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# A NOTE ON RESIDUALS IN GROUPS WITH QUASICENTRAL ABELIAN NORMAL SUBGROUPS

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Abstract. In 1973, I. N. Abramovskiĭ initiated the study of groups in which the transitivity condition is imposed on abelian normal subgroups only. He studied these locally finite groups under the additional restriction of commutativity of their Sylow $p$-subgroups. Much later the groups in which all abelian subnormal subgroups are normal were considered by M. Chaboksavar and F. de Giovanni. In the article we consider a broader class of groups, namely, groups in which every normal abelian subgroup is quasicentral.

Recall that a group $G$ is said to be *a $T$-group* if every subnormal subgroup of $G$ is normal. A group $G$ is a *$\bar{T}$-group*, if every subgroup of $G$ is a $T$-group. E. Best and O. Taussky have introduced these groups in [2]. Finite soluble $T$-groups have been described by W. Gaschütz [4]. In particular, he has found that every finite soluble $T$-group is a $\bar{T}$-group. Infinite soluble $T$-and$\bar{T}$-groups

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have been studied by D. J. S. Robinson [5]. A locally soluble $\bar{T}$-group $G$ has the following structure. If $G$ is not periodic, then $G$ is abelian. If $G$ is periodic and $L$ is the locally nilpotent residual of $G$, then $G/L$ is a Dedekind group (i.e., a group such that all of its subgroups are normal), $\pi(L) \cap \pi(G/L) = \varnothing, 2 \notin \pi(L)$, and every subgroup of $L$ is $G$-invariant. In particular, if $L \neq \langle 1 \rangle$, then $L = [L, G]$.

I. N. Abramovskiĭ in [1] has initiated the study of groups in which the transitivity condition is imposed on abelian normal subgroups only. Actually, he has studied the groups with the transitivity of normality for Dedekind normal subgroups, but since he has considered such locally finite groups with abelian Sylow $p$-subgroups, those Dedekind subgroups are abelian. If for every Dedekind normal subgroup $B$ each its subgroup $A$ is normal in $G$, then we call such a group $G$ a $TD$-group. I. N. Abramovskiĭ has described locally finite $TD$-groups having this $TD$-condition to be inherited on all subgroups. It has turned out that the class of such groups coincides with the class of locally finite $\overline{T}$-groups. He also has characterized soluble locally finite $TD$-groups with all abelian Sylow primary $p$-subgroups. His main result is the following theorem.

**Theorem A** ([1]). *A locally finite group $G$ with abelian Sylow $p$-subgroups is a supersoluble $TD$-group if and only if its derived subgroup $[G, G]$ is quasicentral and $\pi([G, G]), \pi(\zeta(G)) = \varnothing$, i.e., the orders of the elements of the derived subgroup are relatively prime with the orders of the elements of the center $\zeta(G)$ of $G$.*

In the article [3], the authors have considered groups in which all abelian subnormal subgroups are normal. They have denoted such a group as an $AT$-group. They have called the $AT$-groups having an ascending series with abelian factors consisting of subnormal subgroups *subsoluble*. They have proved that such groups are metabelian and their Baer radical coincides with the centalizer of the derived subgroup. They also have found that a subsoluble $AT$-$p$-group is ableian for all odd primes, and described subsoluble $AT$-2-groups which are not $T$-groups.

The current article is dedicated to the more general case of the groups in which every normal abelian subgroup is quasicentral. We call these groups $TA$-groups. The classes of $TD$-groups and $AT$-groups are proper subclasses of the class of $TA$-groups.

Consider some simple examples of such groups.

1. The group $G = \langle x \rangle \leftthreetimes \langle y \rangle$, $|x| = \infty$, $y^2 = 1$, $x^y = x^{-1}$.
2. The $p$-group $G = P \leftthreetimes \langle y \rangle$, where $P$ is the Prüfer $p$-group (the group of all $p^n$-roots of 1, $n \geq 0$), $y^p = 1$, $x^y = x^{p+1}$, $x \in P$.

3. All semisimple and all simple groups.
4. The finite $p$-group $G = \langle x \rangle \rightthreetimes \langle y \rangle$, $|x| = p^3$, $y^p = 1$, $x^y = x^{p+1}$.
5. Groups in which all normal proper abelian subgroups are cyclic.

Note that the class of $TA$-groups is broader that the classes studied in [1] and [3]. For instance, consider the group $G = \langle x, y \rangle \rightthreetimes \langle z \rangle$ where $\langle x, y \rangle$ is the group of quaternions, $|x| = |y| = 4$, $x^2 = y^2 = -1$, $x^y = x^3$, $y^x = y^3$, $|z| = 3$, $x^z = y$, $y^z = y^{-1}x$. The derived subgroup of this $TA$- (but not $TD$- or $AT$-) group is a quaternion group $\langle x, y \rangle$. It also shows that there are non-metabelian $TA$-groups. Since groups of the mentioned subclasses of $TA$-groups described in [1] and [3] are metabelian and since the class of $TA$-groups is quite wide, we shall consider here metabelian $TA$-groups.

**Lemma 1.** *Let $G$ be a $TA$-group of nilpotency class $\leq 2$. Then $G$ is a Dedekind group.*

Proof. If $x$ is an arbitrary element of $G \backslash \zeta(G)$ and $G \neq \zeta(G)$, then $\langle x \rangle$ is an abelian normal subgroup in $G$. It means that $\langle x \rangle$ is normal in $G$. If $G = \zeta(G) \langle x \rangle$, then $G$ is abelian, and $\langle x \rangle$ is normal in $G$. Thus $G$ is a Dedekind group. □

**Lemma 2.** *Let $G$ be a $TA$-group. Then all of its abelian normal subgroups belong to $C_G([G, G])$.*

Proof. Since the group of automorphisms of a cyclic group is abelian, every cyclic normal subgroup centralizes the derived subgroup of the group. □

**Corollary 3.** *Let $G$ be a $TA$-group with a Dedekind derived subgroup. Then $G$ is metabelian.*

Recall that a subgroup $H$ of a group $G$ is called *quasicentral* in $G$ if each subgroup of $H$ is normal in $G$.

**Lemma 4.** *Let $G$ be a metabelian group. Then $G$ is a $TA$-group if and only if $C_G([G, G])$ is a quasicentral Dedekind subgroup in $G$ and all of the abelian normal in $G$ subgroups belong to $C_G([G, G])$.*

Proof. Let $G$ be a metabelian $TA$-group. By Lemma 2, all of the abelian normal subgroups of a $TA$-group $G$ belong to $C_G([G, G])$. If $C_G([G, G]) = [G, G]$, then it is an abelian and quasicentral in $G$ subgroup. If $x$ is an arbitrary element of $C_G([G, G]) \backslash [G, G]$, the subgroup $[G, G] \langle x \rangle$ is an abelian normal subgroup in $G$, and therefore it is quasicentral in $G$. It means that $\langle x \rangle$ is normal in $G$ and $C_G([G, G])$ is a Dedekind group and it is quasicentral in $G$.

Let $G$ be a group, $C_G([G,G])$ be a quasicentral Dedekind subgroup in $G$ containing all of the abelian normal in $G$ subgroups. Then obviously, $G$ is a $TA$-group. □

Consider the product $Q(G)$ of all normal abelian subgroups of a $TA$-group $G$. This subgroup is generated by all cyclic normal subgroups of $G$. Following P. Venzke [6], we call the subgroup $Q(G)$ of a group $G$ generated by all cyclic normal in $G$ subgroups the *weak center* or the *quasicenter* of $G$. It is obvious that this subgroup is a characteristic subgroup of $G$. If $G$ coincides with its quasicenter it is called a *semiabelian group*. In the paper [6], P. A. Venzke has proved that finite semiabelian groups are nilpotent of class $\leq 2$. It is well known that the group of automorphisms of a cyclic subgroup is abelian. It follows that any semiabelian group is a group of class of nilpotency $\leq 2$. Indeed, if $G$ is a semiabelian group generated by the cyclic normal in $G$ subgroups $\langle x\rangle$, $\langle y\rangle$, ..., $\langle z\rangle$, ..., then as all cyclic normal subgroups, the generators $\langle x\rangle$, $\langle y\rangle$, ..., $\langle z\rangle$, ..., centralize the derived subgroup $[G,G]$. It follows that $G$ is a group of nilpotency class $\leq 2$.

**Corollary 5.** *Let $G$ be a metabelian $TA$-group. Then $C_G([G,G]) = Q(G)$ and it is quasicentral in $G$.*

Indeed, every normal cyclic subgroup centralizes the derived subgroup, and therefore in the group $G$ we have $Q(G) = C_G([G,G])$. The rest follows from Lemma 4.

**Lemma 6.** *If the weak center $Q(G) = C_G([G,G])$ of a metabelian $TA$-group $G$ contains a central element of infinite order, then $Q(G) = \zeta(G)$.*

Proof. Note, that if $Q(G)$ includes an element of infinite order, then as a Dedikind non-periodic group it is abelian. Let $c \in Q(G) \cap \zeta(G)$, $|c| = \infty$. Then $c^g = c$ for any $g \in G$. If $x$ is a $p$-element of $[G,G]$, $p$ being prime, then $|cx| = \infty$ because $Q(G)$ is abelian. Since $Q(G)$ is quasicentral $(cx)^g = (cx)^\alpha$ for any $g \in G$ where $\alpha$ is 1 or $-1$. If $\alpha = 1$, then $(cx)^g = cx = c^g x^g = cx^g$, i.e., $x^g = x$. If $\alpha = -1$, then $(cx)^g = (cx)^{-1} = c^{-1}x^{-1} = cx^g$, and $x^{g+1} = c^{-2}$, which is a contradiction. Therefore, any element of finite order of $Q(G)$ belongs to $\zeta(G)$.

Let $y$ be an element of infinite order of $Q(G)\setminus\zeta(G)$. Since $\langle y\rangle \trianglelefteq G$, there is an element $g \in G$ such that $y^g = y^{-1}$.

Here we have two cases: 1. $|cy| < \infty$, and 2. $|cy| = \infty$. In the first case, as we proved above, we have $cy \in \zeta(G)$. Therefore $cy = (cy)^g = cy^{-1}$, $y^2 = 1$, which is a contradiction.

In the second case, $(cy)^g = cy^{-1}$, and if $(cy)^g = (cy)^{-1}$, then $c = c^{-1}$. It contradicts our assumption that $c$ is an element of infinite order. If $(cy)^g = cy$, then $cy^{-1} = cy, y^2 = 1$. But $y$ is an element of infinite order. This contradiction proves the lemma. □

**Lemma 7.** *Let $G$ be a $TA$-group, $c$ be an element of infinite order from $Q(G)\backslash\zeta(G)$. If $g$ is an element of $G$ such that $c^g = c^{-1}$, then for any element $x \in Q(G)$, $x^g = x^{-1}$. In particular,* $\exp(Q(G) \cap \zeta(G)) \le 2$.

Proof. It follows from Lemma 6 that for any element $y$ of infinite order from $Q(G)$ we have $y^g = y^{-1}$. Let $x$ be a $p$-element of $Q(G)$. Since the Dedekind subgroup $Q(G)$ is non-periodic, it is abelian. Therefore $cx$ is an element of infinite order, and $(cx)^g = (cx)^{-1} = c^{-1}x^{-1} = c^{-1}x^g$. So $x^g = x^{-1}$. □

**Lemma 8.** *Let $G$ be a $TA$-group with non-central non-periodic weak center $Q(G) = C_G([G,G])$, and let $g \notin Q(G)$. Then $G = Q(G)\langle g\rangle$, $g^2 \in Q(G)$, and for any $x \in Q(G)$ we have $x^g = x^{-1}$.*

Proof. Since the Dedekind subgroup $Q(G)$ is non-periodic, it is abelian. Since $Q(G) = C_G([G,G])$ is abelian, $Q(G)$ coincides with its centralizer $C_G(Q(G))$. By Lemma 7, for any element $x \in Q(G)$ we have $x^g = x^{-1}$. Let $d$ be another element of $G\backslash Q(G)$. Then by Lemma 7, for any element $x \in Q(G)$ we have $x^d = x^{-1}$. Then $x^{dg} = x$, and therefore $dg \in C_G(Q(G)) = Q(G)$. So there is an element $q \in Q(G)$ such that $d = qg^{-1}$, and then $G = Q(G)\langle g\rangle$. □

**Lemma 9.** *Let $G$ be a group with non-central non-periodic abelian quasicentral weak center $Q(G) = C_G([G,G])$, $G = Q(G)\langle g\rangle$, $g^2 \in Q(G)$, and for any $x \in Q(G)$, $x^g = x^{-1}$. Then $G$ is a $TA$-group, and the derived subgroup $[G,G]$ coincides with $Q^2(G)$.*

Proof. Let $S$ be an abelian normal subgroup of $G$. Let $y$ be an element of $G = Q(G)\langle g\rangle$, and let $y = qg_1 \in S$, where $q \in Q(G)$, $g_1 \in \langle g\rangle\backslash Q(G)$. Without loss of generality we can assume that $g_1 = g$. Then $y^g = q^{-1}g$. For any element $r \in Q(G)$, we have $y^r = y(y^{-1}r^{-1}yr) = yr^2$. It means that $Q^2(G) \le S$. Since the factor-group $G/Q^2(G)$ is abelian, it follows that $S \ge [G,G]$. Since $S$ is abelian, $S \le Q(G) = C_G([G,G])$. It means that any abelian normal subgroup of $G$ is contained in the weak center $Q(G)$, and therefore it is quasicentral in $G$.

To finish the proof we just need to show that $[G,G] \ge Q^2(G)$. Indeed, for any element $q \in Q(G)$, $(q^{-1})^g = q$, and then $[g,q] = q^2 \in [G,G]$. Therefore $[G,G] = Q^2(G)$. □

Note, that in a $TA$-group $G$ with an abelian non-central derived subgroup the weak center $Q(G)$ is not periodic if and only if the derived subgroup is not periodic.

The following theorem follows from Lemmas 6, 8, 9 and Corollary 3.

**Theorem 1.** *Let group $G$ be a group with a non-periodic abelian derived subgroup $[G, G]$. Then $G$ is a $TA$-group if and only if it is either a Dedekind group or a group of the type $G = Q(G)\langle g\rangle$ with a non-central quasicentral abelian weak center $Q(G) = C_G([G, G])$, $g^2 \in Q(G)$, and $x^g = x^{-1}$ for any $x \in Q(G)$.*

We call a normal subgroup $N$ of a group $G$ the *hypocentral strong residual* (*hypercentral strong residual*, *locally nilpotent strong residual*) of $G$, if $N$ defines a hypocentral factor-group (hypercentral factor-group, locally nilpotent factor-group, respectively) $G/N$, and $N$ is contained in each normal subgroup of $G$ defining a hypocentral factor-group (hypercentral factor-group, locally nilpotent factor-group, respectively). Actually, in the expressed above meaning a hypocentral residual (hypercentral residual, locally nilpotent residual) becomes a strong residual if it defines a hypocentral (hypercentral, locally nilpotent) factor-group.

Let $G$ be a metabelian $TA$-group. Let $L$ be a maximal subgroup of the derived subgroup $[G, G]$ such that $L^2 = L$ where $L^2$ is the subgroup generated by all squares of the elements of $L$. In the factor-group $L/L^2$ there is no a nonidentity subgroup of such kind.

**Lemma 10.** *In a $TA$-group with a non-periodic non-central abelian derived subgroup $[G, G]$ the subgroup $Q^2(G)$ coincides with $[G, G]$, and the subgroups $\Gamma_0 = [G, G], \Gamma_1 = [G, [G, G]] = \Gamma_0^2$, $\Gamma_2 = [G, \Gamma_1] = \Gamma_1^2$, $\dots$, $\Gamma_{i+1} = [G, \Gamma_i] = \Gamma_i^2$, $\dots$, $\Gamma_\alpha = \bigcap_{i<\alpha} \Gamma_i$ for limit ordinals $\alpha$, are hypocenters of $G$.*

This statement follows directly from Lemmas 8, 9 and Theorem 1.

**Lemma 11.** *Let $G$ be a $TA$-group with a non-periodic non-central abelian derived subgroup $[G, G]$. Then the weak center $Q(G)$ of $G$ is abelian and the maximal subgroup $L$ of $Q(G)$ having the property $L^2 = L$ is the hypocentral strong residual of $G$.*

P r o o f. Let $G$ be a $TA$-group with non-periodic non-central $[G, G]$. It means that the derived subgroup $[G, G]$ is also a non-periodic and non-central abelian subgroup and $[G, G] = Q^2(G)$. Consider the factor-group $\bar{G} = G/L$ where $L$ is the maximal subgroup of $[G, G]$ having the property $L^2 = L$. The factor-group $\bar{G} = G/L$ has hypocenters $\Gamma_0 = [\bar{G}, \bar{G}], \Gamma_1 = \Gamma_0^2, \Gamma_2 = \Gamma_1^2, \dots, \Gamma_{i+1} = \Gamma_i^2, \dots$ and $\Gamma_\alpha = \bigcap_{i<\alpha} \Gamma_i$, for limit ordinals $\alpha$. By the definition of the subgroup $L$, the lower central series of $\bar{G}$ stabilizes on identity, i.e., $\bar{G}$ is a hypocentral group.

If $N$ is an arbitrary normal subgroup of $G$ defining a hypocentral factor-group $\bar{G} = G/N$, and $N \cap L \neq L$, then in $[\bar{G}, \bar{G}] = [G, G]N/N$ there is a non-trivial normal subgroup $\bar{L} = LN/N \cong L/L \cap N$ such that $\bar{L}^2 = \bar{L}$. This contradiction shows that $N \cap L = L$. □

**Lemma 12.** *Let $G$ be a $TA$-group with a periodic abelian non-central derived subgroup $[G, G]$. Then its weak center $Q(G)$ is a periodic Dedeking group. Let $R$ be a direct product of all Sylow $p$-sibgroups of $Q(G)$ which intersect the center $\zeta(G)$ by identity, $F$ be a direct product of all Sylow $p$-sibgroups of $Q(G)$ which intersect the center $\zeta(G)$ non-trivially and $P$ be the divisible part of $F$. Then $L = R \times P$ is the hypocentral strong residual of $G$.*

Proof. First of all we prove that $Q(G)$ is a periodic Dedekind subgroup. Suppose the contrary. Let $x$ be an element of infinite order of $Q(G)$. If $x \in \zeta(G)$, then by Lemma 6 $[G, G] \leq \zeta(G)$. So $x \notin \zeta(G)$. By Lemma 8, there is an element $g$ such that $G = Q(G) \langle g \rangle$, $g^2 \in Q(G)$, and for any $x \in Q(G)$ we have $x^g = x^{-1}$. It means, that $\langle x^2 \rangle \in [G, G]$ and the latter is not periodic, a contradiction. By Corollary 5, $Q(G)$ is a Dedekind group.

We note that $Q(G) = L \times K$ where $L = R \times P$, $K = F \times T$, $T$ is a complement to $P$ in $F$, $K$ is a direct product of the central Sylow $p$-subgroups of $Q(G)$. If $F$ is an abelian group, it has a divisible part $P$. If $F$ is Hamiltonian, $F = H \times A$ where $H$ its 2-Sylow subgroup, $H$ is a direct product of the group of quaternions and a group of exponent $\leq 2$, $A$ is an abelian periodic group having no involutions. In this case, the divisible part $P$ of $F$ is the divisible part of $A$.

Since for the divisible quasicentral subgroup $P$ we have $[G, P] = P$, it is easy to show that the lower hypocenter of $G$ is $L = R \times P$. Note, that the divisible part of $\bar{Q}(G) = Q(G)/L$ is identity. Since in the factor group $\bar{G} = G/L$ the subgroup $\bar{Q}(G) = Q(G)/L$ is a Dedekind quasicentral subgroup, whose all Sylow $p$-subgroups intersect the center non-trivially, we can show that $\bar{G}$ is hypocentral. Indeed, let $\bar{S}$ be a non-central Sylow $p$-subgroup of $\bar{Q}$. It is quasicentral in $\bar{G}$. If $\exp(\bar{S} \cap \zeta(\bar{G})) = p^n$, $n > 1$, then since $\bar{S}$ is quasicentral in $\bar{G}$, there is an element $y \in \bar{G}$ such that for each element $x \in \bar{S}$, $[y, x] = x^{p^n}$. In this case, one can observe that $[\bar{S}, \bar{G}] = \bar{S}^{p^n}, [\bar{S}^{p^n}, \bar{G}] = \bar{S}^{p^{2n}}, [\bar{S}^{p^{2n}}, \bar{G}] = \bar{S}^{p^{3n}}, \ldots$. It means that $\bar{G}$ is hypocentral.

If $N$ is a normal subgroup of $G$ defining a hypocentral factor-group $G/N$, then $I = N \cap Q \trianglelefteq G$, and by Remak theorem, $G/I$ is hypocentral. Since $I$ is quasicentral in $G$, it means that $I \geq P$, and $I \geq L$, i.e., $I = L$. □

**Theorem 2.** *Let $G$ be a $TA$-group with a periodic abelian derived subgroup $[G, G]$. Then its weak center $Q(G)$ is a periodic Dedekind quasicentral* *subgroup, the direct product $L$ of all of Sylow $p$-subgroups of $Q(G)$ having trivial intersection with the center $\zeta(G)$ is the locally nilpotent strong residual of $G$. Moreover, the locally nilpotent and hypercentral strong residuals of the group $G$ coincide.*

P r o o f. By Corollary 5, $C_G([G,G]) = Q(G)$ and by Lemma 12, it is a periodic Dedekind quasicentral in $G$ subgroup. It is clear that $Q(G) = L \times K$, where $K$ is a direct product of those Sylow $p$-subgroups of the Dedekind periodic group $Q(G)$, which intersect the center $\zeta(G)$ non-trivially. We shall show that $\bar{G} = G/L$ is locally nilpotent. Indeed, let $R$ be a finitely generated subgroup of $\bar{G} = G/L$. Denote by $T$ the product of the normal in $\bar{G}$ subgroup $\bar{K} = (L \times K)/L$ and the subgroup $R$. The group $T/\bar{K} \cong R/\left(R \cap \bar{K}\right)$ is a factor-group of a finitely generated group, and therefore is finitely generated. Since $Q(G) = L \times K \geq [G,G]$, $T/\bar{K} \cong R/\left(R \cap \bar{K}\right)$ is abelian.

Every element $x$ of $\bar{G}$ induces a power automorphism on $\bar{K}$. Lemma 4.1.1 of [5] implies that every such automorphism transforms all of elements of $\bar{K}$ of the same order in the same power. Since every Sylow $p$-subgroup of $\bar{K}$ intersects with $\zeta(\bar{G})$ nontrivially, then Lemma 4.1.1 of [5] implies that all of elements of the lower liar of each Sylow $p$-subgroup of $\bar{K}$ belong to $\zeta(\bar{G})$. Let $y$ be an arbitrary element of order $p^n$ of some Sylow $p$-subgroup of $\bar{K}$. Then Lemma 4.1.1 of [5] implies the existence of a number $\alpha$ depending of $x$ and independent of $y$, such that $[x,y] = y^\alpha$ and $\alpha \equiv 0 \pmod{p}$. From the last congruence it follows the existing in $T$ an ascending central series coming through $\bar{K}$. Thus $T$ is hypercentral, and therefore locally nilpotent. Hence $\bar{G}$ is locally nilpotent.

Let now $F$ be an arbitrary normal subgroup of $G$ defining a locally nilpotent factor group. We show that $F \geq L$. Assume the contrary. Let $F \cap L = C \neq L$. Then clearly $G/C$ is locally nilpotent. Since $L$ is abelian and periodic, there is an element $x \in A \backslash C$ such that $x^p \in C$. We shall show that in the group $G$ there is an element $y$ whose order is not divisible by $p$ such that $[x,y] \neq 1$. Let $z$ be an element of order $p$ from $\langle x \rangle$. For any $p$-element $t$ from $G$, $Z = \langle z \rangle \langle t \rangle$ is a finite $p$-group. Therefore a normal in $G$ subgroup $\langle z \rangle$ belongs to the center of the group $Z$. Thus $[t,z] = 1$. But all of the Sylow $p$-subgroups of $L$ intersect the center $\zeta(G)$ trivially. Then $\zeta(G) \cap \langle z \rangle = 1$. Hence there exists an element $a = by \in G$, where $b$ is an element of order $p^\alpha$ and $(|y|, p) = 1$ such that $[z,a] \neq 1$. Since $[z,b] = 1, [z,a] = [x,y] \neq 1$. Note that $[z,y]$ coincides with a power of $z$, the exponent of which is relatively prime with $z$, i.e., $\langle [x,y] \rangle = \langle z \rangle$. Since $\langle x \rangle$ is normal in $G$, it follows from here that $[y,x] \neq 1$, moreover, $\langle [x,y] \rangle = \langle x \rangle$. It is obvious that $y \notin C$. It means that in the group $G/C$ the elements $Cx$ and $Cy$ have relatively prime orders and do not commute.

However since $\langle Cx \rangle \trianglelefteq G/C$, $\langle Cx, Cy \rangle$ is a finite group. As we showed above, it is non-nilpotent. Thus $G/C$ is not locally nilpotent. This contradiction proves that $C = L$, and $F \geq L$.

We proved that $L$ is locally nilpotent residual of $G$. It is easy to observe that every Sylow $p$-subgroup of $Q(G)$ which trivially intersects with $\zeta(G)$ is a subgroup of $[G, G]$ for all odd primes $p$.

Together with this the first statement of the theorem is proved.

Now we shall prove that a locally nilpotent group with periodic quasicentral derived subgroup is hypercentral. Indeed, consider an element $x$ of prime order of the derived subgroup $[G, G]$. If $y$ is another element of $G$, the subgroup $\langle x, y \rangle$ is nilpotent. Therefore the normal subgroup $\langle x \rangle$ of prime order is central in it. It means that $\langle x \rangle \leq \zeta(G)$. Therefore the subgroup $A_1$ generated by all such elements of prime orders from $[G, G]$ is central. The factor group $G_1 = G/A_1$ is a locally nilpotent group with a quasicentral periodic derived subgroup. Repeating for this group the same arguments as we used above, we easily conclude that the corresponding subgroup $A_2$ generated by all elements of prime orders of $G_1$ is central in this group. Continue in this way, we shall get a hypercentral series coming through the derived subgroup of the group $G$. It means that a locally nilpotent $TA$-group with Dedekind periodic quasicenter $Q(G)$ is hypercentral. By the above arguments, its locally nilpotent strong residual is its hypercentral strong residual, and it is the direct product $L$ of all of Sylow $p$-subgroups of $Q(G)$ having trivial intersection with the center $\zeta(G)$. $\square$

**Corollary 13.** *In a $TA$-group with Dedekind periodic weak center $Q(G)$ the hypercentral strong residual is a subgroup of the hypocentral strong residual. In the notations of Lemma* 12 *it coincides with the subgroup $R$.*

## REFERENCES

[1] I. N. Abramovskiĭ. Subgroups of direct products of groups. In: Some Questions on the Theory of Groups and Rings, Inst. Fiz. im. Kirenskogo Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 3–8, 173 (in Russian).

[2] E. Best, O. Taussky. A class of groups. *Proc. Roy. Irish. Acad. Sect. A.* **47** (1942), 55–62.

[3] M. Chaboksavar, F. de Giovanni. Groups in which every finite subgroup is normal. *Ric. Mat.* **64**, *2* (2015), 331–338.

[4] W. Gaschütz. Gruppen, in denen das Normalreilersein transitiv ist. *J. Reine. Angew. Math.* **198** (1957), 87–92 (in German).

[5] D. J. S. Robinson. Groups in which normality is a transitive relation. *Proc. Cambridge Philos. Soc.* **60** (1964), 21–38.

[6] P. Venzke. A contribution to the theory of finite supersoluble groups. *J. Algebra* **57**, *2* (1979), 567–579.

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