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BÄCKLUND–DARBOUX TRANSFORMATIONS IN SATO'S GRASSMANNIAN

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Abstract. We define Bäcklund–Darboux transformations in Sato's Grassmannian. They can be regarded as Darboux transformations on maximal algebras of commuting ordinary differential operators. We describe the action of these transformations on related objects: wave functions, tau-functions and spectral algebras.

Introduction. Classically, a Darboux transformation [6] of a differential operator L, presented as a product $L = QP$, is defined by exchanging the places of the factors, i.e. $\overline{L} = PQ$. Obviously all versions of Darboux transformations have the property that if $\Phi(x)$ is an eigenfunction of L, i.e. $L\Phi = \lambda\Phi$ then $P\Phi$ is an eigenfunction of $\overline{L}$, i.e. $\overline{L}P\Phi = \lambda P\Phi$. Motivated by this characteristic property we give a version of Darboux transformation directly on wave functions. The plane W of Sato's Grassmannian is said to be a *Bäcklund–Darboux transformation* of the plane V iff the corresponding wave functions are connected by:

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Psi_V(x,z),$$
$$\Psi_V(x,z) = \frac{1}{f(z)} Q(x,\partial_x)\Psi_W(x,z)$$

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for some polynomials f, g and differential operators P, Q, or equivalently –

$$fV \subset W \subset \frac{1}{g}V.$$

To any plane $W \in Gr$ one can associate a maximal algebra $\mathcal{A}_W$ of commuting ordinary differential operators [11, 12, 13] (called a *spectral algebra*). Recall that a *rank* of $\mathcal{A}_W$ is the g.c.d. of the orders of the operators from $\mathcal{A}_W$. We prove that Bäcklund–Darboux transformations preserve the rank of the spectral algebra. Moreover if W is a Bäcklund–Darboux transformation of V such that $\mathcal{A}_V = \mathbb{C}[L_V]$ for some operator L_V then every operator from $\mathcal{A}_W$ is a Darboux transformation (in the sense of eq. (17) below) of an operator from $\mathcal{A}_V$.

In our terminology the set of rational solutions of the KP hierarchy [11, 13] coincides with the set of Bäcklund–Darboux transformations of the simplest plane $H_+ = \text{span}\{z^n | n = 0, 1, \ldots\}$. The corresponding tau-function τ_W is given by the so-called "*superposition low for wobbly solitons*" (see e.g. [13, 15]). We generalize this formula for a Bäcklund–Darboux transformation W of an arbitrary plane V provided that the wave function $\Psi_V(x, z)$ is well defined at all zeros of the polynomial $f(z)g(z)$ (see Theorem 2 below). In a particular but important case which we use it is proven in [1]. The case when $\Psi_V(x, z)$ is not well defined for some zero $z = \lambda$ of $f(z)g(z)$ is even more interesting (cf. [3, 4]). We obtain a formula for τ_W valid in the general situation (see Theorem 1 below).

A geometric interpretation of $\text{Ker}P$ can be given using the so-called conditions C (introduced in [15] for the rational solutions of KP). When the spectral curve $\text{Spec}\mathcal{A}_V = \mathbb{C}$ (i.e. $\mathcal{A}_V = \mathbb{C}[L_V]$) the spectral curve $\text{Spec}\mathcal{A}_W$ of W can be obtained from that of V by introducing singularities at points where the conditions C are supported – see [15] and also [3] (from [5] it is known that $\text{Spec}\mathcal{A}_W$ is an algebraic curve).

This paper may be considered as a part of our project [2]–[4] on the bispectral problem (see [8]). Although here we do not touch the latter, the present paper arose in the process of working on [3, 4]. We noticed that many facts, needed in [3, 4] can be naturally obtained in a more general situation. Apart from the applications to the bispectral problem we hope that some of the results can be useful elsewhere.

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1. Preliminaries on Sato's Grassmannian. The aim of this section is to recollect some facts and notation from Sato's theory of KP-hierarchy [12, 7, 13] needed in the paper. We use the approach of V. Kac and D. Peterson based on infinite wedge products (see e.g. [10]) and the recent survey paper by P. van Moerbeke [14].

Consider the space of formal Laurent series in z^{-1}

$$\mathbb{V} = \Big\{ \sum_{k\in\mathbb{Z}} a_k z^k \,\Big|\, a_k = 0 \text{ for } k \gg 0 \Big\}.$$

We define the fermionic Fock space F consisting of formal infinite sums of semi-infinite monomials

$$z^{i_0} \wedge z^{i_1} \wedge \ldots$$

such that $i_0 < i_1 < \ldots$ and $i_k = k$ for $k \gg 0$. Let gl_∞ be the Lie algebra of all $\mathbb{Z} \times \mathbb{Z}$ matrices having only a finite number of non-zero entries. One can define a representation r of gl_∞ in the fermionic Fock space F as follows

$$r(A)(z^{i_0} \wedge z^{i_1} \wedge \ldots) = Az^{i_0} \wedge z^{i_1} \wedge \ldots + z^{i_0} \wedge Az^{i_1} \wedge \ldots + \cdots . \tag{1}$$

The above defined representation r obviously cannot be continued on the Lie algebra $\widetilde{gl}_\infty$ of all $\mathbb{Z} \times \mathbb{Z}$ matrices with finite number of non-zero diagonals. If we regularize it by

$$\hat{r}(D)(z^{i_0} \wedge z^{i_1} \wedge \ldots) = [(d_{i_0} + d_{i_1} + \cdots) - (d_0 + d_1 + \cdots)](z^{i_0} \wedge z^{i_1} \wedge \ldots) \tag{2}$$

for $D = \mathrm{diag}(\ldots, d_{-1}, d_0, d_1, \ldots)$ and by (1) for an off-diagonal matrix this will give a representation of a central extension $\widehat{gl}_\infty = \widetilde{gl}_\infty \oplus \mathbb{C}c$ of $\widetilde{gl}_\infty$. Here the central charge c acts as multiplication by 1. Introduce also the shift matrices $\Lambda_n (n \in \mathbb{Z})$ representing the multiplication by z^n in the basis $\{z^i\}_{i\in\mathbb{Z}}$ of $\mathbb{V}$. Then $\hat{r}(\Lambda_k)$ generate a representation of the Heisenberg algebra:

$$\Big[\hat{r}(\Lambda_n), \hat{r}(\Lambda_m)\Big] = n\delta_{n,-m}.$$

There exists a unique isomorphism (see [10] for details):

$$\sigma \colon F \to B = \mathbb{C}\,[[t_1, t_2, t_3, \ldots]] \tag{3}$$

$$\sigma(\hat{r}(\Lambda_n)) = \frac{\partial}{\partial t_n}, \quad \sigma(\hat{r}(\Lambda_{-n})) = nt_n, \qquad n > 0, \tag{4}$$

known as the boson-fermion correspondence (B is called a bosonic Fock space).

Sato's Grassmannian Gr [12, 7, 13] consists of all subspaces $W \subset \mathbb{V}$ which have an admissible basis

$$w_k = z^k + \sum_{i<k} w_{ik} z^i, \quad k = 0, 1, 2, \ldots$$

To a plane $W \in Gr$ we associate a state $|W\rangle \in F$ as follows

$$|W\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \ldots$$

A change of the admissible basis results in a multiplication of $|W\rangle$ by a non-zero constant. Thus we define an embedding of Gr into the projectivization of F which is called a Plücker embedding. One of the main objects of Sato's theory is the *tau-function* of W defined as the image of $|W\rangle$ under the boson-fermion correspondence (3):

$$\tau_W(t) = \sigma(|W\rangle) = \langle 0|\, e^{H(t)}\, |W\rangle, \tag{5}$$

where $H(t) = -\sum_{k=0}^{\infty} t_k \hat{r}(\Lambda_k)$. Another important function connected to W is the *Baker* or *wave function*

$$\Psi_W(t,z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau\left(t - [z^{-1}]\right)}{\tau(t)}, \tag{6}$$

where $[z^{-1}]$ is the vector $(z^{-1}, z^{-2}/2, \ldots)$. We often use the notation $\Psi_W(x,z) = \Psi_W(t,z)|_{t_1=x, t_2=t_3=\cdots=0}$.

The Baker function $\Psi_W(x,z)$ contains the whole information about W and hence about τ_W, as the vectors $w_k = \partial_x^k \Psi_W(x,z)|_{x=0}$ form an admissible basis of W. We can expand $\Psi_W(t,z)$ in a formal series as

$$\Psi_W(t,z) = e^{\sum_{k=1}^{\infty} t_k z^k} \left(1 + \sum_{k>0} a_k(t) z^{-k}\right). \tag{7}$$

Introduce also the pseudo-differential operators $K_W(t,\partial_x) = 1 + \sum_{j>0} a_j(t) \partial_x^{-j}$ such that $\Psi_W(t,z) = K_W(t,\partial_x) e^{\sum_{k=1}^{\infty} t_k z^k}$ (the *wave operator*) and $P = K_W \partial_x K_W^{-1}$. Then P satisfies the following infinite system of non-linear differential equations

$$\frac{\partial}{\partial t_k} P = \left[P_+^k, P\right], \tag{8}$$

called the *KP hierarchy* and $\Psi_W(x,z)$ is an eigenfunction of $P(x,\partial_x)$:

$$P\Psi_W(x,z) = z\Psi_W(x,z). \tag{9}$$

A very important object connected to the plane W is the algebra A_W of polynomials $f(z)$ that leave W invariant:

$$A_W = \{f(z) | f(z) W \subset W\}. \tag{10}$$

For each $f(z) \in A_W$ one can show that there exists a unique differential operator $L_f(x,\partial_x)$, the order of L_f being equal to the degree of f, such that

$$L_f \Psi_W(x,z) = f(z) \Psi_W(x,z). \tag{11}$$

Explicitly we have

(12) $$L_f = K_W f(\partial_x) K_W^{-1}.$$

We denote the commutative algebra of these operators by $\mathcal{A}_W$, i.e.

(13) $$\mathcal{A}_W = \{L_f | L_f \Psi_W = f \Psi_W,\ f \in A_W\}.$$

Obviously A_W and $\mathcal{A}_W$ are isomorphic. We call $\mathcal{A}_W$ *spectral algebra* corresponding to the plane W. Following I. Krichever [11] we introduce the *rank* of $\mathcal{A}_W$ to be the dimension of the space of joint eigenfunctions of the operators from $\mathcal{A}_W$. It coincides with the greatest common divisor of the orders of the operators L_f.

We also use the notation $Gr^{(N)} := \{W \in Gr | z^N \in A_W\}$. It coincides with the subgrassmannian of solutions of the so-called N-th reduction of the KP hierarchy.

2. Definitions. In this section we introduce our basic definition of *Bäcklund–Darboux transformation* in the Sato's Grassmannian.

The classical Darboux transformation is defined on ordinary differential operators in the variable x, presented in a factorized form $L = QP$; it exchanges the places of the factors, i.e. the image of L is the operator $\overline{L} = PQ$. The next classical lemma answers the question when the factorization $L = QP$ is possible (see e.g. [9]).

Lemma 1. *L can be factorized as*

(14) $$L = QP \quad \textit{iff} \quad \mathrm{Ker}P \subset \mathrm{Ker}L.$$

In this case

(15) $$\mathrm{Ker}Q = P(\mathrm{Ker}L).$$

A slightly more general construction is the following one. For operators L and P such that the kernel of P is invariant under L, i.e.

(16) $$L(\mathrm{Ker}P) \subset \mathrm{Ker}P$$

we consider the transformation

(17) $$L \mapsto \overline{L} = PLP^{-1}.$$

The fact that $\overline{L}$ is a differential operator follows from Lemma 1. Indeed, $L(\mathrm{Ker}P) \subset \mathrm{Ker}P$ is equivalent to $\mathrm{Ker}P \subset \mathrm{Ker}(PL)$. If h is the characteristic polynomial of the linear operator $L|_{\mathrm{Ker}P}$ then $h(L|_{\mathrm{Ker}P}) = 0$, i.e.

$$\mathrm{Ker}P \subset \mathrm{Ker}h(L).$$

This shows that

$$h(L) = QP, \quad h(\overline{L}) = PQ$$

for some operator Q.

Now we come to our basic definition.

Definition 1. *We say that a plane W (or the corresponding wave function $\Psi_W(x,z)$) is a* Bäcklund–Darboux transformation *of the plane V (respectively wave function $\Psi_V(x,z)$) iff there exist (monic) polynomials $f(z)$, $g(z)$ and differential operators $P(x,\partial_x)$, $Q(x,\partial_x)$ such that*

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Psi_V(x,z), \tag{18}$$

$$\Psi_V(x,z) = \frac{1}{f(z)} Q(x,\partial_x)\Psi_W(x,z). \tag{19}$$

Here necessarily $\mathrm{ord}P = \deg g$ and $\mathrm{ord}Q = \deg f$. The polynomial $g(z)$ can be chosen arbitrary but of the same degree because the wave function $\Psi_W(x,z)$ is determined up to a multiplication by a formal series of the form $1 + \sum_{k=1}^{\infty} a_k z^{-k}$.

Note that a composition of two Bäcklund–Darboux transformations is again a Bäcklund–Darboux transformation. For example the bispectral potentials of [8] can be obtained by one Bäcklund–Darboux transformation in contrast to the finite number of "rational" Darboux transformations of Duistermaat and Grünbaum.

Simple consequences of Definition 1 are the identities

$$PQ\Psi_W(x,z) = f(z)g(z)\Psi_W(x,z), \tag{20}$$

$$QP\Psi_V(x,z) = f(z)g(z)\Psi_V(x,z), \tag{21}$$

i.e. the operator $\overline{L} = PQ \in \mathcal{A}_W$ is a Darboux transformation of $L = QP \in \mathcal{A}_V$. Obviously (18) implies the inclusion

$$gW \subset V. \tag{22}$$

Conversely, if (22) holds there exists P satisfying (18). Therefore a definition equivalent to Definition 1 is the following one.

Definition 2. *A plane W is a Bäcklund–Darboux transformation of a plane V iff*

$$fV \subset W \subset \frac{1}{g} V \tag{23}$$

for some polynomials $f(z)$, $g(z)$.

3. Bäcklund–Darboux transformations on the spectral algebras. In this section we study the behavior of the spectral algebra under Bäcklund–Darboux transformations. The following simple lemma will be useful.

Lemma 2. *In the notation of* (23) *and* (10)

$$fgA_V \subset A_W \subset \frac{1}{fg}A_V. \tag{24}$$

The *proof* is obvious from (23). □

Proposition 1. *The Bäckund–Darboux transformations preserve the rank of the spectral algebras, i.e. if W is a Bäckund–Darboux transformation of V then* $\mathrm{rank}\mathcal{A}_W = \mathrm{rank}\mathcal{A}_V$.

P r o o f. Let $\mathrm{rank}A_V = N$. Then Lemma 2 implies that $fg \in A_V$ and therefore $\deg fg = Nj$, $j \in \mathbb{N}$. The right inclusion in (24) gives $N|\mathrm{rank}A_W$. Because $\mathrm{rank}A_V = N$, A_V contains polynomials of degrees $ka + lb$, for $k, l \in \mathbb{Z}_{\geq 0}$ with $(a, b) = N$. The left inclusion in (24) implies that A_W contains polynomials of degrees $ka + lb + Nj$, for $k, l \in \mathbb{Z}_{\geq 0}$, i.e. $\mathrm{rank}A_W|N$ and therefore $\mathrm{rank}A_W = N$. □

The most important case of algebras $\mathcal{A}_V$ of $\mathrm{rank}N$ is

$$A_V = \mathbb{C}[z^N], \quad \mathcal{A}_V = \mathbb{C}[L_V] \tag{25}$$

for some natural number N and a differential operator L_V of order N (see [3]). This corresponds to the case when the spectral curve $\mathrm{Spec}\mathcal{A}_V$ of V is $\mathbb{C}$. We shall describe $\mathcal{A}_W$ for a Bäcklund–Darboux transformation W of V for which (25) holds. First observe that due to (21) we have

$$f(z)g(z) = h(z^N), \tag{26}$$

$$QP = h(L_V) \tag{27}$$

for some polynomial h.

Proposition 2. *If* $\mathcal{A}_V = \mathbb{C}[L_V]$, $\mathrm{ord}L_V = N$ *then*

$$A_W = \left\{ u \in \mathbb{C}[z^N] \mid u(L_V)\mathrm{Ker}P \subset \mathrm{Ker}P \right\}, \tag{28}$$

$$\mathcal{A}_W = \left\{ Pu(L_V)P^{-1} \mid u \in A_W \right\}. \tag{29}$$

P r o o f. Since $A_W \subset \mathbb{C}[z]$ the right inclusion of (24) with $A_V = \mathbb{C}[z^N]$ and $f(z)g(z) = h(z^N)$ implies $A_W \subset \mathbb{C}[z^N]$. Let $u(z^N) \in A_W$ and L be the corresponding operator from $\mathcal{A}_W$ (see (12)), such that

$$L\Psi_W(x, z) = u(z^N)\Psi_W(x, z).$$

Using (18) we compute

$$L\Psi_W(x,z) = L\frac{1}{g(z)}P\Psi_V(x,z) = \frac{1}{g(z)}LP\Psi_V(x,z)$$

and

$$u(z^N)\Psi_W(x,z) = u(z^N)\frac{1}{g(z)}P\Psi_V(x,z)$$
$$= \frac{1}{g(z)}Pu(z^N)\Psi_V(x,z) = \frac{1}{g(z)}Pu(L_V)\Psi_V(x,z).$$

Therefore

$$LP = Pu(L_V).$$

The operator $L = Pu(L_V)P^{-1}$ is differential iff $u(L_V)\mathrm{Ker}P \subset \mathrm{Ker}P$ and obviously it belongs to $\mathcal{A}_W$. $\square$

Thus the determination of A_W is reduced to the following finite-dimensional problem:

For the linear operator $L_V|_{\mathrm{Ker}h(L_V)}$ *find all polynomials* $u(z^N)$ *such that the subspace* $\mathrm{Ker}P$ *of* $\mathrm{Ker}h(L_V)$ *is invariant under the operator* $u(L_V)$.

4. Bäcklund–Darboux transformations on tau-functions. In this section we shall describe the tau-function of the Bäcklund–Darboux transformation W of V in terms of the tau-function of V and $\mathrm{Ker}P$.

Let $V \in Gr^{(N)}$, i.e. $z^N V \subset V$ and $L_V\Psi_V(x,z) = z^N\Psi_V(x,z)$ for some operator L_V of order N. (We do not suppose that $A_V = \mathbb{C}[z^N]$ but only that $\mathbb{C}[z^N] \subset A_V$.) Let W be a Bäcklund–Darboux transformation of V such that (18, 19, 26, 27) hold. Let us fix a basis $\{\Phi_i(x)\}_{0\le i\le dN-1}$ of $\mathrm{Ker}h(L_V)$ (where $d = \deg h$). The kernel of P is a subspace of $\mathrm{Ker}h(L_V)$. We fix a basis of $\mathrm{Ker}P$

$$f_k(x) = \sum_{i=0}^{dN-1} a_{ki}\Phi_i(x), \quad 0 \le k \le n-1. \tag{30}$$

We can suppose that P and g are monic. Eq. (18) implies (see e.g. [9])

$$\Psi_W(x,z) = \frac{Wr(f_0(x),\ldots,f_{n-1}(x),\Psi_V(x,z))}{g(z)Wr(f_0(x),\ldots,f_{n-1}(x))}, \tag{31}$$

where Wr denotes the Wronski determinant. When we express $f_k(x)$ by (30) we obtain

$$\Psi_W(x,z) = \frac{\sum \det A^I Wr(\Phi_I(x))\Psi_I(x,z)}{\sum \det A^I Wr(\Phi_I(x))}. \tag{32}$$

The sum is taken over all n-element subsets

$$I = \{i_0 < i_1 < \ldots < i_{n-1}\} \subset \{0, 1, \ldots, dN - 1\}$$

and here and further we use the following notation:

$$A = (a_{ki})_{0 \le k \le n-1,\, 0 \le i \le dN-1}$$

is the matrix from (30) and

$$A^I = (a_{k,i_l})_{0 \le k \le n-1,\, 0 \le l \le n-1}$$

is the corresponding minor of A,

$$\Phi_I(x) = \{\Phi_{i_0}(x), \ldots, \Phi_{i_{n-1}}(x)\}$$

is the corresponding subset of the basis $\{\Phi_i(x)\}$ of Ker$h(L_V)$ and

$$\Psi_I(x, z) = \frac{Wr(\Phi_I(x), \Psi_V(x, z))}{g(z) Wr(\Phi_I(x))} \tag{33}$$

is the Bäcklund–Darboux transformation of V with a basis of Ker$P$ $f_k(x) = \Phi_{i_k}(x)$.

So if we know how V transforms when the basis $\{f_k(x)\}$ of Ker$P$ is a subset of the basis $\{\Phi_i(x)\}$ of Ker$h(L_V)$, the formula (32) gives us $\Psi_W(x, z)$ for an arbitrary Bäcklund–Darboux transformation W.

We shall obtain a similar formula for the tau-functions as well.

Let $\tau_V(0)$ and $\tau_W(0)$ be nonzero and let us normalize them to be equal to 1 (recall that the tau-function is defined up to a multiplication by a constant). We set

$$\Delta_I = Wr(\Phi_I(x))|_{x=0}.$$

Denote by τ_I the tau-function corresponding to the wave function (33), also normalized by $\tau_I(0) = 1$. Then we have the following theorem.

Theorem 1. *In the above notation*

$$\tau_W(t) = \frac{\sum \det A^I \Delta_I \tau_I(t)}{\sum \det A^I \Delta_I}. \tag{34}$$

For the *proof* we have to introduce some more terminology. These are the so-called *conditions* C (cf. [15]). They are conditions (or equations) that should be imposed on a vector $v \in V$ in order to belong to gW (recall (22)).

Let us fix an admissible basis $\{v_k\}_{k\geq 0}$ of V and set

$$V_{(n)} = \bigoplus_{k=0}^{n-1} \mathbb{C} v_k$$

(this is independent of the choice of the basis). We define a linear map

$$C\colon V \to V_{(n)}$$

by defining it on the wave function of V

$$C\Psi_V(x,z) = \sum_{k=0}^{n-1} f_k(x) v_k, \tag{35}$$

where $\{f_k(x)\}$ is the basis of Ker$P$. The point is that C acts on the variable z. If we choose $v_k = \partial_x^k \Psi_V(x,z)|_{x=0}$ then

$$Cv_p = C\partial_x^p \Psi_V(x,z)|_{x=0} = \partial_x^p C\Psi_V(x,z)|_{x=0} = \sum_{k=0}^{n-1} f_k^{(p)}(0) v_k.$$

Let V_C be the kernel of C, i.e.

$$V_C = \{v \in V \mid Cv = 0\}.$$

Then the description of gW is straightforward (cf. [15]).

Lemma 3. $W = \dfrac{1}{g} V_C$.

Proof. First we show that $gW \subset V_C$. Indeed,

$$C(g(z)\Psi_W(x,z)) = C(P(x,\partial_x)\Psi_V(x,z)) = P(x,\partial_x)C\Psi_V(x,z)$$

$$= P(x,\partial_x)\sum_{k=0}^{n-1} f_k(x) v_k = 0, \quad \text{because } f_k \in \mathrm{Ker}P.$$

On the other hand the vectors $g(z)\partial_x^j \Psi_W(x,z)|_{x=0}$ can be expressed in the form

$$v_{j+n} + \sum_{k<j+n} d_{jk} v_k, \quad j \geq 0,$$

i.e. the plane gW maps one to one on the plane $\bigoplus_{j\geq n} \mathbb{C} v_j$. But the same is true for the plane V_C as Im$C = V_{(n)}$ (because $\det(C|_{V_{(n)}}) = Wr(f_k(x))|_{x=0} \neq 0$). $\square$

Corollary 1. *W has an admissible basis*

$$w_j = \frac{1}{g(z)} \left(1 - C_{(n)}^{-1} C\right) v_{j+n}, \quad j \geq 0, \tag{36}$$

where $C_{(n)} = C|_{V_{(n)}}$.

Proof. $C\left(1 - C_{(n)}^{-1} C\right) = C - C_{(n)} C_{(n)}^{-1} C = 0$ since $\mathrm{Im} C = V_{(n)}$. □

V_C can also be interpreted as the intersection of the kernels of certain linear functionals on V ($\mathrm{pr}_k \circ C: V \to \mathbb{C} v_k \equiv \mathbb{C}$) which form an n-dimensional linear space. We denote it by abuse of notation again by C.

Lemma 4. *Any condition $c \in C$ gives rise to a function*

$$f(x) = \langle c, \Psi_V(x,z) \rangle \tag{37}$$

from $\mathrm{Ker} P$, *and vice versa.*

The *proof* follows immediately from the definition (35). □

We define linear functionals χ_i and c_k on V by

$$\langle \chi_i, \Psi_V(x,z) \rangle = \Phi_i(x), \quad 0 \leq i \leq dN-1, \tag{38}$$

$$\langle c_k, \Psi_V(x,z) \rangle = f_k(x), \quad 0 \leq k \leq n-1, \tag{39}$$

i.e. $c_k = \sum_{i=0}^{dN-1} a_{ki} \chi_i$.

We can now give the *proof of Theorem 1.* The basis (36) of W can be written as

$$w_j = \frac{1}{g(z)} \Big(v_{j+n} - \sum_{0 \leq k \leq n-1,\, 0 \leq i \leq dN-1} \left(C_{(n)}^{-1} A \right)_{ki} \langle \chi_i, v_{j+n} \rangle v_k \Big), \quad j \geq 0.$$

We use the formula (5)

$$\tau_W(t) = \sigma(w_0 \wedge w_1 \wedge w_2 \wedge \ldots)$$

and expand all w_j:

$$\tau_W(t) = \sum_{r=-1}^{n-1} \sum_{\substack{0 \leq k_s \leq n-1,\, 0 \leq i_s \leq dN-1 \\ \text{for } 0 \leq s \leq r}} \left(C_{(n)}^{-1} A \right)_{k_0 i_0} \cdots \left(C_{(n)}^{-1} A \right)_{k_r i_r} \cdot (-1)^{r+1}$$

$$\times \sum_{n \leq j_0 < \ldots < j_r} \sigma \left(\frac{1}{g} v_n \wedge \frac{1}{g} v_{n+1} \wedge \ldots \wedge \frac{1}{g} \langle \chi_{i_0}, v_{j_0} \rangle v_{k_0} \wedge \ldots \wedge \frac{1}{g} \langle \chi_{i_r}, v_{j_r} \rangle v_{k_r} \wedge \ldots \right)$$

(the term $\frac{1}{g}\langle\chi_{i_s}, v_{j_s}\rangle v_{k_s}$ is on the $(j_s - n + 1)$-st place in the wedge product). Let $(k_0, \dots, k_{n-1})$ be a permutation of $\{0, 1, \dots, n-1\}$. Noting that

$$\sum_{0\le i\le dN-1} \left(C_{(n)}^{-1}A\right)_{ki} \langle\chi_i, v_j\rangle = \delta_{kj} \quad \text{for } 0 \le k, j \le n-1$$

we can insert

$$\sum_{\substack{0\le k_s\le n-1,\, 0\le i_s\le dN-1\\ \text{for } r+1\le s\le n-1}} \left(C_{(n)}^{-1}A\right)_{k_{r+1}i_{r+1}} \cdots \left(C_{(n)}^{-1}A\right)_{k_{n-1}i_{n-1}}$$

$$\times \sum_{0\le j_{r+1}<\ldots<j_{n-1}\le n-1} \langle\chi_{i_{r+1}}, v_{j_{r+1}}\rangle \cdots \langle\chi_{i_{n-1}}, v_{j_{n-1}}\rangle$$

in the above expression for $\tau_W(t)$. Then

$$\tau_W(t) = \sum_{(k_0,\ldots,k_{n-1})} \sum_{0\le i_0,\ldots,i_{n-1}\le dN-1} \left(C_{(n)}^{-1}A\right)_{k_0 i_0} \cdots \left(C_{(n)}^{-1}A\right)_{k_{n-1}i_{n-1}}$$

$$\times\, \sigma\left(R\left(\frac{1}{g}\right) r(\chi_{i_0})\cdots r(\chi_{i_{n-1}})(v_{k_0}\wedge\ldots\wedge v_{k_{n-1}}\wedge v_n\wedge v_{n+1}\wedge\ldots)\right),$$

where the operator $R(\frac{1}{g})$ acts as a group element

$$R\left(\frac{1}{g}\right)(u_0\wedge u_1\wedge\ldots) = \frac{1}{g}u_0\wedge\frac{1}{g}u_1\wedge\ldots$$

and $r(\chi_i)$ is a contracting operator:

$$r(\chi_i)(u_0\wedge u_1\wedge\ldots) = \sum_{j\ge 0}(-1)^j\langle\chi_i, u_j\rangle u_0\wedge u_1\wedge\ldots\wedge\widehat{u}_j\wedge\ldots$$

(the hat on u_j means as usually that it is omitted).

By the antisymmetry we obtain

$$\tau_W(t) = \sum \det\left(C_{(n)}^{-1}A\right)^I R\left(\frac{1}{g}\right) r(\chi_I)\tau_V(t), \tag{40}$$

where the sum is over the subsets

$$I = \{i_0 < \ldots < i_{n-1}\} \subset \{0, 1, \ldots, dN-1\}$$

and

$$r(\chi_I) = r(\chi_{i_0})\cdots r(\chi_{i_{n-1}}).$$

For the special Bäcklund–Darboux transformation $\tau_I(t)$ with $f_k(x) = \Phi_{i_k}(x)$ we have

$$c_k = \chi_{i_k}, \quad A = (\delta_{ii_k})_{\substack{0\le k\le n-1\\ 0\le i\le dN-1}}, \quad \det C_{(n)} = Wr(\Phi_I(x))|_{x=0} = \Delta_I.$$

Now (40) implies

$$\tau_I(t) = \frac{1}{\Delta_I} R\left(\frac{1}{g}\right) r(\chi_I)\tau_V(t).$$

Noting that

$$\left(C_{(n)}^{-1}A\right)^I = C_{(n)}^{-1}A^I$$

and

$$\det C_{(n)} = Wr(f_k(x))|_{x=0} = \sum \det A^I \Delta_I$$

completes the proof of Theorem 1. □

Suppose that linear functionals χ_i can be defined on all z^k, e.g. χ_i are of the form:

(41) $$\chi_i = \sum_{j\ge 0} \alpha_{ij}\partial_z^j|_{z=\lambda_i}$$

for $\lambda_i \neq 0$ (cf. [15]). We recall that (see e.g. [9])

$$\operatorname{Ker}\prod_{i=1}^{r}(L_V - \lambda_i^N)^{d_i} = \operatorname{span}\left\{\partial_z^{k_i}\Psi_V(x,z)\Big|_{z=\varepsilon^j\lambda_i}\right\}_{0\le j\le N-1,\,1\le i\le r,\,0\le k_i\le d_i-1},$$

with $\varepsilon = e^{2\pi i/N}$, when $\Psi_V(x,z)$ is well defined for $z = \varepsilon^j\lambda_i$ (cf. eq. (7)). Then we can put

(42) $$f_k(t) = \langle c_k, \Psi_V(t,z)\rangle, \quad 0\le k\le n-1$$

– $f_k(t)$ can be thought as obtained from $f_k(x)$ by applying the KP flows. In this case τ_W is given by the following theorem.

Theorem 2. *If $f_k(t)$ are as above and $g(z) = z^n$, then*

(43) $$\tau_W(t) = \frac{Wr(f_k(t))}{Wr(f_k(0))}\tau_V(t).$$

P r o o f. Proof uses the *differential Fay identity* [1] (see also [14]):

$$Wr(\Psi_V(t,z_0),\ldots,\Psi_V(t,z_n))\tau_V(t)$$

$$= \prod_{0\le j<i\le n}(z_i - z_j)\cdot\exp\Big(\sum_{k=0}^{\infty}\sum_{i=0}^{n} t_k z_i^k\Big)\tau_V\Big(t - \sum_{i=0}^{n}[z_i^{-1}]\Big),$$

where $[z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)$. After introducing the vertex operator

$$X(t,z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{kz^k}\frac{\partial}{\partial t_k}\right)$$

the RHS can be written in the form

$$z_0^0 z_1^1 \cdots z_n^n X(t,z_n) X(t,z_{n-1}) \cdots X(t,z_0)\tau_V(t)$$
$$= z_n^n X(t,z_n)\Big(Wr(\Psi_V(t,z_0),\ldots,\Psi_V(t,z_{n-1}))\tau_V(t)\Big).$$

We apply the condition c_0 to the variable z_0, c_1 to z_1, …, c_{n-1} to z_{n-1}, and set $z_n = z$. We obtain

$$Wr(f_0(t),\ldots,f_{n-1}(t),\Psi_V(t,z))\tau_V(t)$$
$$= z^n X(t,z)\Big(Wr(f_0(t),\ldots,f_{n-1}(t))\tau_V(t)\Big).$$

But (31) with $g(z) = z^n$ imply

$$\Psi_W(t,z) = \frac{Wr(f_0(t),\ldots,f_{n-1}(t),\Psi_V(t,z))}{z^n Wr(f_0(t),\ldots,f_{n-1}(t))}$$

(KP flows applied to (31)). Because the tau-function is determined from (6) up to a multiplication by a constant, (43) follows (when $\tau_V(0) = \tau_W(0) = 1$). □

Example 1. Let us consider the simplest plane in the Sato's Grassmannian $V = H_+ = \operatorname{span}\{z^i | i = 0,1,\ldots\}$. Then

$$\psi_V(t,z) = \exp\sum t_k z^k, \quad L_V = \partial_x, \quad \tau_V(t) = 1.$$

Every linear functional on H_+ is a linear combination of conditions of the type

$$e(k,\lambda) = \partial_z^k|_{z=\lambda}$$

and $h\left(L_{(0)}\right) = h(\partial_x)$ is an operator with constant coefficients. The set of rational solutions of the KP hierarchy [11, 13] coincides with the set of Bäcklund–Darboux transformations of H_+. The formula (43) with $\tau_V = 1$ is called a "superposition law for wobbly solitons" (cf. [15], eq. (5.7)). The so called polynomial solutions of KP [7, 13] correspond to the case $h(z) = z^d$, i.e. all conditions are supported at 0. □

Without any constraints on conditions C, there is a weaker version of Theorem 2.

Proposition 3. *For* $g(z) = z^n$

$$\tau_W(x) = \frac{Wr(f_k(x))}{Wr(f_k(0))}\tau_V(x),$$

where $\tau_W(x) = \tau_W(x, 0, 0, \ldots)$.

Proof. Formula (6) implies

$$\Psi_W(x,z) = e^{xz}\left(1 - \partial_x \log \tau_W(x) z^{-1} + \cdots\right). \quad (44)$$

On the other hand

$$z^{-n}(\partial_x^n + p_1(x)\partial_x^{n-1} + \cdots + p_0(x))\Psi_V(x,z)$$
$$= e^{xz}(1 - \partial_x \log \tau_V(x) z^{-1} + p_1(x) z^{-1} + \cdots).$$

Comparing the coefficients at z^{-1} and noting that

$$p_1(x) = -\partial_x \log Wr(f_k(x))$$

completes the proof. □

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