# Serdica

## Mathematical Journal

# Сердика

## Математическо списание

Serdica Math. J. **42** (2016), 59–64

**Serdica**
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

# JAMES BOUNDARIES AND MARTIN'S AXIOM

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*Communicated by S. L. Troyanski*

Abstract. Let $X$ be a separable Banach space, and $B$ a subset of the dual unit ball $B_{X^*}$ such that every $x \in X$ attains its norm on $B$. Under Martins's axiom and the negation of continuum hypothesis, it is shown that one of the following statements is true: (a) the dual unit ball $B_{X^*}$ is the norm-closed convex hull of $B$; (b) the set $B$ contains a subset $\Gamma$ which has the cardinality of the continuum, and is equivalent to the canonical basis of $l_1(\Gamma)$. Several consequences of this optimal result are spelled out.

**1. Introduction.** Let $X$ be an arbitrary Banach space. A subset $B$ of the dual unit ball $B_{X^*}$ is called a James boundary, or in short a boundary (see [5]), if for every $x \in X$, there exists $x^* \in B$ such that $x^*(x) = \|x\|$. The set $\mathrm{Ext}(B_{X^*})$ of extreme points of the dual unit ball provides a classical example. It readily follows from Hahn-Banach theorem that $B_{X^*}$ is the weak*-closed convex hull of any boundary $B$. Sometimes, for instance if $X$ is separable and does not contain an isomorphic copy of $l_1(\mathbb{N})$ (see [5]), it can actually be shown that the dual unit ball is the norm-closed convex hull of any boundary $B$. However this is not true in general: if for instance $X = \mathcal{C}([0,1])$ is the space of continuous functions on

2010 *Mathematics Subject Classification:* Primary 46B20, Secondary 46A50.
*Key words:* James boundary, uncountable $l_1$-bases, Martins axiom.

the unit interval, then $B = \{\delta_x : x \in [0,1]\}$ is a boundary, whose norm-closed convex hull consists of discrete measures. We refer to [9] for related results, under assumptions of topological regularity of $B$. We refer to [2] for lineability in the set of norm-attaining functionals, and to [12], [15], [16] for important results on James' theorem and boundaries.

Martin's axiom is a combinatorial statement which allows to extend Cantor's diagonal argument to all cardinals which are strictly less than the continuum $c$. It is therefore an obvious consequence of the continuum hypothesis $(CH)$, but what makes it important is that it is compatible with the negation of continuum hypothesis (see Theorem 11E in [4]). It turns out that if we assume Martin's axiom and the negation of continuum hypothesis (denoted $(MA+\neg CH)$), transfinite biorthogonal systems can be constructed in quite a few spaces (see [21], [1], [7]). This feature of Martin's axiom is confirmed by the present work. On the other hand, the world looks quite different if $(CH)$ is assumed (see [19], [13]).

In this short note, we show that under $(MA+\neg CH)$, the natural example given above, namely the boundary $B = \{\delta_x : x \in [0,1]\}$ in $\mathcal{C}([0,1])^*$ which does not span $\mathcal{C}([0,1])^*$ in norm, is somewhat minimal. This result does not require any assumption of regularity on the boundary $B$. Some consequences of this observation are given.

**2. Results.** The proof of our main result will rely in particular on Simons' inequality [17]. We refer to [6] for various applications of this inequality. A bounded subset $\Gamma$ of cardinality $\tau$ in a Banach space $E$ is said to be equivalent to the natural basis of $l_1(\tau)$ if there exists $m > 0$ such that

$$\left\| \sum \lambda_\gamma \gamma \right\| \geq m \sum |\lambda_\gamma|$$

for any family $(\lambda_\gamma)_{\gamma \in \Gamma}$ of scalars with finitely many non-zero terms. With this notation, our main result reads as follows.

**Theorem 2.1.** $(MA + \neg CH)$: *Let $X$ be a separable Banach space, and let $B \subset B_{X^*}$ be a boundary. If $B$ does not contain a subset equivalent to the natural basis of $l_1(c)$ where $c$ is the cardinality of the continuum, then $B_{X^*}$ is the norm-closed convex hull of $B$.*

P r o o f. We assume that $B$ does not contain a subset equivalent to the natural basis of $l_1(c)$ where $c$ is the cardinality of the continuum. Since $c$ is not of countable cofinality, it follows from Theorem 4 in [20] that if we denote $Z = \overline{\operatorname{span}}(B) \subset X^*$, then this space $Z$ does not contain an isomorphic copy of $l_1(c)$. Hence by [10], it follows from $(MA + \neg CH)$ that if $(z_n^*)$ is any bounded sequence in $Z^*$, there exists a sequence $(c_k^*)$ of successive convex combinations of $(z_n^*)$, such that $(c_k^*)$ is a weak* convergent sequence (when $k \to \infty$).

Therefore, if $(x_n)$ is any bounded sequence in $X$, there exists a sequence $(c_k)$ of successive convex combinations which is pointwise convergent on $B$, and hence, weakly Cauchy since $B$ is a boundary ([17], see [6, Corollary 2]). It follows that $X \not\supseteq l_1$, since obviously the canonical basis of $l_1$ has no weakly Cauchy sequence of convex combinations.

If $B_{X^*}$ is not the norm-closed convex hull of $B$, there exists $F \in B_{X^{**}}$ and $x_0^* \in B_{X^*}$ such that $F(x_0^*) > \sup F(B)$. Let $\sup F(B) < \alpha < F(m_0^*)$. Let $C = \{x \in B_X : x_0^*(x) > \alpha\}$. Clearly $F \in \overline{C}^{w^*}$. Since $X$ is separable and $X \not\supseteq l_1$, the compact space $B_{X^{**}}$ consists of first Baire class functions [14] and thus it is angelic in the sense defined in [3]. In particular, there is a sequence $\{x_n\} \subseteq C$ such that $\lim_{n\to\infty} x^*(x_n) = F(x^*)$ for all $x^* \in B$. Since $B$ is a boundary of $B_{X^*}$ and $\sup F(B) < \alpha$, it follows from Simons' inequality [17] that there is $x \in co(\{x_n\}) \subseteq C$ such that $\alpha > \sup x(B)$. Since we clearly have $\overline{\text{conv}}^{w^*}(B) = B_{X^*}$, this implies $\alpha > \|x\|$. But this contradicts $x_0^*(x) > \alpha$. □

Theorem 2.1 implies of course that if $X$ is separable and $X^*$ does not contain $l_1(c)$, then $B_{X^*}$ is the norm-closed convex hull of any boundary. This result does not request axioms (it is part of Theorem III.1 in [5]), and it is shown by the second half of the above proof.

A duality argument provides the following translation of Theorem 2.1.

**Corollary 2.2.** $(MA + \neg CH)$ *Let $Z$ be a Banach space which does not contain $l_1(c)$ isomorphically. If there is a norm-closed separable subspace $X$ of $Z^*$ which separates $Z$ and consists of norm-attaining linear functionals, then $X$ is an isometric predual of $Z$.*

P r o o f. Indeed let $Q : Z \to X^*$ be the canonical map of restriction to $X$. The set $B = Q(B_Z)$ is a boundary of $B_{X^*}$, and it follows from the lifting property of $l_1$ that the norm-closed linear span of $B$ does not contain $l_1(c)$. Hence $B$ is norm-dense in $B_{X^*}$ by Theorem 2.1. But this implies that $Q$ is an isometry from $Z$ onto $X^*$. □

Note that the space $Z$ is simply assumed to be separating in Corollary 2.2, and since it ends up being an isometric predual it is in particular 1-norming. The example of $X = \mathcal{C}([0,1])$ as separating subspace of $l_1(c)^*$ shows that our assumption on $Z$ is necessary. Along these lines, $l_1(c)^*$ contains a norm-closed separable subspace consisting of norm-attaining functionals which is separating but not norming (Proposition 4 in [8]).

**Corollary 2.3.** $(MA)$ *Let $X$ be a separable Banach space. If there is a* *boundary $B \subset B_{X^*}$ with cardinality strictly less than c, then $X^*$ is separable and $B_{X^*}$ is the norm-closed convex hull of $B$.*

Proof. If we assume $(\neg CH)$, Theorem 2.1 shows that that $B_{X^*}$ is the norm-closed convex hull of $B$. If we assume $(CH)$, then $B$ is countable and again $B_{X^*}$ is the norm-closed convex hull of $B$ by Simons' inequality [17]. In both cases, the density character of $X^*$ is strictly less than $c$. But a Cantor-type construction (or the stronger Stegall's theorem [18]) shows that if $X$ is separable and $X^*$ is not, then the density character of $X^*$ is $c$. Therefore $X^*$ is separable. □

We note that in the above Corollary 2.3, we may merely assume that the norm-density character of the norm-closed linear span is less than $c$ and reach the same conclusion. Assuming norm-attainment on $B$ is of course necessary: indeed if $X$ is any separable Banach space, there exists a 1-norming separable subspace of $X^*$.

We conclude this short note with some remarks and examples, and a natural problem.

**Remarks.** 1) Assuming the separability of $X$ in Theorem 2.1 or in Corollary 2.2 is necessary. For instance let $X = \mathcal{C}(\omega_1)$ be the space of continuous functions on the locally compact ordered space $\omega_1$ of all countable ordinals. The set $B = \{\delta_\alpha : \alpha < \omega_1\}$ is a boundary of $B_{X^*}$ whose norm closed linear span $Z = l_1(\omega_1)$ does not contain $l_1(c)$ (under $(MA + \neg CH)$). However, $\delta_{\omega_1} \in X^*$ does not belong to $Z$.

2) The statement "there exists a separable norm-closed subspace $X$ of $l_\infty(\omega_1)$ which separates $l_1(\omega_1)$ and consists of norm-attaining functionals" is undecidable in $(ZFC)$. Indeed, it is true if we assume $(CH)$ (use a bijective map between $\omega_1$ and $[0,1]$ and take $X = \mathcal{C}([0,1])$). It is false if we assume $(MA + \neg CH)$ since then, Theorem 2.1 would show that $l_1(\omega_1)$ is isometric to $X^*$, but since $l_1(\omega_1)$ has the Radon-Nikodym property and is not separable it cannot be isomorphic to the dual of a separable space.

3) It follows from the proof of Theorem 2.1 and Proposition 11 in [15] that in (ZFC) and with the assumptions and notation of Theorem 2.1, if $B_{X^*}$ is not the norm-closed convex hull of $B$, then the norm closed linear span of $B$ contains asymptotic copies of $l_1(\mathbb{N})$. Note that it follows from Theorem 6 in [20] that if a Banach space $Z$ contains isomorphically $l_1(c)$, then it contains asymptotic copies of $l_1(\mathbb{N})$.

4) Corollary 2.2 easily implies the following James-type theorem, under $(MA + \neg CH)$: let $Z$ be a Banach space which does not contain isomorphically

$l_1(c)$. Let $X$ be a separating subspace of $Z^*$, and denote by $\tau_X$ the locally convex topology of pointwise convergence on $X$. We assume that the topology $\tau_X$ is metrizable on $B_Z$. Then the convex set $(B_Z, \tau_X)$ is compact if and only if every $\tau_X$-continuous affine function on $B_Z$ attains its supremum. The usual example of $X = \mathcal{C}([0,1]) \subset l_1(c)^*$ shows that the assumption made on $Z$ is necessary.

5) Is Theorem 2.1 a result from $(ZFC)$, or is an axiom necessary? In the proof, $(MA + \neg CH)$ has been used for showing the existence of weak* convergent sequences of convex combinations in $Z^*$, and this completely fails under $(CH)$ since for instance, there exists ([19]) under $(CH)$ a Grothendieck space $\mathcal{C}(K)$ which does not contain $l_1(c)$, and in the norm-closed space $l_1(K)$ the weak* convergent sequences are actually norm-convergent. Hence it might be so that Theorem 2.1 is not provable in $(ZFC)$. However, under a mild topological assumption on the boundary $B$, one can dispense with axioms: indeed, assume that $B$ is weak* analytic (in the sense of Suslin). It is in particular so if $B$ is a weak* Borel subset of $X^*$. The restriction to $B$ of $B_{X^{**}}$ is a pointwise compact set $\mathcal{K}$ of functions on the topological space $A = (B, w^*)$, which contains a dense subset of continuous functions – namely, the restriction of $B_X$ to $B$. By [3], the compact set $\mathcal{K}$ is angelic or contains a copy of $\beta\mathbb{N}$. If $\mathcal{K}$ is angelic, the end of the proof of Theorem 1 shows that $B_{X^*}$ is the norm-closed convex hull of $B$. If $\mathcal{K}$ contains a copy of $\beta\mathbb{N}$ and if we denote by $Z$ the norm-closed linear span of $B$, then $B_{Z^*}$ contains a weak* homeomorphic copy of $\beta\mathbb{N}$, and thus by [20], the space $Z$ contains $l_1(c)$. This argument also works as soon as $B$ contains a weak* analytic boundary, e.g. the range of a norm-to-weak* Borel selector of the support mapping (see [11]).

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*Received March 4, 2015*