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A NOTE ON COERCIVITY OF LOWER SEMICONTINUOUS FUNCTIONS AND NONSMOOTH CRITICAL POINT THEORY

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0. Introduction. The first motivation for this note is to obtain a general version of the following result: let E be a Banach space and $f : E \to \mathbb{R}$ be a differentiable function, bounded below and satisfying the Palais-Smale condition; then, f is coercive, i.e., $f(x)$ goes to infinity as $\|x\|$ goes to infinity. In recent years, many variants and extensions of this result appeared, see [3], [5], [6], [9], [14], [18], [19] and the references therein.

A general result of this type was given in [3, Theorem 5.1] for a lower semicontinuous function defined on a Banach space, through an approach based on an abstract notion of subdifferential operator, and taking into account the "smoothness" of the Banach space. Here, we give (Theorem 1) an extension in a metric setting, based on the notion of slope from [11] and coercivity is considered in a generalized sense, inspired by [9]; our result allows to recover, for example, the coercivity result of [19], where a weakened version of the Palais-Smale condition is used. Our main tool (Proposition 1) is a consequence of Ekeland's variational principle extending [12, Corollary 3.4], and deals with a function f which is, in some sense, the "uniform" Γ-limit of a sequence of functions.

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Our coercivity result also contains the result of [14], dealing, somewhat, with the most general class of functions (considering the references given above), see Section 3. However, this class of functions is too general from the point of view of critical point theory (hence, of possible applications to variational problems) for which some additional continuity property of the function is needed. The second purpose of this note is to show that, under such a hypothesis, the general nonsmooth critical point theory developed in [12], [8], based on the notion of weak slope of [12], applies to a class of lower semicontinuous functions containing the ones considered in [2] and [22]. The proofs follow the lines of corresponding results in [12] and illustrate how the abstract theory is linked to more "concrete" settings. The main result in this part (Theorem 2) is of general interest.

1. Preliminaries. Let X be a metric space endowed with the metric d and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. We recall two notions of "slope" of f at a point $u \in \mathrm{dom}\,(f) = \{u \in X : f(u) < +\infty\}$.

According to [11], define

$$|\nabla f|(u) = \begin{cases} \limsup\limits_{v \to u} \dfrac{f(u) - f(v)}{d(u,v)} & \text{if } u \text{ is not a local minimum of } f \\ 0 & \text{if } u \text{ is a local minimum of } f. \end{cases}$$

The extended real number $|\nabla f|(u)$ is called the (*strong*) *slope* of f at u.

We now recall from [12] the notion of *weak slope*. In a first step, the definition is given for $f : X \to \mathbb{R}$ *continuous*: the *weak slope* of f at a point $u \in X$, denoted by $|df|(u)$, is defined as the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and $\eta : B(u; \delta) \times [0, \delta] \to X$ continuous with

$$d(\eta(v,t), v) \le t$$

$$f(\eta(v,t)) \le f(v) - \sigma t,$$

where $B(u; \delta)$ denotes the closed ball of radius δ centered at u.

In the case of a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the weak slope of f at $u \in \mathrm{dom}\,(f)$ is defined in the following indirect way: let

$$\mathrm{epi}\,(f) = \{(v, \xi) \in X \times \mathbb{R} : f(v) \le \xi\}$$

denote the epigraph of f, that we consider as a metric space with the metric

$$d((v,\xi),(w,\mu)) = d(v,w) + |\xi - \mu|;$$

set ([11]):

$$\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}, \quad \mathcal{G}_f(v, \xi) = \xi,$$

and define

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{1 - |d\mathcal{G}_f|(u, f(u))} & \text{if } \ |d\mathcal{G}_f|(u, f(u)) < 1 \\ +\infty & \text{if } \ |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

Notice that the function $\mathcal{G}_f$ being Lipschitz continuous of constant 1, $|d\mathcal{G}_f|(u, \xi) \leq 1$ for every $(u, \xi) \in \mathrm{epi}\,(f)$.

Indeed, the latter definition is not quite the one given in [12] since we chose a different (equivalent) metric on epi(f); of course, it is still consistent with the former definition whenever f is continuous: see [12, Proposition 2.3] – just changing metric. Furthermore, the following lower estimate of $|df|(u)$ still holds. For $b \in \mathbb{R}$, we let, as usual,

$$f^b := \{v \in X \ : \ f(v) \leq b\}.$$

Proposition 0 (see [12, Proposition 2.5]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathrm{dom}\,(f)$. Assume that there exist $\delta > 0$, $b > f(u)$, $\sigma > 0$ and a continuous $\eta : (B(u; \delta) \cap f^b) \times [0, \delta] \to X$ such that*

$$d(\eta(v, t), v) \leq t$$

$$f(\eta(v, t)) \leq f(v) - \sigma t.$$

Then, $|df|(u) \geq \sigma$.

It readily follows from the definitions that $|df|(u) \leq |\nabla f|(u)$ for each u. Equality holds for many important classes of functions; still, the weak slope seems to be the suitable notion in order to develop a general critical point theory, see [12], [8] and Section 3 of this note. We say that u is a *critical point* of f if $|df|(u) = 0$.

Let us mention that the notion of weak slope, as defined in the continuous case, was introduced independently in [17], after a similar definition was given in [16].

Throughout this note we apply the usual convention $\inf \emptyset = +\infty$.

2. A coercivity result. Let (X, d) be a metric space and $f, f_h : X \to \mathbb{R} \cup \{+\infty\}$, $h \in \mathbb{N}$, be lower semicontinuous functions such that

(1) for all $u \in \mathrm{dom}\,(f)$, there exists $u_h \to u$ with $f_h(u_h) \to f(u)$

and

$$\liminf_{h\to\infty}\left(\inf_X f_h\right) \geq \inf_X f \tag{2}$$

(this inequality is indeed an equality, because of (1)).

The following consequence of Ekeland's variational principle (see, e.g., [2]) extends [12, Corollary 3.4].

Proposition 1. *Let X be a complete metric space and $f, f_h : X \to \mathbb{R}\cup\{+\infty\}$, $h \in \mathbb{N}$, be lower semicontinuous functions such that* (1) *and* (2) *hold. Let Y be a nonempty subset of X and $\varepsilon > 0$, $\lambda > 0$ be such that*

$$\inf_Y f < \inf_X f + \varepsilon\lambda.$$

Then, for any $h_0 \in \mathbb{N}$ there exist $h \geq h_0$ and $u_h \in X$ such that

$$|f_h(u_h) - \inf_X f| < \varepsilon\lambda,$$

$$d(u_h, Y) < \lambda,$$

$$|\nabla f_h|(u_h) < \varepsilon.$$

Proof. Let $0 < \varepsilon' < \varepsilon$ and $u \in Y$ such that $f(u) < \inf_X f + \varepsilon'\lambda$ and $\varepsilon'' \in]\varepsilon', \varepsilon[$ be fixed. Let $n \in \mathbb{N}$, $h \geq h_0$ and $v_h \in \mathrm{dom}\,(f_h)$ with $0 < n\varepsilon''/(n-1) < \varepsilon$, and

$$\inf_X f_h \geq \inf_X f - (\varepsilon'' - \varepsilon')\lambda/2,$$

$$d(v_h, u) < \lambda/n, \quad f_h(v_h) \leq f(u) + (\varepsilon'' - \varepsilon')\lambda/2,$$

according to (1) and (2). Then,

$$f_h(v_h) < \inf_X f_h + \varepsilon''\lambda;$$

according to Ekeland's variational principle, there exists $u_h \in X$ with

$$f_h(u_h) \leq f_h(v_h) \implies |f_h(u_h) - \inf_X f| < \varepsilon\lambda,$$

$$d(u_h, v_h) \leq \frac{n-1}{n}\lambda \implies d(u_h, Y) \leq d(u_h, u) \leq d(u_h, v_h) + d(v_h, u) < \lambda,$$

and

$$f_h(v) \geq f_h(u_h) - \frac{n}{n-1}\varepsilon'' d(v, u_h) \qquad \text{for all } v \in X,$$

so that $|\nabla f_h|(u_h) \leq \dfrac{n}{n-1}\varepsilon'' < \varepsilon$. $\square$

Given f and (f_h) as above, we say that f satisfies *condition* $(PSB)^*$ (resp., *condition* $(PS)^*$) if whenever (f_{h_k}) is a subsequence of (f_h) and $(u_k) \subset X$ are such that $(f_{h_k}(u_k))$ is bounded and $|\nabla f_{h_k}|(u_k) \to 0$ then (u_k) is bounded (resp., (u_k) has a convergent subsequence).

In the next results we consider the following strengthening of condition (2):

$$\text{(3)} \qquad \text{for all closed } Y \subset X, \qquad \liminf_{h\to\infty}\left(\inf_Y f_h\right) \geq \inf_Y f.$$

Conditions (1) and (3) are a kind of "uniform Γ-convergence" of (f_h) to f (for the notion of Γ-convergence, see [1], [10]).

We also assume given a function $F : X \to \mathbb{R}$, bounded on bounded subsets of X, and with the property that there exist $\gamma_1, \gamma_2 > 0$ such that

$$\text{(4)} \qquad d(u,v) < \gamma_1 \implies |F(u) - F(v)| < \gamma_2;$$

for example, F could be Lipschitz continuous, or, in the case when X is a Banach space, uniformly continuous. Following [9], we say that the function f is *F-bounded from below* if $f^a \subset F^b$ for some $a, b \in \mathbb{R}$ (this is obviously the case if f is bounded below), and that f is *F-coercive* if $f(u) \to +\infty$ as $F(u) \to +\infty$.

Theorem 1. *Let X be a complete metric space, $f, f_h : X \to \mathbb{R} \cup \{+\infty\}$, $h \in \mathbb{N}$, be lower semicontinuous functions satisfying* (1) *and* (3)*, and $F : X \to \mathbb{R}$ be a function bounded on bounded subsets of X and satisfying* (4)*. If f is F-bounded from below and satisfies condition $(PSB)^*$, then f is F-coercive.*

Proof. Let $a, b \in \mathbb{R}$ be such that $f^a \subset F^b$. Define a nonincreasing sequence of subsets of X by setting

$$X_k := \overline{X \setminus F^{b+(2k+1)\gamma_2}}, \quad k = 0, 1, 2, \ldots\ .$$

Of course, we may assume that F is not bounded above — otherwise there is nothing to prove — so that the X_k's are nonempty. Since $X_0 \subset X \setminus F^b$, because of (4), the function f is bounded below on each X_k. We may further assume that $\inf_{X_k} f < +\infty$ for all k — otherwise we are done. Set:

$$\alpha_k := \inf_{X_{k+1}} f - \inf_{X_k} f \geq 0$$

and apply Proposition 1 to f with $X := X_k$, $Y := X_{k+1}$, $\varepsilon := \alpha_k + 1/k$ and $\lambda := \gamma_1$ to obtain sequences (f_{h_k}), (u_k) with $h_k \to \infty$ as $k \to \infty$, $u_k \in X_k$ and

$$\text{(5)} \qquad f_{h_k}(u_k) < \inf_{X_{k+1}} f + \gamma_1/k,$$

$$d(u_k, X_{k+1}) < \gamma_1, \tag{6}$$

and

$$|\nabla(f_{h_k|X_k})|(u_k) < \alpha_k + 1/k. \tag{7}$$

According to (4), $F(v) > b + (2k+2)\gamma_2$ for all $v \in X_{k+1}$; from (6) and (4) again, we deduce that $F(u) > b + (2k+1)\gamma_2$ for u in a neighborhood of u_k, and it follows that

$$|\nabla(f_{h_k|X_k})|(u_k) = |\nabla f_{h_k}|(u_k);$$

also, $F(u_k) \to +\infty$, which implies that (u_k) is unbounded.

Now, assuming that f is not F-coercive, so that $(\inf_{X_k} f)$ is convergent and $\alpha_k \to 0$, we get from (5) and (7) that

$$(f_{h_k}(u_k)) \text{ is bounded and } \quad |\nabla f_{h_k}|(u_k) \to 0,$$

contradicting condition $(PSB)^*$. □

Corollary 1. *Let $(E, \|\cdot\|)$ be a Banach space and $f, f_h : E \to \mathbb{R} \cup \{+\infty\}$, $h \in \mathbb{N}$, be lower bounded, lower semicontinuous functions satisfying* (1) *and* (3)*. Then, f is $\|\cdot\|$-coercive if and only if f satisfies condition* $(PSB)^*$.

Proof. The "if" part is a special case of Theorem 1, the "only if" part follows easily from (3). □

Remarks. (i) Corollary 1 contains the coercivity result of [19] which is stated for X a Banach space, f is C^1, f_h is the restriction of f to a linear subspace X_h with $X = \overline{\cup X_h}$ and condition $(PS)^*$ holds; indeed, we have $|\nabla f_h|(u) = \|f_h'(u)\|_{X_h^*}$ for all $u \in X_h$, see e.g. [12] (see also Proposition 2 below) and it is clear that this is still true if we extend f_h to all of X by giving it the value $+\infty$ outside of X_h. Variants of condition $(PS)^*$ in such a context have been used by several authors, starting with [4], [20].

(ii) In the case when $f_h \equiv f$ for all $h \in \mathbb{N}$, we shall speak of condition (PSB) instead of $(PSB)^*$. Let us note that condition (PSB) is equivalent to the condition that whenever $(u_k) \subset X$ is such that $(f(u_k))$ is bounded and $|\nabla(f)|(u_k) \to 0$ then (u_k) has a bounded *subsequence* — this kind of condition is used in [3], [9]. The same remark is valid for condition $(PSB)^*$.

(iii) See [9] for various relevant choices of the function F, in relation with applications to PDE'S. In that respect, let us note that, reasoning in a similar way as in the proof of Theorem 1, one shows the following result, in the spirit of [9, Proposition 2F].

Theorem 1′. *Let X be a complete metric space, $f, f_h : X \to \mathbb{R} \cup \{+\infty\}$, $h \in \mathbb{N}$, be lower semicontinuous functions satisfying* (1) *and* (3) *and $F : X \to \mathbb{R}$ be a continuous function satisfying* (4)*. Assume that for some $c \in \mathbb{R}$, it holds:*

$$\forall a < c, \quad \exists\, r \in \mathbb{R}, \quad f^a \subset F^r; \qquad \forall a > c, \quad \forall\, r \in \mathbb{R}, \quad f^a \not\subset F^r.$$

Then, there exist a subsequence (f_{h_k}) of (f_h) and a sequence $(u_k) \subset X$ such that

$$f_{h_k}(u_k) \to c, \quad |\nabla f_{h_k}|(u_k) \to 0 \quad \textit{and} \quad F(u_k) \to +\infty.$$

As a matter of fact, this result implies Theorem 1, but we preferred to state the less general result, the meaning of which is easier to grasp.

3. A class of nonsmooth functions. Let E be a Banach space with (topological) dual E^* and $f : E \to \mathbb{R} \cup \{+\infty\}$ a function, not identically equal to $+\infty$. Recall that the Gâteaux subdifferential of f at $u \in \mathrm{dom}\,(f)$ is defined by

$$\partial^G f(u) = \{\alpha \in E^* \ : \ \liminf_{t \to 0^+} \frac{f(u + tv) - f(u)}{t} \geq \langle \alpha, v \rangle \text{ for all } \ v \in X\}.$$

We shall assume that f is of the form

(F) $f = \phi + \psi$ with $\phi : E \to \mathbb{R}$ Gâteaux differentiable and $\psi : E \to \mathbb{R} \cup \{+\infty\}$ convex.

In this case, $\partial^G f = \phi' + \partial\psi$, where ψ' is the Gâteaux derivative and $\partial\psi$ the Fenchel subdifferential of Convex Analysis. Setting

$$\ell(u) := \{\inf \|\phi'(u) + \alpha\| \ : \ \alpha \in \partial\psi(u)\}$$

we show that $|\nabla f|(u) \geq \ell(u)$ for $u \in \mathrm{dom}\,(f)$, so that Corollary 1 contains the coercivity result of [14], because then, if every sequence $(u_h) \subset E$ with $(f(u_h))$ bounded and $\ell(u_h) \to 0$ is bounded, then f satisfies condition (PSB). We appeal to the following separation lemma, which is essentially [22, Lemma 2.1] (with a simpler proof).

Lemma. *Let E be a Banach space, $g : E \to \mathbb{R} \cup \{+\infty\}$ be convex and $\sigma > 0$ with*

$$g(0) = 0 \quad \textit{and} \quad g(u) + \sigma\|u\| \geq 0 \textit{ for all } u \in E.$$

Then there exists $\alpha \in E^*$ such that

$$\langle \alpha, u \rangle \leq g(u) \quad \textit{for all} \quad u \in E \quad \textit{and} \quad \|\alpha\| \leq \sigma.$$

P r o o f. It is an immediate consequence of the Mazur-Orlicz Theorem (see e.g., [15, Theorem p. 27]). □

Proposition 2. *Let E be a Banach space, $f : E \to \mathbb{R} \cup \{+\infty\}$ be a function of the form* **(F)**, *and $u \in \mathrm{dom}\,(f)$. Then, $|\nabla f|(u) \geq \ell(u)$.*

P r o o f. We may assume $\ell(u) > 0$ (in particular, u is not a local minimum of f). Let $\sigma > 0$ with $\ell(u) > \sigma$; using the lemma (with $g = \langle \phi'(u), \cdot \rangle + \psi(\cdot + u) - \psi(u)$), we find $w \in E$ such that

$$\langle \phi'(u), w - u \rangle + \psi(w) - \psi(u) < -\sigma \|w - u\|. \tag{8}$$

Using the definition of Gâteaux derivative and the convexity of ψ, it follows that for $t > 0$ small enough

$$\frac{f(u + t(w-u)) - f(u)}{t\|w - u\|} \leq -\sigma,$$

and from the definition of slope that $|\nabla f|(u) \geq \sigma$; hence the conclusion. □

Under the assumption that ϕ and ψ were lower semicontinuous, the class of functions of the form **(F)** was termed "smooth" in [3], for the reason that all results therein could be obtained through Ekeland's variational principle instead of using the smooth variational principle of Borwein and Preiss. From the point of view of critical point theory, however, some additional regularity seems necessary in order to obtain a "good" class of functions. In what follows, we shall assume that f is of the form **(F)** and satisfies

(H) the Gâteaux derivative $\phi' : E \to E^*$ of ϕ is norm-to-weak* continuous and ψ is lower semicontinuous.

Results in critical point theory are established in [22] for functions of the type $f = \phi + \psi$ with $\phi \in C^1(E, \mathbb{R})$, ψ convex lower semicontinuous, and in [2], [13] for continuous ϕ with a norm-to-weak* continuous Gâteaux derivative, by means of the variational principle.

It is shown in [12] that the general nonsmooth critical point theory developed in [8], [12] applies to the class of functions considered in [22] (see also [7], [21]). This theory is based on a general deformation theorem for continuous functions defined on metric spaces and reduction of the lower semicontinuous case to the continuous one by means of the function $\mathcal{G}_f$ whenever

$$\inf\{|d\mathcal{G}_f|(u, \xi) : \xi > f(u)\} > 0.$$

The next results show that this theory applies to a function f verifying **(F)** and **(H)**.

Let us mention first that, if we assume that the Gâteaux derivative of the function ϕ is norm-to-weak* continuous, then it is locally bounded, which implies that ϕ is actually locally Lipschitz.

Proposition 3. *Let E be a Banach space, $f : E \to \mathbb{R} \cup \{+\infty\}$ a function of the form* **(F)** *satisfying* **(H)**, *and $u \in \mathrm{dom}\,(f)$. Then, $|df|(u) \geq \ell(u)$.*

Proof. As in the proof of Proposition 2, we may assume that $\ell(u) > 0$ and, given $\sigma \in]0, \ell(u)[$, we find $w \in E$ such that (8) above holds. Because of hypothesis **(H)**, there exists $\delta > 0$ such that $w \notin B(u; 2\delta)$ and

$$\langle \phi'(v), w - v \rangle + \psi(w) - \psi(v) \leq -\sigma \|w - v\| \quad \text{for all} \quad v \in B(u; 2\delta).$$

Set:

$$\eta(v, t) = v + t \frac{w - v}{\|w - v\|}, \quad v \in B(u; \delta), \; t \in [0, \delta].$$

It is now easy to verify (in particular, using the fact that η is a "flow") that

$$f(\eta(v, t)) \leq f(v) - \sigma t \quad \text{for } v \in B(u; \delta), \; t \in [0, \delta].$$

By Proposition 0 we have $|df|(u) \geq \sigma$ and the conclusion follows. □

Observe that if $f : E \to \mathbb{R} \cup \{+\infty\}$ satisfies **(F)** and **(H)** and $(u_h) \subset E$ is such that $\ell(u_h) \to 0$, $f(u_h) \to c$ and $u_h \to u$, then $\ell(u) = 0$ and $f(u) = c$, which suggests that f is well-behaved from the point of view of critical point theory. This fact is expressed, in a different way, by Proposition 3 and Corollary 2 below (thanks to the theory of [12], [8]).

Theorem 2. *Let X be a metric space, $f_0 : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, $f_1 : X \to \mathbb{R}$ be locally Lipschitz, $f := f_0 + f_1$ and $(u, \xi) \in \mathrm{epi}\,(f)$. Then,*

$$|d\mathcal{G}_{f_0}|(u, \xi - f_1(u)) = 1 \quad \Longrightarrow \quad |d\mathcal{G}_f|(u, \xi) = 1.$$

Proof. Assume that f_1 is Lipschitz of constant k in $B(u; \delta_0)$ $(k, \delta_0 > 0)$. Let $\varepsilon \in]0, 1]$ be fixed and let $\delta \in]0, \delta_0]$ and $\eta = (\eta_1, \eta_2) : B((u, \xi - f_1(u)); \delta) \times [0, \delta] \to \mathrm{epi}\,(f_0)$ continuous with

$$d(\eta((v, \mu), t), (v, \mu)) = d(\eta_1((v, \mu), t), v) + |\eta_2(v, \mu) - \mu| \leq t,$$

$$|d\mathcal{G}_{f_0}|(\eta((v, \mu), t) - |d\mathcal{G}_{f_0}|(v, \mu) = \eta_2((v, \mu), t) - \mu \leq -(1 - \varepsilon)t.$$

Then, $d(\eta_1((v, \mu), t), v) \leq \varepsilon t$.

Let $\delta' \in]0,\delta]$ be such that $(v, \mu - f_1(v)) \in B((u, \xi - f_1(u)); \delta)$ whenever $(v,\mu) \in B((u,\xi);\delta')$ and define $\tilde\eta : B((u,\xi);\delta') \times [0,\delta'] \to \operatorname{epi}(f)$ by

$$\begin{aligned}\tilde\eta((v,\mu),t) \;&=\; \Big(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}), \eta_2((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}) + \\ &\qquad + f_1(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}))\Big).\end{aligned}$$

Clearly, $\tilde\eta$ is well-defined and continuous, and

$$\begin{aligned}&d(\tilde\eta((v,\mu),t),(v,\mu)) = \\ &\quad = d(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}), v) + |\eta_2((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}) - \mu + \\ &\qquad\qquad + f_1(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}))| \\ &\quad \le d(\eta((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}), (v,\mu - f_1(v))) + \\ &\qquad\qquad + |f_1(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon})) - f_1(v)| \\ &\quad \le \frac{t}{1+k\varepsilon} + k\varepsilon\frac{t}{1+k\varepsilon} = t.\end{aligned}$$

Also, it holds:

$$\begin{aligned}&\mathcal{G}_f(\tilde\eta((v,\mu),t)) = \\ &\quad = \eta_2((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon}) + f_1(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon})) \\ &\quad \le \mu - f_1(v) - (1-\varepsilon)\frac{t}{1+k\varepsilon} + f_1(\eta_1((v,\mu - f_1(v)), \frac{t}{1+k\varepsilon})) \\ &\quad \le \mu - (1-\varepsilon)\frac{t}{1+k\varepsilon} + k\varepsilon\frac{t}{1+k\varepsilon} = \mathcal{G}_f(v,\mu) - \frac{1-\varepsilon-k\varepsilon}{1+k\varepsilon}t.\end{aligned}$$

Hence,

$$|d\mathcal{G}_f|(u,\xi) \ge \frac{1-\varepsilon-k\varepsilon}{1+k\varepsilon}$$

and the conclusion follows from the arbitrariness of $\varepsilon > 0$. □

Remark. The implication in Theorem 2 is indeed an equivalence, the reverse implication being obtained from it replacing f_0 by $f = f_0 + f_1$ and f_1 by $-f_1$.

Corollary 2. *Let E be a Banach space, $f : E \to \mathbb{R} \cup \{+\infty\}$, $f = \phi + \psi$, satisfy* **(F)** *and* **(H)** *and let $(u,\xi) \in \operatorname{epi}(f)$ with $\xi > f(u)$. Then,*

$$|d\mathcal{G}_f|(u,\xi) = 1.$$

P r o o f. It is shown in [12, Theorem (3.13)] that the result holds for convex lower semicontinuous functions, so that $|d\mathcal{G}_\psi|(u, \xi - \phi(u)) = 1$. As mentioned before, ϕ is locally Lipschitz on E and the conclusion follows from Theorem 2. □

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