Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **42** (2016), 301–310

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

TRANSITION TO CANONICAL PRINCIPAL PARAMETERS ON MAXIMAL SPACELIKE SURFACES IN MINKOWSKI SPACE*

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Communicated by O. Mushkarov

Abstract. The first author has recently proposed to use special geometric parameters in the study of maximal spacelike surfaces in Minkowski 3-space. In canonical principal parameters any maximal spacelike surface is determined up to its position in the space by the normal curvature of the surface. Here we prove a theorem that permits a transition from general isothermal parameters to canonical principal parameters and we make some applications on parametric polynomial maximal spacelike surfaces. Thus we show that this approach implies an effective method to prove the coincidence of two maximal spacelike surfaces given in isothermal coordinates by different parametric equations.

2010 *Mathematics Subject Classification:* Primary 53A10, Secondary 53A05.
Key words: Maximal surfaces, canonical principal parameters.
*The first author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract DFNI-I 02/14.
The third author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under Grant DFNI-T 01/0001, 2012.

1. Introduction. In the differential geometry of curves in Euclidean space $\mathbb{R}^3$ the notion of a natural parameter plays an essential role. In natural parameters the two natural equations of a regular curve determine the curve uniquely up to a motion.

Analogous parameters in the general theory of surfaces in $\mathbb{R}^3$ are not known.

Recently Ganchev and Mihova [2] introduced natural parameters for a wide class of surfaces, namely the class of Weingarten surfaces.

In [3] Ganchev studied maximal spacelike surfaces in the three-dimensional Minkowski space with respect to canonical principal parameters. These parameters are determined up to renumbering, sign and additive constants. Then the normal curvature satisfying the natural PDE determines locally the surface up to a motion in $\mathbb{R}^3_1$.

Weierstrass formulas for maximal spacelike surfaces in isothermal parameters were given in [5].

In the present paper we study the question how to make the transition from general isothermal parameters on a maximal spacelike surface to canonical principal parameters. We find the differential equation that allows us to realize such a transition. Then we make some applications of this result.

Considering a holomorphic function that generates a maximal spacelike surface S, we find all holomorphic functions that generate the same surface. Thus we obtain a correspondence between a maximal spacelike surface and a class of holomorphic functions.

2. Preliminaries. Let $\mathbb{R}^3_1$ be the three-dimensional Minkowski space with the standard flat metric $\langle\, , \,\rangle$ of signature $(2,1)$. We assume that the following orthonormal coordinate system $Oe_1e_2e_3:\ e_1^2 = e_2^2 = -e_3^2 = 1,\ e_i\, e_j = 0,\ i \neq j$ is fixed and gives the orientation of the space.

All considerations in this paper are local and all functions are supposed to be of class $\mathcal{C}^\infty$.

Let S be a regular spacelike surface in $\mathbb{R}^3_1$ defined by the parametric equation

$$\mathbf{x} = \mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)); \qquad (u,v) \in U \subset \mathbb{R}^2.$$

Denote the derivatives of the vector function $\mathbf{x} = \mathbf{x}(u,v)$ by

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} \qquad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}.$$

The coefficients of the first fundamental form of S are given by

$$E = \mathbf{x}_u^2 > 0, \qquad F = \mathbf{x}_u \mathbf{x}_v, \qquad G = \mathbf{x}_v^2 > 0 \ .$$

Denote by $\mathbf{U}$ the unit normal to the surface, such that the triple $\{\mathbf{x}_u,\, \mathbf{x}_v,\, \mathbf{U}\}$ is right oriented. The coefficients of the second fundamental form are given by

$$L = \mathbf{U}\,\mathbf{x}_{uu} \qquad M = \mathbf{U}\,\mathbf{x}_{uv} \qquad N = \mathbf{U}\,\mathbf{x}_{vv}.$$

Suppose that the principal lines of S are parametric lines and the surface has no umbilic points. Then $F = M = 0$ and the principal curvatures ν_1 and ν_2 are expressed by

$$\nu_1 = \frac{L}{E}\ , \qquad \nu_2 = \frac{N}{G}.$$

The Gauss curvature K and the mean curvature H of a surface S are defined by

$$K = \frac{LN - M^2}{EG - F^2} = \nu_1 \nu_2 \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\nu_1 + \nu_2}{2}.$$

The surface S is said to be *maximal* if its mean curvature vanishes identically. In this case the Gauss curvature is negative and the principal curvatures are related by $\nu_1 + \nu_2 = 0$. Then the *normal curvature* ν of S is defined to be the function $\nu = \sqrt{-K}$, i.e. the normal curvature is the positive principal curvature [3].

In the different aspects of the study of a surface one can consider special parameters, closely referred to the problem. We recall two of these kinds of special parameters, that we shall use:

— the parameters are called isothermal, if $E = G,\ F = 0$;

— the parameters are called principal, if the parametric lines are principal lines.

It is always possible to change the parameters (u, v) so that the resulting parametrization is isothermal or principal.

Let us now consider a maximal spacelike surface parameterized by isothermal parameters. Then very often it is convenient to use complex functions to investigate this surface. We recall briefly this procedure.

Let $f(z)$ and $g(z)$ be two holomorphic functions defined in a region $D \subset \mathbb{C}$. Assume that $|g(z)| - 1$ never vanishes. Then $f(z)$ and $g(z)$ define a Weierstrass curve as follows [5]:

$$\Psi(z) = \int_{z_0}^{z} \left(\frac{1}{2}\, f(z)(1 + g^2(z)), \frac{i}{2}\, f(z)(1 - g^2(z)), f(z)g(z) \right)\, dz.$$

The real and imaginary parts $\mathbf{x}(u,v)$ and $\mathbf{y}(u,v)$ define two conjugate maximal space-like surfaces in isothermal parametrizations.

For example the analogue of the classical Enneper surface is defined by

$$\mathbf{x}(u,v) = \left(\frac{u^3}{6} - \frac{uv^2}{2} + \frac{u}{2}, -\frac{v^3}{6} + \frac{u^2 v}{2} - \frac{v}{2}, \frac{u^2 - v^2}{2}\right).$$

This surface is generated by the Weierstrass formula with the functions $f(z) = 1,\ g(z) = z$.

Conversely, every maximal spacelike surface can be obtained at least locally in this way. Note however that a maximal spacelike surface can be generated via the Weierstrass formula by different pairs of holomorphic functions $f(z)$, $g(z)$.

It is easy to see that the coefficients E and G of the first fundamental form of the surface defined via the Weierstrass formula with functions $f(z)$, $g(z)$ are given by

$$E = G = \frac{1}{4}|f|^2(1 - |g|^2)^2. \tag{2.1}$$

Analogously the normal curvature is

$$\nu = \frac{4|g'|}{|f|(1 - |g|^2)^2}, \tag{2.2}$$

see [4, Theorem 22.33].

Recently Ganchev [3] proposed maximal spacelike surfaces in Minkowski space to be investigated in special principal parameters, namely the *canonical principal parameters*. If the surface is parameterized with such parameters, then the coefficients of the I and II fundamental forms are expressed only by the invariant ν:

$$E = G = \frac{1}{\nu} > 0 \qquad F = 0$$

$$L = 1 \qquad M = 0 \qquad N = -1.$$

This idea leads to the special Weierstrass curve

$$\Phi(z) = \int_{z_0}^{z} \left(\frac{1 + g^2(z)}{2g'(z)}, \frac{i(1 - g^2(z))}{2g'(z)}, \frac{g(z)}{g'(z)}\right) dz \tag{2.3}$$

and the real part of this curve is a maximal spacelike surface with canonical principal parametrization. We shall use also the following theorem:

Theorem A ([3]). *If a surface is parameterized with canonical principal parameters, then its normal curvature satisfies the PDE*

$$\Delta \ln \nu - 2\nu = 0.$$

Conversely for any solution $\nu(u,v)$ of this PDE there exists locally a unique (up to position in the space) maximal space-like surface with normal curvature $\nu(u,v)$, (u,v) being canonical principal parameters.

The canonical principal parameters (u,v) are determined uniquely up to the following transformations [3]

$$\begin{aligned} u &= \varepsilon\bar{u} + a \\ v &= \varepsilon\bar{v} + b \end{aligned} \qquad \varepsilon = \pm 1, \quad a = \text{const}\,, \quad b = \text{const}. \tag{2.4}$$

3. Transformation of the isothermal parameters to canonical ones. Suppose the maximal spacelike surface S is defined as the real part of the Weierstrass maximal curve

$$\Psi(z) = \int_{z_0}^{z} \left(\frac{1}{2} f(z)(1 + g^2(z)), \frac{i}{2} f(z)(1 - g^2(z)), f(z)g(z) \right) dz. \tag{3.1}$$

We look for a transformation $z = z(w)$ so that the resulting curve $\Phi(w)$ has the form

$$\Phi(z) = \int_{z_0}^{z} \left(\frac{1 + \tilde{g}^2(z)}{2\tilde{g}'(z)}, \frac{i(1 - \tilde{g}^2(z))}{2\tilde{g}'(z)}, \frac{\tilde{g}(z)}{\tilde{g}'(z)} \right) dz$$

for a holomorphic function $\tilde{g}(w)$. The real part of this curve will be a canonical principal representation of the given surface S. The equality $\Psi(z(w)) = \Phi(w)$ gives $\Psi'(z(w))z'(w) = \Phi'(w)$. Hence it is easy to derive

$$f(z(w))z'(w) = \frac{1}{\tilde{g}'(w)}, \qquad g(z(w)) = \tilde{g}(w). \tag{3.2}$$

The last equation implies

$$\tilde{g}'(w) = g'(z(w))z'(w)$$

and now from the first equation of (3.2) we obtain

$$(z'(w))^2 = \frac{1}{f(z(w))g'(z(w))}. \tag{3.3}$$

Now we know also the function $\tilde{g}(w) = g(z(w))$.

So we have the following

Theorem 3.1. *Let the maximal spacelike surface S be defined by the real part of* (3.1)*. Any solution of the differential equation* (3.3) *defines a transformation of the isothermal parameters of S to canonical principal parameters. Moreover the function $\tilde{g}(w)$ that defines S via the formula* (2.3) *is given by $\tilde{g}(w) = g(z(w))$.*

Corollary 3.2 ([3])**.** *The canonical principal parameters of a maximal spacelike surface are determined up to the transformations* (2.4)*.*

As an application of Theorem 3.1 consider the maximal surface S generated by the functions

$$(3.4) \qquad f(z) = a, \qquad g(z) = bz + c, \qquad a,\, b,\, c \in \mathbb{C}, \qquad a,\, b \neq 0$$

via the Weierstrass formula. Equation (3.3) has the form

$$(z'(w))^2 = \frac{1}{ab}$$

and its solution is

$$z(w) = \pm \frac{w}{\sqrt{a}\sqrt{b}} + \text{const.}$$

According to Corollary 3.2 and Theorem 3.1 we may replace z in $g(z)$ with $\dfrac{z}{\sqrt{a}\sqrt{b}} - \dfrac{c}{b}$ and we will obtain a parametrization of the surface S in canonical principal parameters via the formula (2.3) and the function

$$\tilde{g}(z) = g\left(\frac{z}{\sqrt{a}\sqrt{b}} - \frac{c}{b}\right) = \frac{\sqrt{b}}{\sqrt{a}}\, z.$$

We find directly its normal curvature:

$$\nu = \frac{4\left|\frac{b}{a}\right|}{\left(1 - \left|\frac{b}{a}\right|(u^2 + v^2)\right)^2}$$

Remark that the obtained result shows that the surface S_0 generated via the Weierstrass formula by the functions

$$f(z) = \frac{|a|}{|b|}, \qquad g(z) = z$$

has the same normal curvature in canonical principal parameters. So because of Theorem A we may identify S with S_0. On the other hand the Weierstrass formula implies that S_0 is homothetic to the standard Enneper surface ($a = b = 1$).

So as in the case of minimal surfaces in the Euclidean space [1] we have

Corollary 3.3. *The maximal surface generated by the functions* (3.4) *coincide with the Enneper surface up to position in the space and homothety.*

4. Holomorphic functions generating a maximal space-like surface. It is known that a maximal spacelike surface is generated by different pairs of holomorphic functions via the Weierstrass formula. For example the Enneper surface is generated by the pair

$$f(z) = 1 \qquad g(z) = z$$

but also by

$$f(z) = e^z \qquad g(z) = e^z$$

and of course many others. It is natural to ask: under what condition do two pairs of holomophic functions give rise to the same maximal spacelike surface via the Weierstrass representation? It is not difficult to prove the following:

Proposition 4.1. *Suppose the pairs* $(\tilde{f}(z), \tilde{g}(z))$ *and* $(f(w), g(w))$ *generate two maximal spacelike surfaces via the Weierstrass formula. Then these surfaces coincide (up to translation) iff there exists a function* $w = w(z)$*, such that*

$$\tilde{g}(z) = g(w(z)) \qquad \text{and} \qquad \tilde{f}(z) = f(w(z))w'(z).$$

For the two pairs above, that generate the Enneper surface, the function $w(z) = e^z$.

Analogously the following question related to the formula (2.3) arises: how are related the functions that generate a maximal spacelike surface in canonical principal parameters? We have the following result in this direction:

Theorem 4.2. *Let the holomorphic function* $g(z)$ *generate a maximal spacelike surface in canonical principal parameters, i.e. via* (2.3)*. Then for an arbitrary complex number* α *with* $|\alpha| \neq 1$ *and for an arbitrary real number* φ *any of the functions*

$$e^{i\varphi}\frac{\alpha + g(z)}{1 + \bar{\alpha}g(z)} \qquad \text{and} \qquad e^{i\varphi}\frac{1}{g(z)} \tag{4.1}$$

generates the same surface in canonical principal parameters (up to position in the space). Conversely any function that generates this surface (up to position in the space) in canonical principal parameters has the above form.

P r o o f. Let us consider the function

$$\widetilde{g}(z) = e^{i\varphi}\frac{\alpha + g(z)}{1 + \bar{\alpha}g(z)}.$$

Denote by S the maximal spacelike surface generated via the formula (2.3) by $g(z)$ and by $\Psi(z)$ the corresponding complex curve. Analogously we define $\widetilde{S}$ and $\widetilde{\Psi}(z)$.

We may prove that S and $\widetilde{S}$ coincide (up to motion) by a direct computation of their normal curvatures using the formula

$$\nu = \frac{4|g'|^2}{(1 - |g|^2)^2}$$

and applying Theorem A. Now we show another possibility for the proof, thus clarifying the relation between S, $\widetilde{S}$ and the transformation (4.1). We have

$$\Psi'(z) = \left(\frac{1}{2}\frac{1 + g^2(z)}{g'(z)}, \frac{i}{2}\frac{1 - g^2(z)}{g'(z)}, \frac{g(z)}{g'(z)}\right),$$

$$\widetilde{\Psi}'(z) = \left(\frac{1}{2}\frac{1 + \widetilde{g}^2(z)}{\widetilde{g}'(z)}, \frac{i}{2}\frac{1 - \widetilde{g}^2(z)}{\widetilde{g}'(z)}, \frac{\widetilde{g}(z)}{\widetilde{g}'(z)}\right).$$

Let $\alpha = a + bi,\ a, b \in \mathbb{R}$. Consider the matrices

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \dfrac{1 + a^2 - b^2}{1 - a^2 - b^2} & \dfrac{2ab}{1 - a^2 - b^2} & \dfrac{2a}{1 - a^2 - b^2} \\ \dfrac{2ab}{1 - a^2 - b^2} & \dfrac{1 - a^2 + b^2}{1 - a^2 - b^2} & \dfrac{2b}{1 - a^2 - b^2} \\ \dfrac{2a}{1 - a^2 - b^2} & \dfrac{2b}{1 - a^2 - b^2} & \dfrac{1 + a^2 + b^2}{1 - a^2 - b^2} \end{pmatrix}.$$

A straightforward verification shows that these matrices belong to $SO(2,1)$ and that

$$A\,B\,\Psi'(z) = \widetilde{\Psi}'(z).$$

The last equality implies up to translation

$$A\,B\,\mathbf{x}(u,v) = \tilde{\mathbf{x}}(u,v).$$

Hence the considered transformation of the function $g(z)$ corresponds to a motion applied to the surface S.

Now for the converse we note that every motion can be presented as an $SO(2,1)$-rotation, i.e. a product AB of two matrices of the above form and then a translation. □

Remark 1. The second transformation (4.1) can be considered as a special case of the first one with $\alpha = \infty$.

Remark 2. The group of transformations (4.1) is $SO(2,1)$ and it is not a connected group. Its identity component $SO^+(2,1)$ consists of the transformations with $|\alpha| < 1$.

Remark 3.

1. When we use as above the Weierstrass representation in the form (3.1), the identity component $SO^+(2,1)$ of the transformation group (4.1) acting on g and preserving S may be writen also in the form

$$\widetilde{g}(z) = \frac{\bar{a}g(z) + \bar{b}}{bg(z) + a}\ , \quad |a|^2 - |b|^2 = 1\ , \quad a, b \in \mathbb{C}.$$

This is the group $\mathbf{SU}(1,1)$ of unitary matrices with respect to the indefinite Hermitian dot product in $\mathbb{C}^2_1$.

2. Analogously if we use the Weierstrass representation in the form

$$\Psi(z) = \int_{z_0}^{z} \left(\frac{1}{2} f(z)(1 - g^2(z)), f(z)g(z), \frac{1}{2} f(z)(1 + g^2(z)) \right) dz,$$

then we obtain the following group of linear fractional transformations of g:

$$\widetilde{g}(z) = \frac{ag(z) + b}{cg(z) + d}\ , \quad ad - bc = 1\ , \quad a, b, c, d \in \mathbb{R}.$$

This is the special linear group $\mathbf{SL}(\mathbb{R}, 2)$ of real matrices.

3. If we use the Weierstrass representation in the form

$$\Psi(z) = \int_{z_0}^{z} \left(\frac{1}{2} f(z)(1 + g^2(z)), if(z)g(z), \frac{1}{2} f(z)(1 - g^2(z)) \right) dz.$$

then we obtain the following group of linear fractional transformations of g:

$$\widetilde{g}(z) = \frac{ag(z) + ib}{icg(z) + d}\ , \quad ad + bc = 1\ , \quad a, b, c, d \in \mathbb{R},$$

This is the group of $\mathbf{SL}(\mathbb{C}, 2)$ matrices, which are real along the main diagonal and imaginary along the antidiagonal.

Of course these groups are isomorphic.

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Received June 22, 2016