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ON A VARIATIONAL APPROACH TO SOME QUASILINEAR PROBLEMS

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Abstract. We prove some multiplicity results concerning quasilinear elliptic equations with natural growth conditions. Techniques of nonsmooth critical point theory are employed.

1. Introduction. In this paper, we will be concerned with two problems related to a quasilinear elliptic equation of the form

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum\limits_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) + \omega & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $a_{ij}(x,s) = a_{ji}(x,s)$. As we pointed out in [7, 8, 9], the first difficulty is that classical critical point theory fails in the case of quasilinear equations with natural growth conditions. In fact, let us consider the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega G(x,u)\,dx - \langle \omega, u\rangle,$$

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where $G(x,s) = \int_0^s g(x,t)\,dt$. Under reasonable assumptions on a_{ij} and g, it is possible to prove that f is continuous and that for every $u \in H_0^1(\Omega)$ and $v \in C_0^\infty(\Omega)$

$$\lim_{t\to 0} \frac{f(u+tv)-f(u)}{t} = \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v\,dx +$$

$$+\frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u \right) v\,dx - \int_\Omega g(x,u) v\,dx - \langle \omega, v\rangle.$$

However, we cannot expect f to be of class C^1 or even locally Lipschitz continuous.

On the other hand, for similar reasons

$$\left\{ u \mapsto -\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u - g(x,u) \right\}$$

is not well defined as an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and the topological methods, applied so far in the literature, cannot be directly adapted to this setting.

We will use a variational method based on the nonsmooth critical point theory of [10, 11]. Similar abstract techniques have been developed also in [13, 14]. We will prove an Ambrosetti-Rabinowitz type result for a symmetric superlinear problem and an Ambrosetti-Prodi type result for a jumping problem. We will essentially follow [7, 8, 9], but we will impose a weaker form of assumption $(a.4)$ below. For the convenience of the reader we repeat the relevant material from [7, 8, 9] without proof, thus making our exposition self-contained.

Finally, let us mention that different techniques of nonsmooth critical point theory have been applied to quasilinear equations in [3, 20].

2. Functionals of the calculus of variations. Let Ω be a bounded open subset of $\mathbb{R}^n$. For the sake of simplicity, let us suppose $n \geq 3$. Let $f : H_0^1(\Omega) \to \mathbb{R}$ be a functional of the form

$$f(u) = \int_\Omega L(x,u,Du)\,dx - \langle \omega, u\rangle. \tag{2.1}$$

The associated Euler equation is formally given by the quasilinear problem

$$\begin{cases} -\sum_{j=1}^n D_{x_j}(D_{\xi_j} L(x,u,Du)) + D_s L(x,u,Du) = \omega & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{2.2}$$

Assume that $\omega \in H^{-1}(\Omega)$ and that

$$L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

is such that:

$$\text{(2.3)} \qquad \begin{cases} \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^n & L(x,s,\xi) \text{ is measurable with respect to } x, \\ \text{for a.e. } x \in \Omega & L(x,s,\xi) \text{ is of class } C^1 \text{ with respect to } (s,\xi). \end{cases}$$

Assume also the following growth conditions:
there exist $a_0 \in L^1(\Omega)$, $b_0 \in \mathbb{R}$, $a_1 \in L^1_{loc}(\Omega)$, $b_1 \in L^\infty_{loc}(\Omega)$ such that
for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ we have

$$|L(x,s,\xi)| \leq a_0(x) + b_0(|s|^{\frac{2n}{n-2}} + |\xi|^2), \tag{2.4}$$

$$|D_s L(x,s,\xi)| \leq a_1(x) + b_1(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2), \tag{2.5}$$

$$|D_{\xi_j} L(x,s,\xi)| \leq a_1(x) + b_1(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2). \tag{2.6}$$

Under these conditions, it is readily seen that f is continuous and for every $u \in H^1_0(\Omega)$:

$$D_s L(x,u,Du) \in L^1_{loc}(\Omega), \qquad D_{\xi_j} L(x,u,Du) \in L^1_{loc}(\Omega).$$

Definition 2.1. *We say that u is* a weak solution *of* (2.2), if $u \in H^1_0(\Omega)$ and

$$-\sum_{i=j}^{n} D_{x_j}(D_{\xi_j} L(x,u,Du)) + D_s L(x,u,Du) = \omega$$

in $D'(\Omega)$.

In order to apply variational methods, let us introduce a natural adaptation of the Palais-Smale condition.

Definition 2.2. *Let $c \in \mathbb{R}$. A sequence (u_h) in $H^1_0(\Omega)$ is said to be* a concrete Palais-Smale sequence at level c ($(CPS)_c$-sequence*, for short*) *for f, if* $\lim_h f(u_h) = c$*,*

$$-\sum_{j=1}^{n} D_{x_j}(D_{\xi_j} L(x,u_h,Du_h)) + D_s L(x,u_h,Du_h) \in H^{-1}(\Omega)$$

eventually as $h \to \infty$ and

$$\left(-\sum_{j=1}^{n} D_{x_j}(D_{\xi_j} L(x,u_h,Du_h)) + D_s L(x,u_h,Du_h) - \omega \right) \to 0$$

strongly in $H^{-1}(\Omega)$.

We say that f *satisfies* the concrete Palais-Smale condition at level c ($(CPS)_c$ *for short), if every* $(CPS)_c$*-sequence for* f *admits a strongly convergent subsequence in* $H_0^1(\Omega)$.

The next results are adaptations to the functional f of some classical theorems of mountain pass type (see [2, 17, 21]).

Theorem 2.3. *Let* (D, S) *be a compact pair, let* $\psi : S \to H_0^1(\Omega)$ *be a continuous map and let*

$$\Phi = \left\{ \varphi \in C(D, H_0^1(\Omega)) : \varphi_{|S} = \psi \right\}.$$

Assume that there exists a closed subset A *of* $H_0^1(\Omega)$ *such that*

$$\inf_A f \geq \max_{\psi(S)} f,$$

$$A \cap \psi(S) = \emptyset \text{ and } A \cap \varphi(D) \neq \emptyset \text{ for all } \varphi \in \Phi.$$

If f *satisfies the concrete Palais-Smale condition at level*

$$c = \inf_{\varphi \in \Phi} \max_{\varphi(D)} f,$$

then there exists a weak solution u *of* (2.2) *with* $f(u) = c$. *Furthermore, if* $\inf_A f \geq c$, *then there exists a weak solution* u *of* (2.2) with $f(u) = c$ and $u \in A$.

Proof. The case $\omega = 0$ can be found in [9, Theorem 2.1.5]. The extension to the general case is straightforward. □

Theorem 2.4. *Let* $v_0, v_1 \in H_0^1(\Omega)$. *Suppose that there exists* $r > 0$ *such that* $\|v_1 - v_0\| > r$ *and*

$$\inf\{f(u) : u \in H_0^1(\Omega), \|u - v_0\| = r\} > \max\{f(v_0), f(v_1)\}.$$

Let

$$\Gamma = \{\gamma : [0,1] \to H_0^1(\Omega) \text{ continuous with } \gamma(0) = v_0, \gamma(1) = v_1\}$$

and assume that f *satisfies the concrete Palais-Smale condition at the two levels*

$$c_1 = \inf_{\overline{B(v_0,r)}} f, \quad c_2 = \inf_{\gamma \in \Gamma} \max_{[0,1]} (f \circ \gamma).$$

Then $c_1 < c_2$ *and there exist a weak solution* u_1 *of* (2.2), *with* $\|u_1 - v_0\| < r$ *and* $f(u_1) = c_1$, *and a second weak solution* u_2 *with* $f(u_2) = c_2$.

P r o o f. See [8, Theorem 1.3]. $\square$

Theorem 2.5. *Suppose that* $\omega = 0$ *and that*

$$L(x, -s, -\xi) = L(x, s, \xi)$$

for a.e. $x \in \Omega$ *and every* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. *Assume also that*

(a) *there exist* $\rho > 0$, $\alpha > f(0)$ *and a subspace* $V \subset H_0^1(\Omega)$ *of finite codimension such that*

$$\forall u \in V : \|u\| = \rho \Rightarrow f(u) \geq \alpha;$$

(b) *for every finite dimensional subspace* $W \subset H_0^1(\Omega)$, *there exists* $R > 0$ *such that*

$$\forall u \in W : \|u\| > R \Rightarrow f(u) \leq f(0);$$

(c) f *satisfies* $(CPS)_c$ *for any* $c \geq \alpha$.

Then there exists a sequence (u_h) *of weak solutions of* (2.2) *with*

$$\lim_h f(u_h) = +\infty.$$

P r o o f. See [9, Theorem 2.1.6]. $\square$

3. Homogeneous quadratic functionals of the gradient. In this section, we restrict our attention to the case:

$$L(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, s) \xi_i \xi_j.$$

Let $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \leq i,\, j \leq n)$ be such that

$$(a.1) \qquad \begin{cases} \forall s \in \mathbb{R} & a_{ij}(\cdot, s) \text{ is measurable,} \\ \text{for a.e. } x \in \Omega & a_{ij}(x, \cdot) \text{ is of class } C^1, \\ \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ 1 \leq i, j \leq n & a_{ij}(x, s) = a_{ji}(x, s); \end{cases}$$

there exists $C > 0$ such that for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $1 \leq i,\, j \leq n$,

$$(a.2) \qquad |a_{ij}(x, s)| \leq C, \quad |D_s a_{ij}(x, s)| \leq C;$$

there exists $\nu > 0$ such that for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$,

$$(a.3) \qquad \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2;$$

there exists $R > 0$ such that for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$,

$$(a.4) \qquad |s| \geq R \Longrightarrow \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \geq 0.$$

Because of $(a.1)$ and $(a.2)$, conditions (2.3), (2.4), (2.5) and (2.6) are clearly satisfied. Moreover, because of $(a.2)$ and $(a.3)$, there exists $M > 0$ such that

$$(3.1) \qquad \frac{1}{2}\left|\sum_{i,j=1}^{n} D_s a_{ij}(x,s)\xi_i\xi_j\right| \leq M \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j$$

for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$.

Let us begin with a consequence of the Brezis-Browder Theorem.

Theorem 3.1. *Let $\omega \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be a weak solution of*

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,u)D_{x_i}u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u = \omega.$$

Let $v \in H_0^1(\Omega)$ be such that

$$\left[\left(\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u\right)v\right]^- \in L^1(\Omega).$$

Then we have

$$\left(\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u\right)v \in L^1(\Omega)$$

and

$$\int_\Omega \left\{\sum_{i,j=1}^{n} a_{ij}(x,u)D_{x_i}uD_{x_j}v + \left[\frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u\right]v\right\} dx = \langle\omega, v\rangle.$$

Proof. The assertion follows by the result of [6]. $\square$

Now, we will state some regularity results.

Lemma 3.2. *Given $A_{ij} \in L^\infty(\Omega)\,(1 \le i,j \le n)$ with*

$$\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2,$$

let $\omega \in W^{-1,q}(\Omega)$, $\mu \in L^1_{loc}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that

$$\int_\Omega \sum_{i,j=1}^n A_{ij} D_{x_i} u D_{x_j} v\, dx = \int_\Omega \mu v\, dx + \langle \omega, v\rangle \quad \forall v \in C_0^\infty(\Omega).$$

Assume that there exist $\alpha \in L^r(\Omega), c > 0$ such that

$$\mu(x)u(x) \le \alpha(x)(u(x))^2 \quad a.e.\ in\ \Omega\ when\ |u(x)| \ge c.$$

Then the following facts hold:

(a) if $2 \le q < n$ and $r \ge \frac{n}{2}$, we have $u \in L^{\frac{nq}{n-q}}(\Omega)$;

(b) if $q > n$ and $r > \frac{n}{2}$, we have $u \in L^\infty(\Omega)$.

Proof. Take $v_1 = (u-\rho)^+$ and $v_2 = -(u+\rho)^-$ with $\rho \ge c$. Then $v_k \in H^1_0(\Omega)$ and we have

$$\mu(x)v_k(x) \le \alpha(x)u(x)v_k(x) \in L^1(\Omega).$$

By the Brezis-Browder Theorem [6], it follows that $(\mu v_k) \in L^1(\Omega)$ and

$$\int_\Omega \sum_{i,j=1}^n A_{ij} D_{x_i} u D_{x_j} v_k\, dx = \int_\Omega \mu v_k\, dx + \langle \omega, v_k\rangle \le \int_\Omega \alpha u v_k\, dx + \langle \omega, v_k\rangle.$$

Now, by well known techniques of regularity theory (see e.g. [15, 19]) the assertion follows. $\square$

Theorem 3.3. *Let $\alpha \in L^r(\Omega)$, $\omega \in W^{-1,q}(\Omega)$ and let $u \in H^1_0(\Omega)$ be a weak solution of*

$$-\sum_{i,j=1}^n D_{x_j}(a_{ij}(x,u)D_{x_i}u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u + \alpha u = \omega.$$

Then the following facts hold:

(*a*) *if* $2 \le q < n$ *and* $r \ge \frac{n}{2}$, *we have* $u \in L^{\frac{nq}{n-q}}(\Omega)$;

(*b*) *if* $q > n$ *and* $r > \frac{n}{2}$, *we have* $u \in L^{\infty}(\Omega)$.

Proof. Set

$$A_{ij} = a_{ij}(x, u),$$

$$\mu = -\frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u) D_{x_i} u D_{x_j} u - \alpha u.$$

By $(a.4)$, we have

$$\mu u = -\frac{1}{2} \sum_{i,j=1}^{n} u D_s a_{ij}(x, u) D_{x_i} u D_{x_j} u - \alpha u^2 \le -\alpha u^2 \quad \text{a.e. in } \Omega \text{ when } |u(x)| \ge R.$$

By the previous lemma the assertion follows. □

We point out that, if a weak solution u belongs to $H_0^1(\Omega) \cap L^{\infty}(\Omega)$, one can apply the regularity results contained in [15].

Now we come to some compactness properties.

Lemma 3.4. *Let* (u_h) *be a bounded sequence in* $H_0^1(\Omega)$ *such that*

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x, u_h) D_{x_i} u_h) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h$$

belongs to $H^{-1}(\Omega)$ *and is strongly convergent in* $H^{-1}(\Omega)$.

Then it is possible to extract a subsequence (u_{h_k}) *strongly convergent in* $H_0^1(\Omega)$.

Proof. Let

$$\omega_h = -\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x, u_h) D_{x_i} u_h) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h.$$

Up to a subsequence, u_h is convergent to some u weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Moreover, by [4, Theorem 2.1] we have, up to a further subsequence, $Du_h \to Du$ a.e. in Ω.

At first, let us prove that

$$\text{(3.2)} \quad \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x, u) D_{x_i} u D_{x_j} u \, dx + \frac{1}{2} \int_{\Omega} \left(\sum_{i,j=1}^{n} D_s a_{ij}(x, u) D_{x_i} u D_{x_j} u \right) u \, dx = \langle \omega, u \rangle$$

where $\omega \in H^{-1}(\Omega)$ is the limit of ω_h.

We will use the same device of [5]. We consider the test functions

$$v_h = \varphi \exp\{-M(u_h+R)^+\}$$

where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, u_h^+ is the positive part of u_h, and $M > 0$ is defined in (3.1). By Theorem 3.1 v_h is an admissible test function, so that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{-M(u_h+R)^+\} D_{x_i}u_h D_{x_j}\varphi\, dx +$$

$$+\int_\Omega \left[\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h - M\sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}(u_h+R)^+\right]$$
$$\varphi \exp\{-M(u_h+R)^+\}\, dx - \langle \omega_h, \varphi \exp\{-M(u_h+R)^+\}\rangle = 0.$$

From $(a.4)$ and (3.1), we deduce that

$$\left[\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h - M\sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}(u_h+R)^+\right]$$
$$\varphi \exp\{-M(u_h+R)^+\} \leq 0,$$

and, by Fatou's lemma, we get

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) \exp\{-M(u+R)^+\} D_{x_i}u D_{x_j}\varphi\, dx +$$

$$+\int_\Omega \left[\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i}u D_{x_j}u - M\sum_{i,j=1}^n a_{ij}(x,u) D_{x_i}u D_{x_j}(u+R)^+\right]$$
$$\varphi \exp\{-M(u+R)^+\}\, dx \geq$$

$$\geq \langle \omega, \varphi \exp\{-M(u+R)^+\}\rangle \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \varphi \geq 0. \tag{3.3}$$

Now, we consider the test functions

$$\varphi_k = \varphi H(\frac{1}{k}u) \exp\{M(u+R)^+\}$$

with $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and

$$H : \mathbb{R} \to \mathbb{R},\ H \in C^1(\mathbb{R}),\ 0 \leq H \leq 1,$$

$$H = 1 \text{ on } [-1/2, 1/2], \quad H = 0 \text{ on }]-\infty, -1] \cup [1, \infty[.$$

Putting them in (3.3), we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i} u D_{x_j}(\varphi H(1/k\, u))\, dx +$$

$$+\frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u \right) \varphi H(1/k\, u)\, dx \geq$$

$$\geq \langle \omega, \varphi H(1/k\, u) \rangle \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{3.4}$$

Passing to the limit as $k \to \infty$ in (3.4), we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i} u D_{x_j} \varphi\, dx + \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u \right) \varphi\, dx \geq$$

$$\geq \langle \omega, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

In a similar way, by considering the test functions $v_h = \varphi \exp\{-M\,(u_h - R)^-\}$, it is possible to prove the opposite inequality. It follows:

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i} u D_{x_j} \varphi\, dx + \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u \right) \varphi\, dx =$$

$$= \langle \omega, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega). \tag{3.5}$$

By (3.5), $(a.4)$ and Theorem 3.1, we deduce (3.2).

Now, let us consider the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined in the following way

$$\zeta(s) = \begin{cases} Ms & 0 < s < R \\ MR & s \geq R \\ -Ms & -R < s < 0 \\ MR & s \leq -R \end{cases}$$

and let us prove that

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u_h) \exp\{\zeta(u_h)\} D_{x_i} u_h D_{x_j} u_h \leq$$

$$\leq \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) \exp\{\zeta(u)\} D_{x_i}u D_{x_j}u. \tag{3.6}$$

By $(a.4)$ and Theorem 3.1, the test functions $u_h \exp\{\zeta(u_h)\}$ are also admissible, so that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{\zeta(u_h)\} D_{x_i}u_h D_{x_j}u_h \, dx +$$

$$+ \int_\Omega \left[\frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h + \zeta'(u_h) \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \right]$$

$$u_h \exp\{\zeta(u_h)\} \, dx - \langle \omega_h, u_h \exp\{\zeta(u_h)\} \rangle \; = \; 0.$$

By $(a.4)$ and (3.1)

$$\left[\frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h + \zeta'(u_h) \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \right]$$

$$u_h \exp\{\zeta(u_h)\} \geq 0,$$

and, by Fatou's lemma, we get

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{\zeta(u_h)\} D_{x_i}u_h D_{x_j}u_h \, dx =$$

$$= \limsup_h \int_\Omega \left[-\frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h - \zeta'(u_h) \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \right]$$

$$u_h \exp\{\zeta(u_h)\} \, dx + \langle \omega_h, u_h \exp\{\zeta(u_h)\} \rangle \leq$$

$$\leq \int_\Omega \left[-\frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u) D_{x_i}u D_{x_j}u - \zeta'(u) \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i}u D_{x_j}u \right]$$

$$u \exp\{\zeta(u)\} \, dx + \langle \omega, u \exp\{\zeta(u)\} \rangle =$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) \exp\{\zeta(u)\} D_{x_i}u D_{x_j}u \, dx.$$

Thus, (3.6) is proved.

Finally, let us show that u_h converges to u in the strong topology of $H_0^1(\Omega)$. Let us observe that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}(u_h-u)D_{x_j}(u_h-u)\,dx =$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}u_h D_{x_j}u_h\,dx +$$

$$-2\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}u D_{x_j}u_h\,dx +$$

$$+\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}u D_{x_j}u\,dx. \tag{3.7}$$

For every $j = 1,\dots,n$, we have:

$$\lim_h \sum_{i=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}u = \sum_{i=1}^n a_{ij}(x,u)\exp\{\zeta(u)\}D_{x_i}u$$

in the strong topology of $L^2(\Omega)$. Then, by (3.6) we get:

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}(u_h-u)D_{x_j}(u_h-u)\,dx =$$

$$= \limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}u_h D_{x_j}u_h\,dx +$$

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)\exp\{\zeta(u)\}D_{x_i}u D_{x_j}u\,dx \le 0. \tag{3.8}$$

Using (3.8) and hypothesis $(a.3)$, we conclude that:

$$\nu \limsup_h \|Du_h - Du\|_{L^2}^2 \le$$

$$\le \nu \limsup_h \int_\Omega \exp\{\zeta(u_h)\}|D(u_h-u)|^2\,dx \le$$

$$\le \limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}(u_h-u)D_{x_j}(u_h-u)\,dx \le 0.$$

Then the assertion is proved. $\square$

In the last part of this section, we add the following assumption: there exists a uniformly Lipschitz continuous bounded function $\vartheta : \mathbb{R} \to [0, +\infty[$ such that

$$(a.5) \qquad \frac{1}{2} \sum_{i,j=1}^{n} s D_s a_{ij}(x,s) \xi_i \xi_j \le s\vartheta'(s) \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j$$

for a.e. $x \in \Omega$, a.e. $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$. Without loss of generality, we can assume that

$$\lim_{s\to-\infty} \vartheta(s) = \lim_{s\to+\infty} \vartheta(s)$$

and denote by $\overline{\vartheta}$ the common value. Let us also set

$$A_{ij}^{\pm}(x) = \lim_{s\to\pm\infty} a_{ij}(x,s)$$

(these limits exist by $(a.4)$).

Lemma 3.5. *Let (v_h) be a sequence weakly convergent to v in $H_0^1(\Omega)$ and (γ_h) a sequence weakly convergent to γ in $L^{\frac{n}{2}}(\Omega)$ with $|\gamma_h(x)| \le c(x)$ for some $c \in L^{\frac{n}{2}}(\Omega)$.*

Then $(\gamma_h v_h)$ is strongly convergent to γv in $H^{-1}(\Omega)$.

Proof. See [8, Lemma 3.1]. □

Lemma 3.6. *Let (u_h) be a sequence in $H_0^1(\Omega)$ and (ρ_h) a sequence in $]0, +\infty[$ with $\rho_h \to +\infty$ such that $(v_h) = \left(\frac{u_h}{\rho_h}\right)$ is weakly convergent to v in $H_0^1(\Omega)$. Let (γ_h) be a sequence weakly convergent to γ in $L^{\frac{n}{2}}(\Omega)$ with $|\gamma_h(x)| \le c(x)$ for some $c \in L^{\frac{n}{2}}(\Omega)$. Let (μ_h) be a sequence strongly convergent to μ in $L^{\frac{2n}{n+2}}(\Omega)$ and (δ_h) a sequence strongly convergent in $H^{-1}(\Omega)$ such that*

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}\varphi\, dx + \frac{1}{2}\int_\Omega \left(\sum_{i,j=1}^{n} D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \right) \varphi\, dx =$$

$$(3.9) \qquad = \int_\Omega \gamma_h u_h \varphi\, dx + \rho_h \int_\Omega \mu_h \varphi\, dx + \langle \delta_h, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega).$$

Then, it holds:

(a) (v_h) is strongly convergent to v in $H_0^1(\Omega)$;

(b) $(\gamma_h v_h)$ is strongly convergent to γv in $H^{-1}(\Omega)$;

(*c*) *there exist* η^+, $\eta^- \in L^\infty(\Omega)$ *such that*

$$\eta^+(x) = \begin{cases} \exp\{-\overline{\vartheta}\} & v(x) > 0 \\ \exp\{MR\} & v(x) < 0 \end{cases}$$

and $\exp\{-\overline{\vartheta}\} \leq \eta^+(x) \leq \exp\{MR\}$ *if* $v(x) = 0$,

$$\eta^-(x) = \begin{cases} \exp\{-\overline{\vartheta}\} & v(x) < 0 \\ \exp\{MR\} & v(x) > 0 \end{cases}$$

and $\exp\{-\overline{\vartheta}\} \leq \eta^-(x) \leq \exp\{MR\}$ *if* $v(x) = 0$,
and such that for every $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$:

$$\int_\Omega \sum_{i,j=1}^n A_{ij}\eta^+ D_{x_i}v D_{x_j}\varphi\, dx \geq \int_\Omega \gamma\eta^+ v\varphi\, dx + \int_\Omega \mu\eta^+\varphi\, dx,$$

$$\int_\Omega \sum_{i,j=1}^n A_{ij}\eta^- D_{x_i}v D_{x_j}\varphi\, dx \leq \int_\Omega \gamma\eta^- v\varphi\, dx + \int_\Omega \mu\eta^-\varphi\, dx,$$

where $A_{ij}(x) = \begin{cases} A_{ij}^+(x) & v(x) > 0 \\ A_{ij}^-(x) & v(x) < 0 \end{cases}$.

P r o o f. Up to a subsequence, v_h is convergent to v a.e. in Ω. From the previous lemma, it follows that $\gamma_h v_h$ is strongly convergent to γv in $H^{-1}(\Omega)$. Let $\zeta(s)$ be the function defined in Lemma 3.4. By $(a.4)$, the result in [6] allows us to put $\varphi = v_h \exp\{\zeta(u_h)\}$ in (3.9), yielding

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\} D_{x_i}u_h D_{x_j}v_h\, dx +$$

$$\int_\Omega \left[\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h + \zeta'(u_h)\sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h\right]$$

$$v_h \exp\{\zeta(u_h)\}\, dx =$$

$$= \int_\Omega \gamma_h u_h v_h \exp\{\zeta(u_h)\}\, dx + \rho_h \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\}\, dx + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle$$

By $(a.4)$ and (3.1) we have

$$\rho_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\} D_{x_i}v_h D_{x_j}v_h\, dx \leq$$

$$\leq \rho_h \int_\Omega \gamma_h v_h^2 \exp\{\zeta(u_h)\}\,dx + \rho_h \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\}\,dx + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle.$$

After division by ρ_h and using hypotheses on γ_h, μ_h and δ_h, we obtain

$$\lim_h \left(\int_\Omega \gamma_h v_h^2 \exp\{\zeta(u_h)\}\,dx + \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\}\,dx + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle\right) =$$

$$= \exp\{MR\}\left(\int_\Omega \gamma v^2\,dx + \int_\Omega \mu v\,dx\right)$$

and

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{\zeta(u_h)\} D_{x_i} v_h D_{x_j} v_h\,dx \leq$$

$$\leq \exp\{MR\}\left(\int_\Omega \gamma v^2\,dx + \int_\Omega \mu v\,dx\right). \tag{3.10}$$

Now, let us define

$$\vartheta_1(s) = \begin{cases} \vartheta(s) & s \geq 0 \\ Ms & -R \leq s \leq 0 \\ -MR & s \leq -R \end{cases}$$

and consider as test functions $(v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}$ $(k \in \mathbb{N})$. Putting them in (3.9), we get:

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{-\vartheta_1(u_h)\} D_{x_i} v_h D_{x_j}(v^+ \wedge k)\,dx +$$

$$+\frac{1}{\rho_h}\int_\Omega \left[\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u_h)\; D_{x_i}u_h D_{x_j}u_h \; -\vartheta_1'(u_h)\sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h\right]$$

$$(v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\,dx =$$

$$= \int_\Omega \gamma_h v_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\,dx + \int_\Omega \mu_h (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\,dx +$$

$$+\frac{1}{\rho_h}\langle \delta_h, (v^+ \wedge k)\exp\{-\vartheta_1(u_h)\}\rangle. \tag{3.11}$$

By $(a.4)$, (3.1) and $(a.5)$, we have:

$$\frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h - \vartheta_1'(u_h)\sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \leq 0.$$

On the other hand, we have:

$$\lim_h a_{ij}(x,u_h)\exp\{-\vartheta_1(u_h)\} D_{x_j}(v^+ \wedge k) = A_{ij}^+ \exp\{-\overline{\vartheta}\} D_{x_j}(v^+ \wedge k)$$

strongly in $L^2(\Omega)$,

$$\lim_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} = (v^+ \wedge k) \exp\{-\overline{\vartheta}\}$$

strongly in each $L^p(\Omega)$ with $p < \infty$,

$$\lim_h v_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} = v(v^+ \wedge k) \exp\{-\overline{\vartheta}\}$$

strongly in $L^{\frac{n}{n-2}}(\Omega)$ and

$$\lim_h \frac{1}{\rho_h} (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} = 0$$

weakly in $H_0^1(\Omega)$.

Letting $h \to +\infty$ in (3.11), we get

$$\int_\Omega \sum_{i,j=1}^n A_{ij}^+ \exp\{-\overline{\vartheta}\} D_{x_i} v D_{x_j} (v^+ \wedge k)\, dx \geq$$

$$\geq \int_\Omega \gamma v (v^+ \wedge k) \exp\{-\overline{\vartheta}\}\, dx + \int_\Omega \mu (v^+ \wedge k) \exp\{-\overline{\vartheta}\}\, dx.$$

Letting now $k \to +\infty$, after division by $\exp\{-\overline{\vartheta}\}$, we have

$$\int_\Omega \sum_{i,j=1}^n A_{ij}^+ D_{x_i} v^+ D_{x_j} v^+\, dx \geq \int_\Omega \gamma (v^+)^2\, dx + \int_\Omega \mu v^+\, dx. \tag{3.12}$$

Analogously, let us define the function

$$\vartheta_2(s) = \begin{cases} \vartheta(s) & s \leq 0 \\ -Ms & 0 \leq s \leq R \\ -MR & s \geq R \end{cases}$$

and consider as test functions $(v^+ \wedge k) \exp\{-\vartheta_2(u_h)\}$ $(k \in \mathbb{N})$. We obtain

$$\int_\Omega \sum_{i,j=1}^n A_{ij}^- D_{x_i} v^- D_{x_j} v^-\, dx \geq \int_\Omega \gamma (v^-)^2\, dx - \int_\Omega \mu v^-\, dx. \tag{3.13}$$

Thus, (3.12) and (3.13) give

$$\int_\Omega \sum_{i,j=1}^n A_{ij} D_{x_i} v D_{x_j} v\, dx \geq \int_\Omega \gamma v^2\, dx + \int_\Omega \mu v\, dx. \tag{3.14}$$

It follows from (3.10) and (3.14):

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}v_h D_{x_j}v_h\,dx \le$$

$$\le \exp\{MR\}\int_\Omega \sum_{i,j=1}^n A_{ij}D_{x_i}vD_{x_j}v\,dx. \tag{3.15}$$

Now, let us show that v_h converges to v in the strong topology of $H_0^1(\Omega)$. Let us observe that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}(v_h-v)D_{x_j}(v_h-v)\,dx =$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}v_h D_{x_j}v_h\,dx +$$

$$-2\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}vD_{x_j}v_h\,dx +$$

$$+\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}vD_{x_j}v\,dx \tag{3.16}$$

and

$$\lim_h \sum_{i=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}v = \exp\{MR\}\sum_{i=1}^n A_{ij}D_{x_i}v \qquad \forall j=1,\dots,n$$

strongly in $L^2(\Omega)$.

Then, passing to the lim sup in (3.16), we have by (3.15)

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)\exp\{\zeta(u_h)\}D_{x_i}(v_h-v)D_{x_j}(v_h-v)\,dx =$$

$$= \limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h\exp\{\zeta(u_h)\})D_{x_i}v_h D_{x_j}v_h\,dx +$$

$$-\exp\{MR\}\int_\Omega \sum_{i,j=1}^n A_{ij}D_{x_i}vD_{x_j}v\,dx \le 0. \tag{3.17}$$

By (3.17) and $(a.3)$, we conclude that:

$$\nu \limsup_h \|Dv_h - Dv\|_{L^2}^2 \leq$$

$$\leq \limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) \exp\{\zeta(u_h)\} D_{x_i}(v_h - v) D_{x_j}(v_h - v)\, dx \leq 0.$$

So v_h converges strongly to v in $H_0^1(\Omega)$.

Up to a subsequence, $\exp\{-\vartheta_1(u_h)\}$ is weakly* convergent in $L^\infty(\Omega)$ to some η^+. Of course, we have:

$$\eta^+(x) = \begin{cases} \exp\{-\overline{\vartheta}\} & v(x) > 0 \\ \exp\{MR\} & v(x) < 0 \end{cases}$$

and $\exp\{-\overline{\vartheta}\} \leq \eta^+(x) \leq \exp\{MR\}$ if $v(x) = 0$. Then, let us consider as test functions $\varphi \exp\{-\vartheta_1(u_h)\}$ with $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Let us observe that we have

$$\begin{aligned}
&\lim_h a_{ij}(x,u_h)\exp\{-\vartheta_1(u_h)\} D_{x_i} v_h = A_{ij}\eta^+ D_{x_i} v && \text{strongly in } L^2(\Omega),\\
&\lim_h v_h \varphi \exp\{-\vartheta_1(u_h)\} = v\varphi\eta^+ && \text{strongly in } L^{\frac{n}{n-2}}(\Omega),\\
&\lim_h \mu_h \varphi = \mu\varphi && \text{strongly in } L^1(\Omega),\\
&\lim_h \frac{1}{\rho_h}\varphi \exp\{-\vartheta_1(u_h)\} = 0 && \text{weakly in } H^{-1}(\Omega).
\end{aligned}$$

Therefore, putting the test functions in (3.9), we get like in the previous argument,

$$\int_\Omega \sum_{i,j=1}^n A_{ij}\eta^+ D_{x_i} v D_{x_j}\varphi\, dx \geq \int_\Omega \gamma\eta^+ v\varphi\, dx + \int_\Omega \mu\eta^+\varphi\, dx.$$

Then, this inequality holds for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$.

In a similar way, by means of the test functions $\varphi \exp\{-\vartheta_2(u_h)\}$, we get

$$\int_\Omega \sum_{i,j=1}^n A_{ij}\eta^- D_{x_i} v D_{x_j}\varphi\, dx \leq \int_\Omega \gamma\eta^- v\varphi\, dx + \int_\Omega \mu\eta^-\varphi\, dx,$$

where η^- is the weak* limit of some subsequence of $\exp\{-\vartheta_2(u_h)\}$. $\square$

4. Quadratic functionals of the gradient. This section contains the main tools we need, in order to improve the results of [7, 8, 9]. We consider the case

$$L(x,s,\xi) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j - G(x,s),$$

where $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \le i, j \le n)$ satisfy the conditions $(a.1)$, $(a.2)$, $(a.3)$ and $(a.4)$ of the previous section, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $G(x,s) = \int_0^s g(x,t)\,dt$. We assume that there exist $a \in L^r(\Omega)$, $r \ge \frac{2n}{n+2}$, and $b \in \mathbb{R}$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ we have

$$|g(x,s)| \le a(x) + b|s|^{\frac{n+2}{n-2}}. \tag{4.1}$$

Because of $(a.1)$, $(a.2)$ and (4.1), conditions (2.3), (2.4), (2.5) and (2.6) are satisfied.

Theorem 4.1. *Let $\omega \in W^{-1,q}(\Omega)$ and let $u \in H_0^1(\Omega)$ be a weak solution of*

$$-\sum_{i,j=1}^n D_{x_j}(a_{ij}(x,u)D_{x_i}u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u = g(x,u) + \omega.$$

Then the following facts hold:

(a) if $\frac{2n}{n+2} \le r < \frac{n}{2}$ and $q \ge \frac{nr}{n-r}$, we have $u \in L^{\frac{nr}{n-2r}}(\Omega)$;

(b) if $r > \frac{n}{2}$ and $q > n$, we have $u \in L^\infty(\Omega)$.

Proof. It is sufficient to follow the argument of [9, Theorem 2.2.5], with [9, Theorem 2.2.3] substituted by Theorem 3.3. $\square$

Definition 4.2. *We say that g is a nonlinearity with* subcritical growth, *if for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega)$ such that*

$$|g(x,s)| \le a_\varepsilon(x) + \varepsilon|s|^{\frac{n+2}{n-2}} \tag{4.2}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

Of course, (4.2) implies (4.1) with $r = \frac{2n}{n+2}$.

Now let $\omega \in H^{-1}(\Omega)$ and let us consider the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_{x_i}uD_{x_j}u\,dx - \int_\Omega G(x,u)\,dx - \langle \omega, u\rangle.$$

Let us provide some results we will use dealing with $(CPS)_c$ condition.

Theorem 4.3. *Assume that g has subcritical growth. Then for any $c \in \mathbb{R}$ the following facts are equivalent:*

(a) f satisfies $(CPS)_c$;

(b) every $(CPS)_c$-sequence for f is bounded in $H_0^1(\Omega)$.

P r o o f. It is sufficient to follow the argument of [9, Theorem 2.2.8], with [9, Theorem 2.2.4] substituted by Lemma 3.4. $\square$

Theorem 4.4. *Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence for f. Then for every $\rho > 0$ and $\varepsilon > 0$ there exists $K(\rho, \varepsilon) > 0$ such that for all $h \in \mathbb{N}$,*

$$\int_{\{|u_h| \leq \rho\}} \sum_{i,j=1}^{n} a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq \varepsilon \int_{\{|u_h| > \rho\}} \sum_{i,j=1}^{n} a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K(\rho, \varepsilon).$$

P r o o f. Let

$$\omega_h = -\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x, u_h) D_{x_i} u_h) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, dx - g(x, u_h),$$

let $\sigma > 0$ and let

$$\vartheta_1(s) = \begin{cases} s & \text{if } |s| < \sigma \\ -s + 2\sigma & \text{if } \sigma \leq s < 2\sigma \\ -s - 2\sigma & \text{if } -2\sigma < s \leq -\sigma \\ 0 & \text{if } |s| \geq 2\sigma \end{cases}.$$

Then we have

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x, u_h) D_{x_i} u_h D_{x_j}(\vartheta_1(u_h)) \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, \vartheta_1(u_h) \, dx \leq$$

$$\leq \int_\Omega g(x, u_h) \vartheta_1(u_h) \, dx + \|\omega_h\|_{H^{-1}} \|\vartheta_1(u_h)\|_{H_0^1}.$$

Taking into account (4.1), it follows

$$\int_{\{|u_h| \leq \sigma\}} \sum_{i,j=1}^{n} a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, dx - \int_{\{\sigma < |u_h| \leq 2\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, dx +$$

$$+ \frac{1}{2} \int_{\{|u_h| \leq \sigma\}} \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, \vartheta_1(u_h) \, dx +$$

$$+ \frac{1}{2} \int_{\{\sigma < |u_h| \leq 2\sigma\}} \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_{x_i} u_h D_{x_j} u_h \, \vartheta_1(u_h) \, dx \leq$$

$$\leq \int_\Omega \left(a(x) + b|2\sigma|^{\frac{n+2}{n-2}}\right) \sigma \, dx + \frac{1}{\nu}\|\omega_h\|^2_{H^{-1}} + \frac{\nu}{4}\|\vartheta_1(u_h)\|^2_{H^1_0}.$$

There exists $K_0 > 0$ such that $\|\omega_h\|_{H^{-1}} \leq K_0$. Then, observing that

$$\|\vartheta_1(u_h)\|^2_{H^1_0} \leq \int_{\{|u_h|\leq\sigma\}} |Du_h|^2 \, dx + \int_{\{\sigma<|u|\leq 2\sigma\}} |Du_h|^2 \, dx \leq$$

$$\leq \frac{1}{\nu}\int_{\{|u_h|\leq\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx +$$

$$+\frac{1}{\nu}\int_{\{\sigma<|u_h|\leq 2\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx,$$

from (3.1) we deduce that

$$\left(1 - \sigma M - \frac{1}{4}\right) \int_{\{|u_h|\leq\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx \leq$$

$$\leq \left(1 + \sigma M + \frac{1}{4}\right) \int_{\{\sigma<|u_h|\leq 2\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx +$$

$$+ \int_\Omega \left(a(x) + b|2\sigma|^{\frac{n+2}{n-2}}\right) \sigma \, dx + \frac{K_0^2}{\nu}.$$

If we set $\sigma = \dfrac{1}{2M}$, we easily find an inequality of the form

$$\int_{\{|u_h|\leq\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx \leq$$

$$\leq K_1 \int_{\{\sigma<|u_h|\leq 2\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i}u_h D_{x_j}u_h \, dx + K_2.$$

If we reapply the same argument, taking $\vartheta_2(s)$ defined in such a way

$$\vartheta_2(s) = \begin{cases} 0 & \text{if } |s| \leq \sigma \\ s - \sigma & \text{if } \sigma < s < 2\sigma \\ s + \sigma & \text{if } -2\sigma < s < -\sigma \\ -s + 3\sigma & \text{if } 2\sigma \leq s < 3\sigma \\ -s - 3\sigma & \text{if } -3\sigma < s \leq -2\sigma \\ 0 & \text{if } |s| \geq 3\sigma \end{cases},$$

we get

$$\int_{\{\sigma<|u_h|\leq 2\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq K_1' \int_{\{2\sigma<|u_h|\leq 3\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_2',$$

hence

$$\int_{\{|u_h|\leq 2\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq K_1'' \int_{\{2\sigma<|u_h|\leq 3\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_2''.$$

Iterating this argument, we get for any $k \geq 1$

$$\int_{\{|u_h|\leq k\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq K_1(k) \int_{\{k\sigma<|u_h|\leq (k+1)\sigma\}} \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_2(k). \tag{4.3}$$

Now, let $k \geq 1$ be such that $k\sigma \geq \rho$ and $k\sigma \geq R$. Take $\delta \in]0,1[$ and let

$$\vartheta_\delta(s) = \begin{cases} 0 & \text{if } |s| \leq k\sigma \\ s - k\sigma & \text{if } k\sigma < s < (k+1)\sigma \\ s + k\sigma & \text{if } -(k+1)\sigma < s < -k\sigma \\ -\delta s + \sigma + \delta(k+1)\sigma & \text{if } (k+1)\sigma \leq s < (k+1)\sigma + \frac{\sigma}{\delta} \\ -\delta s - \sigma - \delta(k+1)\sigma & \text{if } -(k+1)\sigma - \frac{\sigma}{\delta} < s \leq -(k+1)\sigma \\ 0 & \text{if } |s| \geq (k+1)\sigma + \frac{\sigma}{\delta} \end{cases}.$$

As before, we get

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_h) D_{x_i} u_h D_{x_j}(\vartheta_\delta(u_h)) \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} D_s a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, \vartheta_\delta(u_h) \, dx \leq$$

$$\leq \int_\Omega g(x,u_h) \vartheta_\delta(u_h) \, dx + \|\omega_h\|_{H^{-1}} \|\vartheta_\delta(u_h)\|_{H_0^1} \leq$$

$$\leq \int_\Omega g(x,u_h) \vartheta_\delta(u_h) \, dx + \frac{1}{4\delta} \|\omega_h\|_{H^{-1}}^2 + \delta \|\vartheta_\delta(u_h)\|_{H_0^1}^2.$$

Now, by $(a.4)$ we deduce that

$$\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, \vartheta_\delta(u_h)\, dx \geq 0.$$

Moreover

$$\|\vartheta_\delta(u_h)\|^2_{H_0^1} \leq \int_{\{k\sigma<|u_h|\leq(k+1)\sigma\}} |\nabla u_h|^2\, dx + \int_{\{|u_h|>(k+1)\sigma\}} |\nabla u_h|^2\, dx \leq$$

$$\leq \frac{1}{\nu} \int_{\{k\sigma<|u_h|\leq(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx +$$

$$+ \frac{1}{\nu} \int_{\{|u_h|>(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx.$$

Then, by (4.1) it follows

$$\left(1-\frac{\delta}{\nu}\right) \int_{\{k\sigma<|u_h|\leq(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq \left(\delta+\frac{\delta}{\nu}\right) \int_{\{|u_h|>(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx +$$

$$+ \int_\Omega \left(a(x) + b\left|(k+1)\sigma + \frac{\sigma}{\delta}\right|^{\frac{n+2}{n-2}}\right) \sigma\, dx + \frac{K_0^2}{4\delta},$$

hence

$$\int_{\{k\sigma<|u_h|\leq(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq \frac{\nu\delta+\delta}{\nu-\delta} \int_{\{|u_h|>(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_3(k,\delta).$$

Combining this inequality with (4.3), we get

$$\int_{\{|u_h|\leq\rho\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq \int_{\{|u_h|\leq k\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx \leq$$

$$\leq K_1(k) \int_{\{k\sigma<|u_h|\leq(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_2(k) \leq$$

$$\leq K_1(k) \frac{\nu\delta+\delta}{\nu-\delta} \int_{\{|u_h|>(k+1)\sigma\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_1(k) K_3(k,\delta) + K_2(k) \leq$$

$$\leq K_1(k) \frac{\nu\delta+\delta}{\nu-\delta} \int_{\{|u_h|>\rho\}} \sum_{i,j=1}^n a_{ij}(x,u_h) D_{x_i} u_h D_{x_j} u_h \, dx + K_1(k) K_3(k,\delta) + K_2(k).$$

If we take δ such that

$$K_1(k) \frac{\nu\delta+\delta}{\nu-\delta} \leq \varepsilon,$$

the assertion follows. □

5. The superlinear case. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $n \geq 3$, let $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \leq i, j \leq n)$ satisfy the conditions $(a.1)$, $(a.2)$, $(a.3)$ and $(a.4)$, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth as in Definition 4.2 and let $G(x,s) = \int_0^s g(x,t)\, dt$.

We shall consider the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_{x_i} u D_{x_j} u \, dx - \int_\Omega G(x,u)\, dx$$

and the associated Euler equation

$$\text{(5.1)} \quad \begin{cases} -\sum\limits_{i=1}^n D_{x_j}(a_{i,j}(x,u) D_{x_i} u) + \frac{1}{2} \sum\limits_{i=1}^n D_s a_{i,j}(x,u) D_{x_i} u D_{x_j} u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} .$$

Let us make the following further assumptions:
there exist $q > 2$, $\gamma \in]0, q-2[$ and $R' > 0$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ we have

$$\text{(5.2)} \qquad |s| \geq R' \Longrightarrow 0 < qG(x,s) \leq sg(x,s),$$

$$\text{(5.3)} \qquad |s| \geq R' \Longrightarrow \sum_{i=1}^n s D_s a_{i,j}(x,s) \xi_i \xi_j \leq \gamma \sum_{i=1}^n a_{i,j}(x,s) \xi_i \xi_j.$$

Assumption (5.2) means that g is superlinear at infinity in the sense of [2, 17, 21]. Because of $(a.2)$ and $(a.3)$, condition (5.3) seems not to be particularly restrictive.

We can now formulate the main result of this section, which is an extension to the quasilinear case of a well-known theorem of Ambrosetti and Rabinowitz (see [2, 17, 21]).

Theorem 5.1. *Assume that*

$$a_{i,j}(x,-s) = a_{i,j}(x,s), \quad g(x,-s) = -g(x,s)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $i, j = 1, \dots, n$.

Then there exists a sequence (u_h) of weak solutions of (5.1) *with*

$$\lim_h f(u_h) = +\infty.$$

Moreover, if g satisfies (4.1) *with $r > \frac{n}{2}$, all these solutions are in $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. It is sufficient to follow the argument of [9], with [9, Theorems 2.2.5, 2.2.8 and 2.2.9] substituted by Theorems 4.1, 4.3 and 4.4, respectively. □

6. A jumping problem. Let Ω be a connected bounded open subset of $\mathbb{R}^n$ with $n \geq 3$, let $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \leq i,\ j \leq n)$ satisfy the conditions $(a.1)$, $(a.2)$, $(a.3)$, $(a.4)$ and $(a.5)$, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let $\omega \in H^{-1}(\Omega)$.

Let us make the following further assumptions:
there exist $a \in L^{\frac{2n}{n+2}}(\Omega)$ and $b \in L^{\frac{n}{2}}(\Omega)$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$

$$|g(x,s)| \leq a(x) + b(x)|s|; \tag{6.1}$$

there exist α, $\beta \in \mathbb{R}$ such that for a.e. $x \in \Omega$:

$$\lim_{s \to -\infty} \frac{g(x,s)}{s} = \alpha, \quad \lim_{s \to +\infty} \frac{g(x,s)}{s} = \beta. \tag{6.2}$$

Finally, setting

$$A_{ij}^{\pm}(x) = \lim_{s \to \pm\infty} a_{ij}(x,s),$$

let λ_k [resp. $\tilde{\lambda}_k$] denote the eigenvalues of the linear operator $-\sum D_{x_j}(A_{ij}^{+} D_{x_i} u)$ [resp. $-\sum D_{x_j}(A_{ij}^{-} D_{x_i} u)$] with homogeneous Dirichlet condition. Let φ_1 [resp. $\tilde{\varphi}_1$] be a nonnegative eigenfunction corresponding to λ_1 [resp. $\tilde{\lambda}_1$].

We are interested in a jumping problem of Ambrosetti-Prodi type [1]. For further results in the semilinear case, see [12, 16, 18] and references therein.

Theorem 6.1. *Assume that $\alpha > \tilde{\lambda}_1$ and $\beta < \lambda_1$. Then there exists $\hat{t} \in \mathbb{R}$ such that for every $t > \hat{t}$ the equation*

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,u) D_{x_i} u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_{x_i} u D_{x_j} u =$$

$$= g(x,u) + t\varphi_1 + \omega$$

has at least two weak solutions in $H_0^1(\Omega)$.

Moreover, if $\omega \in W^{-1,p}(\Omega)$ *for some* $p > n$ *and* $a, b \in L^r(\Omega)$ *with* $r > \frac{n}{2}$, *such solutions belong to* $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 6.2. *Let* α *and* β *be as in the previous theorem. Then there exists* $\tilde{t} \in \mathbb{R}$ *such that for every* $t < \tilde{t}$ *the equation*

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,u)D_{x_i}u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u =$$

$$= g(x,u) + t\tilde{\varphi}_1 + \omega$$

has no weak solutions in $H_0^1(\Omega)$.

Corollary 6.3. *Let* α *and* β *be as in the previous theorem. Let us suppose that* $A_{ij}^+(x) = A_{ij}^-(x)$ *for a.e.* $x \in \Omega$.

Then there exist $\overline{t} \in \mathbb{R}$ *and* $\underline{t} \in \mathbb{R}$ *such that the equation*

$$-\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x,u)D_{x_i}u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_{x_i}uD_{x_j}u =$$

$$= g(x,u) + t\varphi_1 + \omega$$

has at least two weak solutions in $H_0^1(\Omega)$ *for every* $t > \overline{t}$ *and no weak solutions in* $H_0^1(\Omega)$ *for every* $t < \underline{t}$.

There results can be proved as in [8]. We have only to substitute [8, Proposition 1.4] with Theorem 4.3 and [8, Lemma 3.2] with Lemma 3.6. The L^∞-regularity of u follows from Theorem 4.1.

REFERENCES

[1] A. Ambrosetti, G. Prodi. On the inversion of some differentiable mappings with singularities between Banach spaces. *Ann. Mat. Pura. Appl.* **93** (1972), 231-247.

[2] A. Ambrosetti, P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349-381.

[3] D. Arcoya, L. Boccardo. Critical points for multiple integrals of the Calculus of Variations. *Arch. Rational Mech. Anal.*, in press.

[4] L. Boccardo, F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* **19** (1992), 581-597.

[5] L. Boccardo, F. Murat, J.-P. Puel. Existence de solutions non bornées pour certaines équations quasi-linéaires. *Portugaliae Math.* **41** (1982), 507-534.

[6] H. Brezis, F. E. Browder. Sur une propriété des espaces de Sobolev. *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978), A113-A115.

[7] A. Canino. Multiplicity of solutions for quasilinear elliptic equations. *Topol. Methods Nonlinear Anal.*, in press.

[8] A. Canino. On a jumping problem for quasilinear elliptic equations. *Math. Z.*, in press.

[9] A. Canino, M. Degiovanni. Nonsmooth critical point theory and quasilinear elliptic equations. *Topological Methods in Differential Equations and Inclusions* (Montreal, 1994), 1–50, *NATO ASI Series*, **C 472**, Kluwer Academic Publishers, Dordrecht, 1995.

[10] J.-N. Corvellec, M. Degiovanni, M. Marzocchi. Deformation properties for continuous functionals and critical point theory. *Topol. Methods Nonlinear Anal.* **1** (1993), 151-171.

[11] M. Degiovanni, M. Marzocchi. A critical point theory for nonsmooth functionals. *Ann. Mat. Pura Appl. (4)* **167** (1994), 73-100.

[12] A. R. Domingos, M. Ramos. Remarks on a class of elliptic problems with asymmetric nonlinearities. *Nonlinear Anal.*, in press.

[13] A. Ioffe, E. Schwartzman. Metric critical point theory 1. Morse regularity and homotopic stability of a minimum. *J. Math. Pures Appl.* **75** (1996), 125-153.

[14] G. Katriel. Mountain pass theorems and global homeomorphism theorems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11** (1994), 189-209.

[15] O. A. Ladyzhenskaya, N. N. Ural'tseva. Linear and Quasilinear Elliptic Equations. Nauka Press, Moscow 1964; Academic Press, New York, 1968.

[16] A. Marino, A. M. Micheletti, A. Pistoia. Some variationals results on semilinear elliptic problems with asymptotically nonsymmetric behaviour. *Nonlinear Analysis. A tribute in honour of Giovanni Prodi*, 243-256, Scuola Normale Superiore, Pisa, 1991.

[17] P. H. Rabinowitz. Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Series Math., vol. **65**, Amer. Math. Soc., Providence, R.I., 1986.

[18] S. Solimini. Some remarks on the number of solutions of some nonlinear elliptic problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2** (1985), 143-156.

[19] G. Stampacchia. Équations elliptiques du second ordre à coefficients discontinus. Séminaire de Mathématiques Supérieures, vol. **16**, Montréal, 1966.

[20] M. Struwe. Quasilinear elliptic eigenvalue problems. *Comment. Math. Helv.* **58** (1983), 509-527.

[21] M. Struwe. Variational Methods. Springer-Verlag, Berlin, 1990.

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