iii) There exist $f^n_s \in span(D)$ with $\|f^n_s\|_{X^*} < 1+\varepsilon$ such that

a) $f^n_s\lceil_{\varphi(B^n_s)} \;\geq\; 1 - \frac{\varepsilon}{2^n}$

b) $|f^n_s\lceil_{\varphi(B^n_{s'})}| \;\leq\; \frac{\varepsilon}{2^n}$ whenever $s \neq s'$ in $\mathbf{2}^n$.

We therefore obtain in ω^ω a Cantor set $\Delta = \bigcap_{n\in\omega} A_n$. We now claim the following:

Claim. *For all $\delta > 0$, $b \in \mathbf{2}^\omega$, $l \in \omega$, there are $m_b > n_b > l$ in ω and f_b, g_b in X^* of norm less than $1+\varepsilon$ of the form $f_b = \sum_{l<n<n_b} \alpha_n\, f^n_{(b\lceil_n)}$ and $g_b = \sum_{n_b<n<m_b} \beta_n\, f^n_{(b\lceil_n)}$ where $0 \leq \alpha_n,\; \beta_n \leq 1$ and $\sum \alpha_n = \sum \beta_n = 1$, and such that*

$$2\|f_b\|^2_{X^*} + 2\|g_b\|^2_{X^*} - \|f_b + g_b\|^2_{X^*} < \delta.$$

P r o o f. Set $M_b(n) = \inf\{\|f\|_{X^*}, f \in \bigcup_{m\in\omega} A^m_n\}$ where

$$A^m_n = \{f = \sum_{n<i<m} \gamma_i\; f^i_{(b\lceil_i)} \mid 0 \leq \gamma_i \leq 1, \sum \gamma_i = 1\}.$$

Clearly, $M_b(n) \leq 1+\varepsilon$ and M_b is a non-decreasing function on ω. For every $\rho > 0$, one can find n_ρ greater than l that satisfies $M_b(n_\rho) \geq \sup_{k\in\omega} M_b(k) - \rho$. We choose now $m_b > n_b > n_\rho$ and $f_b \in A^{n_b}_{n_\rho}$, $g_b \in A^{m_b}_{n_b}$ with $\|f_b\|_{X^*} < M_b(n_\rho) + \rho$ and $\|g_b\|_{X^*} < M_b(n_\rho) + 2\rho$ (since $M_b(m_b) \leq \sup_{k\in\omega} M_b(k) \leq M_b(n_\rho) + \rho$). We have then $\|f_b + g_b\|_{X^*} \geq 2M_b(n_\rho)$. Hence, if ρ is small enough, $2\|f_b\|^2_{X^*} + 2\|g_b\|^2_{X^*} - \|f_b + g_b\|^2_{X^*} < \delta$. This concludes the proof of the claim. □

Let us choose now a sequence $\delta_n \searrow 0$. For b^1 in $\mathbf{2}^\omega$, we take n_1, m_1, f_1, g_1 respectively equal to n_{b_1}, m_{b_1}, f_{b_1}, g_{b_1} from the claim with $l = 1$ and $\delta = \delta_1$. We construct by induction n_i, m_i in ω, f_i, g_i in X^* and b^i in $\mathbf{2}^\omega$ as follows: if these elements are chosen for $i \leq k$, we find b^{k+1} in $\mathbf{2}^\omega$ such that $b^{k+1}\lceil_{n_k} = b^k\lceil_{n_k}$ and $b^{k+1}(n_k+1) \neq b^k(n_k+1)$ and n_{k+1}, m_{k+1}, f_{k+1}, g_{k+1} are again given by the claim for $b = b^{k+1}$, $l = n_k$ and $\delta = \delta_{k+1}$. We then get: $2\|f_k\|^2_{X^*} + 2\|g_k\|^2_{X^*} - \|f_k + g_k\|^2_{X^*} \to 0$.

If $b \in \mathbf{2}^\omega$ is the sequence such that $b\lceil_{n_k} = b^k\lceil_{n_k}$ for all $k \in \omega$, then let $\sigma_b \in \Delta \subseteq \omega^\omega$ be such that $\sigma_b \in \bigcap_{n\in\omega} B^n_{b\lceil_n}$ and let $x_b = \varphi(\sigma_b) \in X$. We have:

$$f_k(x_b) - g_k(x_b) \;=\; \sum_{n_{k-1}<n<n_k} \alpha_n\, f^n_{(b^k\lceil_n)}(x_b) - \sum_{n_k<n<m_k} \beta_n\, f^n_{(b^k\lceil_n)}(x_b)$$

$$\geq \sum_{n_{k-1}<n<n_k} \alpha_n(1-\frac{\varepsilon}{2^n}) - \sum_{n_k<n<m_k} \beta_n \frac{\varepsilon}{2^n} \geq 1-\varepsilon.$$

We deduce that $\| \ \|_{X^*}$ cannot be W*UR, so that $\| \ \|_X$ is not UG. □

In [5], biorthogonal subsets of cardinality c have been constructed in similar classes of closed subspaces of $\ell_\infty(\omega)$, assuming supplementary determinacy axioms. If $\mathrm{Det}(\mathbf{\Pi}^1_n)$ is the axiom meaning that every $\mathbf{\Pi}^1_n$ game on the integers is determined, the proof above still applies to show:

Corollary. *In ZFC + Det($\mathbf{\Pi}^1_n$), if X is any non-separable subspace of $\ell_\infty(\omega)$ which is $\mathbf{\Sigma}^1_n$ for the w*-topology $\sigma(\ell_\infty, \ell_1)$, then X does not have an equivalent UG norm.*

Examples. Let $\{f_n\}_{n\in\omega}$ be a dense family in the unit ball of $C(\mathbf{2}^\omega)$. The mapping $(\lambda_s)_{s\in\mathbf{2}^\omega} \mapsto (\sum_{s\in\mathbf{2}^\omega} \lambda_s f_n(s))_{n\geq 1}$ defines an isometry u from $\ell_1(\mathbf{2}^\omega)$ to $\ell_\infty(\omega)$ which is w^*-continuous, with the identification $\ell_1(\mathbf{2}^\omega) \subseteq C(\mathbf{2}^\omega)^*$. Let S be a subset of $\mathbf{2}^\omega$. We identify $\mathbf{2}^\omega$ and the set of Dirac measures $\Delta = \{\delta_s, s \in \mathbf{2}^\omega\} \subseteq (C(\mathbf{2}^\omega)^*, w^*)$. Since $X_S = \overline{span}^{\| \ \|}(u(S))$ satisfies $X_S \cap u(\Delta) = u(S)$, if we choose a subset S of $\mathbf{2}^\omega$ that is $\mathbf{\Sigma}^1_n$ and not $\mathbf{\Sigma}^1_{n-1}$, we obtain a closed subspace X_S of $\ell_\infty(\omega)$ that belongs to the same class.

On the other hand, assuming the continuum hypothesis (independent from $\mathrm{Det}(\mathbf{\Pi}^1_n)$), let K_0 be Kunen's compact set. Recall that under CH, K_0 is a separable non-metrizable compact set such that if F is any uncountable family of $C(K_0)$, then there is $f \in F$ such that $f \in \overline{conv}^{\| \ \|}(F \setminus \{f\})$ (see [11] and references therein; see also in [5] Remark 2.6). Hence $C(K_0)$ does not contain any biorthogonal subset of cardinality c. However, since K_0 is scattered, we know from [10] that $C(K_0)$ admits no equivalent UG norm.

Remark. By the transfer method, we know ([3] Theorem II.6.8) that if X and Y are Banach spaces such that there is a bounded linear map $T : X \to Y$ with dense range, and if X has a UG norm, then Y has a UG norm. Since for any set Γ, the space $c_0(\Gamma)$ has a UG norm, then there is no such map from $c_0(\Gamma)$ to X, for any nonseparable representable space X.

3. Application to Rosenthal compact sets. We recall (see [1]):

Definition. A compact set K is called *Rosenthal compact* if it is homeomorphic to a set of first Baire-class functions on a Polish space, with the pointwise topology τ_p.

For any Banach space X that does not contain $\ell_1(\omega)$, the space $(B_{X^{**}}, w^*)$ is Rosenthal compact. Other examples are the 'two-arrow space' : $[0,1] \times \{0,1\}$ with the lexicographical order topology, Helly's compact space of non-decreasing functions from $[0,1]$ to $[0,1]$, or the space constructed in [9]. It has been shown in [4] that if K is a

Rosenthal compact set, then $C(K)$ is representable. We give here a proof, for the sake of completeness:

Let $K \subseteq \mathcal{B}_1(P)$ be a τ_p-compact set of first Baire-class functions on a Polish space P and let $D = \{u_n\}_{n\in\omega} \subset P$ be a dense countable subset. For $n \in \omega$, let ψ_n be the mapping from $P^n \times \omega$ to $\mathbf{2}^{D\times D}$ defined by $\psi_n(x_1, x_2, \dots, x_n, k) = \{(u_l, u_p), |u_l(x_i) - u_p(x_i)| \le \frac{1}{k} \ \ \forall i \le n\}$. Since basic open sets of $\mathbf{2}^{D\times D}$ are of the form $O_{(l,p)} = \{X \subseteq D \times D, (u_l, u_p) \in X\}$ or $U_{(l,p)} = \{X \subseteq D \times D, (u_l, u_p) \notin X\}$, it is not difficult to see that the ψ_n's are Borel functions, since the u_n's are Borel. Hence, the sets $\psi_n(P^n \times \omega)$ are analytic in $\mathbf{2}^{D\times D}$. So is their union $\mathcal{A}$. Let $\mathcal{U}_D$ be the filter on $\mathbf{2}^{D\times D}$ generated by $\mathcal{A}$. Using the continuity of intersection and the fact that $\{(X,Y), X \subseteq Y\}$ is closed in $\mathbf{2}^{D\times D}$, we deduce that $\mathcal{U}_D$ itself is an analytic subset of $\mathbf{2}^{D\times D}$.

We now want to show that $C(K)$ is analytic in the σ_D-topology. The mapping $\phi : f \mapsto (f(u_n))_{n\in\omega}$ defines an isomorphism from $C(K)$ to a closed subspace C_D of $\ell_\infty(D)$ that transforms σ_D into the product topology τ_p $(= \sigma(\ell_\infty, \ell_1))$. It is enough to show that C_D is analytic in $(\ell_\infty(D), \tau_p)$. But C_D is the set of elements f of $\ell_\infty(D)$ that are $\mathcal{U}_D$-uniformly continuous, in the sense that for all $\varepsilon > 0$, there exists $U \in \mathcal{U}_D$ such that if $(u,v) \in U$, then $|f(u) - f(v)| < \varepsilon$ (since K is compact).

Let $L_n = [-n,n]^D \times \mathbf{2}^{D\times D}$. The set $F_k = \{(f,X) \in L_n \ , |f(s) - f(t)| \le \dfrac{1}{k} \ \ \forall (s,t) \in X\}$ is closed in L_n. We have that $f \in C_D \cap [-n,n]^D$ if and only if for all k, there is $U \in \mathcal{U}_D$ such that $(f,U) \in F_k$. Hence, calling π_1 the natural projection from $\ell_\infty(D) \times \mathbf{2}^{D\times D}$ to $\ell_\infty(D)$, we obtain that $C_D \cap [-n,n]^D = \bigcap_{k\ge 1} \pi_1(([-n,n]^D \times \mathcal{U}_D) \cup F_k)$ is analytic. Whence $C_D = \bigcup_{n\ge 1} (C_D \cap [-n,n]^D)$ is analytic.

From this result and from the theorem, we obtain immediately:

Corollary. *If K is a separable non-metrizable Rosenthal compact set, then $C(K)$ does not have any equivalent uniformly Gâteaux-smooth norm.*

Remark. The separability assumption is needed. If K is the one-point compactification of a discrete set Γ of cardinality c, then K is a non-separable Rosenthal compact space. But $C(K)$ is isomorphic to $c_0(\Gamma)$ and thus has an equivalent UG norm.

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Equipe d'Analyse:
Tour 46-0 – Boîte 186 – Université Paris-6
4, place Jussieu
75252 Paris Cedex 05
France
e-mail: `juf@ccr.jussieu.fr`

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