

Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica

Mathematical Journal

Сердика

Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal <http://www.math.bas.bg/~serdica>
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

SOLITARY PULSE SOLUTIONS OF A COUPLED NONLINEAR SCHRÖDINGER SYSTEM ARISING IN OPTICS

Sharad Silwal

Communicated by J. Albert

ABSTRACT. We prove the existence of travelling-wave solutions for a system of coupled nonlinear Schrödinger equations arising in nonlinear optics. Such a system describes second-harmonic generation in optical materials with $\chi^{(2)}$ nonlinearity. To prove the existence of travelling waves, we employ the method of concentration compactness to prove the relative compactness of minimizing sequences of the associated variational problem.

1. Introduction. The coupled nonlinear Schrödinger system we are studying in this paper has applications in nonlinear optics. Telecommunications and computer technology are prime examples where modern nonlinear optics plays a major role. The system we are considering has $\chi^{(2)}$ nonlinearity, which arises in an optical process where a phenomenon of frequency doubling is exhibited due to interactions between certain nonlinear materials. We refer the reader

2010 *Mathematics Subject Classification:* 35A15, 35C08, 35Q51, 35Q55.

Key words: nonlinear Schrödinger, solitary pulse solutions, optics, concentration compactness, existence.

to [12] for a detailed description of $\chi^{(2)}$ nonlinearity, which, in fact, has gained the focus of many mathematicians and physicists in recent years, in an attempt to increase the speed and efficiency of optical fibres for data transmission. The reader may consult [8, 12, 14] and the references therein for more information about the physics and the engineering applications of the system studied in this paper.

In this paper, we consider a system of two-coupled nonlinear Schrödinger equations in the form

$$(1.1) \quad \begin{cases} i\frac{\partial W}{\partial t} + \frac{\partial^2 W}{\partial x^2} - \alpha W + \overline{W}V = 0, \\ i\mu\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial x^2} - \beta V + \frac{1}{2}W^2 = 0, \end{cases}$$

where W and V are complex-valued functions, α and β are real numbers, and $\mu > 0$. System (1.1) is obtained from the basic $\chi^{(2)}$ second-harmonic generation equations (SHG) of type I (see [12]). Physically, the complex functions W and V represent packets of amplitudes of the first and second harmonics of an optical wave respectively. The constant μ measures the ratios of the dispersions/diffractions.

In this paper, we are interested in the existence problem of travelling solitary wave solutions of (1.1). The travelling waves we are interested in are of the form

$$(1.2) \quad (W(x, t), V(x, t)) = (e^{i\sigma t}\Phi(x), e^{2i\sigma t}\Psi(x)),$$

where $\Phi, \Psi : \mathbb{R} \rightarrow \mathbb{R}$. We specify the boundary conditions $\Phi, \Psi \rightarrow 0$ as $x \rightarrow \pm\infty$ and call these solutions pulses. We show that there exists a nontrivial smooth solution with exponential decay at infinity for the system

$$(1.3) \quad \begin{cases} \Phi'' - \alpha_0\Phi + \Phi\Psi = 0, \\ \Psi'' - \beta_0\Psi + \frac{1}{2}\Phi^2 = 0, \end{cases}$$

with $\alpha_0 = \alpha + \sigma > 0$ and $\beta_0 = \beta + 2\mu\sigma > 0$. It is well-known that when $\alpha_0 = \pm 1 = \beta_0$ and $\Phi = \pm\sqrt{2}\Psi$, system (1.3) has the explicit solutions of the form

$$\Phi(x) = \pm\frac{3}{\sqrt{2}}\operatorname{sech}^2(x/2), \quad \Psi(x) = \pm\frac{3}{2}\operatorname{sech}^2(x/2).$$

From the mathematical point of view, the existence of at least one solution homoclinic to the origin (corresponding physically to a pulse) for (1.3) was proved in [14] for all $\beta_0 > 0$ and $\alpha_0 = 1$. Their method is of variational nature and

uses the mountain pass theorem, along with some convergence arguments. Localized solutions were characterized as critical points which are local minima of an energy functional. In [2], using the concentration compactness technique, the existence of pulses was established for a more general version of system (1.3). Their method uses variational characterizations of pulses subject to two independent constraints. Solutions were characterized as critical points which are global minima of an energy functional, with α_0, β_0 appearing as Lagrange multipliers associated with the constraints. To establish the compactness result for every minimizing sequence, they showed the subadditivity condition through a new idea based on rearrangement inequalities. In [15], a description of the profile of solutions for (1.3) was given by using the framework of homoclinic bifurcation theory. In [3], the existence of periodic pulses of (1.3) as well as the stability and instability of such solutions were studied. See also [5] where the author obtained positive solutions (Φ, Ψ) of (1.3) satisfying the additional condition $\|\Phi\|_2^2 = a$ and $\|\Psi\|_2^2 = b$ for any $a > 0$ and $b > 0$.

In the present paper, we study a different variational problem than those considered in [2, 14] and establish the existence of travelling pulse solutions to (1.3) for all $\alpha_0 > 0$ and $\beta_0 > 0$. Our method exhibits a new one-parameter family of travelling solitary-wave solutions than those obtained in [2, 14]. The key tool in our analysis is the concentration compactness lemma of [11]. To rule out the dichotomy case while applying the concentration compactness lemma, we use an argument developed in [10]. Similar techniques have been used previously in [7] to prove the existence of travelling-wave solutions to Boussinesq systems and in [4] to study solitary waves for an equation of short and long dispersive waves arising in two-layer fluids.

We now state our main results. Our existence result is proved by using a variational approach. Precisely, let the functional $I : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{R}$ be defined by

$$(1.4) \quad I(f, g) = \int_{\mathbb{R}} ((f')^2 + (g')^2 + \alpha_0 f^2 + \beta_0 g^2) dx,$$

where $H^1(\mathbb{R})$ denotes the L^2 -based Sobolev space of first order. For any $\lambda > 0$, we consider the minimization problem

$$(1.5) \quad M_\lambda = \inf \left\{ I(f, g) : (f, g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}), \int_{\mathbb{R}} f^2 g dx = \lambda \right\}.$$

The following is our existence result.

Theorem 1.1. *For any $\lambda > 0$, define*

$$(1.6) \quad \mathcal{S}_\lambda = \left\{ (f, g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) : I(f, g) = M_\lambda, \int_{\mathbb{R}} f^2 g \, dx = \lambda \right\}.$$

Then the following statements hold.

(i) *The infimum M_λ defined in (1.5) satisfies $0 < M_\lambda < \infty$.*

(ii) *For every sequence $\{(f_n, g_n)\}$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ such that*

$$(1.7) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n^2 g_n = \lambda \quad \text{and} \quad \lim_{n \rightarrow \infty} I(f_n, g_n) = M_\lambda,$$

there exists a sequence of real numbers $\{y_k\}$ such that $\{(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))\}$ converges strongly in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ to some (ϕ, ψ) in \mathcal{S}_λ . In particular, the set \mathcal{S}_λ is non-empty.

(iii) *Each function (ϕ, ψ) in \mathcal{S}_λ , after multiplying by a constant, is a solution of (1.3), and hence, when substituted into (1.2), gives a one-parameter family of travelling-wave solutions to the system (1.1).*

(iv) *If $(\phi, \psi) \in \mathcal{S}_\lambda$, then $\phi, \psi \in H^\infty(\mathbb{R})$ and ϕ, ψ decay exponentially at infinity.*

We now provide some notations that will be used throughout the paper.

Notation. For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all measurable functions f on \mathbb{R} for which the norm $\|f\|_p$ is finite, where

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and $\|f\|_\infty$ is the essential supremum of $|f|$ on \mathbb{R} . For $s \geq 0$, we denote by $H_{\mathbb{C}}^s(\mathbb{R})$ the Sobolev space of all complex-valued functions f in L^2 for which the norm

$$\|f\|_s = \left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty.$$

We will always view $H_{\mathbb{C}}^s(\mathbb{R})$ as a vector space over the reals, with inner product given by

$$\langle f_1, f_2 \rangle = \operatorname{Re} \int_{-\infty}^{\infty} (1 + |\xi|^2)^s \widehat{f}_1(\xi) \overline{\widehat{f}_2(\xi)} \, d\xi.$$

The space of all real-valued functions f in $H_{\mathbb{C}}^s(\mathbb{R})$ will be denoted by $H^s(\mathbb{R})$. In particular, we use $\|f\|$ to denote the L^2 or $H^0(\mathbb{R})$ norm of a function f . We define the space X to be $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ provided with the product norm. The letter C will frequently be used to denote various constants whose actual value is not important for our purposes.

2. Proof of the existence result. In this section we provide the proof of our existence result. Our strategy here is to employ the concentration compactness method to solve the variational problem (1.5). In this approach one obtains travelling solitary waves as global minimizers of the problem. When the method works, it shows not only that global minimizers exist, but also that every minimizing sequence is relatively compact up to translation. In the last couple of decades, this method was applied by many authors to prove the existence of solutions for a variety of dispersive evolution equations (see, for example [1, 2, 5, 7, 9, 10, 13]).

As usual in this method, we take any minimizing sequence $\{(f_n, g_n)\}$ for M_λ and consider a sequence of nondecreasing functions $P_n : [0, \infty) \rightarrow [0, \lambda]$ defined by

$$P_n(\omega) = \sup_{y \in \mathbb{R}} \int_{y-\omega}^{y+\omega} \rho_n(x) dx.$$

where $\rho_n(x) = (f'_n(x))^2 + (g'_n(x))^2 + f_n^2(x) + g_n^2(x)$. As $P_n(\omega)$ is a uniformly bounded sequence of nondecreasing functions in ω , one can show that it has a subsequence, which we will again denote by P_n , that converges pointwise to a nondecreasing limit function $P(\omega) : [0, \infty) \rightarrow [0, \lambda]$. Define

$$(2.1) \quad \gamma = \lim_{\omega \rightarrow \infty} P(\omega).$$

Then γ satisfies $0 \leq \gamma \leq \lambda$. From Concentration-Compactness Lemma of P. L. Lions, there are three possibilities for the value of γ :

- (a) Case 1 : (*Vanishing*) $\gamma = 0$. Since $P(\omega)$ is non-negative and nondecreasing, this case is equivalent to saying

$$P(\omega) = \lim_{n \rightarrow \infty} P_n(\omega) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-\omega}^{y+\omega} \rho_n(x) dx = 0,$$

for all $\omega < \infty$, or

- (b) Case 2 : (*Dichotomy*) $\gamma \in (0, \lambda)$, or

- (c) Case 3 : (*Compactness*) $\gamma = \lambda$, that is, there exists a sequence $\{y_n\}$ of real numbers such that $\rho_n(\cdot + y_n)$ is tight, namely, for all $\varepsilon > 0$, there exists $\omega < \infty$ such that for all $n \in \mathbb{N}$,

$$\int_{y_n-\omega}^{y_n+\omega} \rho_n(x) dx \geq \lambda - \varepsilon.$$

The goal here is to show that the only possibility is the case of compactness, so that the minimizing sequence $\{(f_n, g_n)\}$ for M_λ has a subsequence which, up to translations in the underlying spatial domain, converges strongly in X . We first rule out the vanishing case.

I. The vanishing case does not occur. We begin with some preliminary lemmas.

Lemma 2.1. *If $g \in C^1(\mathbb{R})$, $i \in \mathbb{Z}$ and $U_i = [i - 1/2, i + 1/2]$, then*

$$\sup_{r \in U_i} |g(r)| \leq \int_{U_i} [g'(s) + g(s)] ds.$$

Proof. Given any $r \in U_i$ and $s \in U_i$, we have

$$g(r) = g(s) + \int_s^r g'(x) dx.$$

This implies, for all $i \in \mathbb{Z}$, that

$$|g(r)| \leq |g(s)| + \int_{U_i} |g'(x)| dx = |g(s)| + \int_{U_i} |g'(s)| ds.$$

Integrating both sides with respect to $s \in U_i$ yields

$$|g(r)| |U_i| \leq \int_{U_i} |g(s)| + \int_{U_i} |g'(s)| ds |U_i|.$$

Now, using $|U_i| = i + 1/2 - (i - 1/2) = 1$ and taking supremum over $r \in U_i$, the result follows. \square

Lemma 2.2. *There exists a $\gamma_0 \in (0, \gamma]$ such that*

$$\lim_{n \rightarrow \infty} P_n(1/2) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \geq \gamma_0.$$

Proof. Suppose that we have

$$\lim_{n \rightarrow \infty} P_n(1/2) = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx = 0.$$

Letting $U_i = [i - 1/2, i + 1/2]$, $i \in \mathbb{Z}$ and using Lemma 2.1 with g replaced by g_n^2 ,

we have

$$\begin{aligned}
(\sup_{x \in U_i} |g_n(x)|)^2 &\leq \int_{U_i} [2|g_n(s)||g'_n(s)| + (g_n(s))^2] ds \\
&\leq \int_{U_i} \left[2 \left(\frac{1}{2} (|g_n(s)|^2 + |g'_n(s)|^2) \right) + (g_n(s))^2 \right] ds \\
&\leq \int_{i-1/2}^{i+1/2} [(g'_n(s))^2 + 2(g_n(s))^2] ds \\
&\leq C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
0 < \lambda &= \left| \int_{\mathbb{R}} f_n^2(x) g_n(x) dx \right| = \left| \sum_{i=-\infty}^{\infty} \int_{U_i} f_n^2(x) g_n(x) dx \right| \\
&\leq \sum_{i=-\infty}^{\infty} \sup_{x \in U_i} |g_n| \int_{U_i} f_n^2(x) dx \\
&\leq \left(C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \right)^{1/2} \int_{\mathbb{R}} f_n^2(x) dx \\
&\leq \|f_n\|^2 \left(C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \right)^{1/2} \rightarrow 0,
\end{aligned}$$

as $n \rightarrow \infty$, which is a contradiction. Finally,

$$\gamma = \lim_{\omega \rightarrow \infty} P(\omega) \geq P(1/2) = \lim_{n \rightarrow \infty} P_n(1/2) \geq \gamma_0 > 0. \quad \square$$

We now rule out the case of vanishing.

II. The dichotomy case does not occur. As in the case of vanishing, we first prove some technical lemmas.

Proposition 2.3. *For any $\sigma \in \mathbb{R}, 0 < M_\sigma < \infty$. Moreover, for any $\sigma_1, \sigma_2 > 0, M_{\sigma_1} = \left(\frac{\sigma_1}{\sigma_2}\right)^{2/3} M_{\sigma_2}$.*

Proof. Clearly, $0 \leq I(f, g) < \infty$ for $(f, g) \in X$. Now, $I(f, g) > 0$ if either $f \neq 0$ or $g \neq 0$. So, the proposition will hold true after we have shown that given any minimizing sequence (f_n, g_n) for M_σ , the sequences $\{f_n\}$ and $\{g_n\}$

cannot vanish. That is, there exist constants $\gamma_g, \gamma_f > 0$ such that

$$\lim_{n \rightarrow \infty} \|f_n\| \geq \gamma_f \text{ and } \lim_{n \rightarrow \infty} \|g_n\| \geq \gamma_g.$$

In order to prove this statement, we suppose $\lim_{n \rightarrow \infty} \|f_n\| = 0$, let $U_i = [i - 1/2, i + 1/2]$, $i \in \mathbb{Z}$, and use the same argument as in Lemma 2.2 to get a contradiction. In the proof of Lemma 2.2, we obtained

$$0 < \lambda \leq \|f_n\|^2 \left(C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \right)^{1/2} \rightarrow 0,$$

as $n \rightarrow \infty$, which is a contradiction. Next, we suppose $\lim_{n \rightarrow \infty} \|g_n\| = 0$ and argue similarly. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 0 < \lambda &= \left| \int_{\mathbb{R}} f_n^2(x) g_n(x) dx \right| = \left| \sum_{i=-\infty}^{\infty} \int_{U_i} f_n^2(x) g_n(x) dx \right| \\ &\leq \left| \sum_{i=-\infty}^{\infty} \left(\int_{U_i} (f_n(x))^4 dx \right)^{1/2} \left(\int_{U_i} (g_n(x))^2 dx \right)^{1/2} \right| \\ &\leq \sum_{i=-\infty}^{\infty} \left(\sup_{x \in U_i} |f_n(x)| \right)^2 \left(\int_{U_i} (g_n(x))^2 dx \right)^{1/2} \\ &\leq \left(C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \right) \left(\int_{\mathbb{R}} (g_n(x))^2 dx \right)^{1/2} \\ &\leq \|g_n\| \left(C \sup_{y \in \mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x) dx \right) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, which is a contradiction. Thus, we have $M_\sigma > 0$ for any $\sigma \in \mathbb{R}$.

Finally, let $\sigma_1, \sigma_2 > 0$ and set $\xi = \left(\frac{\sigma_1}{\sigma_2} \right)^{1/3}$. Clearly, by homogeneity of the functionals, $M_{\sigma_2} = \frac{1}{\xi^2} M_{\xi^3 \sigma_2}$, which completes the proof. \square

In order to state our next lemma, we will define some new functions. For any $\epsilon > 0$, we will first choose a large $\omega \in \mathbb{R}$ and then a large $n \in \mathbb{N}$, and construct the functions $\eta_{n,\omega}^{(i)}, u_{n,\omega}^{(i)}, i = 1, 2$ as follows:

Given any $\epsilon > 0$, we first find $\omega = \omega(\epsilon) \in \mathbb{R}$ large enough such that $\frac{1}{\omega} \leq \epsilon$ and

$$\gamma - \epsilon/2 < P(\omega) \leq P(2\omega) \leq \gamma.$$

Next, we choose $N \in \mathbb{N}$ large enough such that, for all $n \geq N$, we have

$$\gamma - (2\epsilon/3) < P_n(\omega) \leq P_n(2\omega) \leq \gamma + (2\epsilon/3).$$

For each $n \geq N$, we can find y_n such that

$$\int_{y_n-\omega}^{y_n+\omega} \rho_n(x) dx > \gamma - \epsilon \quad \text{and} \quad \int_{y_n-\omega}^{y_n+\omega} \rho_n(x) dx < \gamma + \epsilon.$$

Let $\phi \in C_0^\infty[-2, 2]$ and $\psi \in C^\infty(\mathbb{R})$ be such that $\phi \equiv 1$ on $[-1, 1]$ and $\phi^2 + \psi^2 = 1$ on \mathbb{R} . Set

$$\phi_\omega(x) = \phi\left(\frac{x}{\omega}\right), \quad \psi_\omega(x) = \psi\left(\frac{x}{\omega}\right),$$

and

$$\tilde{\phi}_{n,\omega}(x) = \phi_\omega(x - y_n), \quad \tilde{\psi}_{n,\omega}(x) = \psi_\omega(x - y_n).$$

Finally, we define

$$\begin{aligned} \eta_{n,\omega}^{(1)}(x) &= \tilde{\phi}_{n,\omega}(x)g_n(x), & \eta_{n,\omega}^{(2)}(x) &= \tilde{\psi}_{n,\omega}(x)g_n(x), \\ u_{n,\omega}^{(1)}(x) &= \tilde{\phi}_{n,\omega}(x)f_n(x), & u_{n,\omega}^{(2)}(x) &= \tilde{\psi}_{n,\omega}(x)f_n(x), \end{aligned}$$

and

$$\rho_n^{(i)}(x) = \left(u_{n,\omega}^{(i)'}(x)\right)^2 + \left(\eta_{n,\omega}^{(i)'}(x)\right)^2 + \left(u_{n,\omega}^{(i)}(x)\right)^2 + \left(\eta_{n,\omega}^{(i)}(x)\right)^2, \quad i = 1, 2.$$

Lemma 2.4. *For every $\epsilon > 0$, there exist a large $\Omega \in \mathbb{R}$ and a large $N \in \mathbb{N}$ such that, for all $n \geq N, \omega \geq \Omega$, we have*

$$(i) \quad I(f_n, g_n) = I\left(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}\right) + I\left(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}\right) + O(\epsilon),$$

$$(ii) \quad \int_{\mathbb{R}} f_n^2 g_n = \int_{\mathbb{R}} \left(u_{n,\omega}^{(1)}\right)^2 \eta_{n,\omega}^{(1)} + \int_{\mathbb{R}} \left(u_{n,\omega}^{(2)}\right)^2 \eta_{n,\omega}^{(2)} + O(\epsilon).$$

Proof. Spelling out and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & I\left(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}\right) \\ &= \int_{\mathbb{R}} \left[\left(u_{n,\omega}^{(1)'}\right)^2 + \left(\eta_{n,\omega}^{(1)'}\right)^2 + \alpha_0 \left(u_{n,\omega}^{(1)}\right)^2 + \beta_0 \left(\eta_{n,\omega}^{(1)}\right)^2 \right] \\ &= \int_{\mathbb{R}} \left[\tilde{\phi}_{n,\omega}^2 (g_n')^2 + (\tilde{\phi}_{n,\omega}')^2 g_n^2 + 2\tilde{\phi}_{n,\omega} g_n' \tilde{\phi}_{n,\omega}' g_n + \tilde{\phi}_{n,\omega}^2 (f_n')^2 + (\tilde{\phi}_{n,\omega}')^2 f_n^2 \right. \\ &\quad \left. + 2\tilde{\phi}_{n,\omega} f_n' \tilde{\phi}_{n,\omega}' f_n + \alpha_0 \tilde{\phi}_{n,\omega}^2 g_n^2 + \beta_0 \tilde{\phi}_{n,\omega}^2 f_n^2 \right] \\ &\leq \|\tilde{\phi}_{n,\omega} g_n'\|^2 + \|\tilde{\phi}_{n,\omega}' g_n\|^2 + 2\|\tilde{\phi}_{n,\omega} g_n'\| \|\tilde{\phi}_{n,\omega}' g_n\| + \|\tilde{\phi}_{n,\omega} f_n'\|^2 \\ &\quad + \|\tilde{\phi}_{n,\omega}' f_n\|^2 + 2\|\tilde{\phi}_{n,\omega} f_n'\| \|\tilde{\phi}_{n,\omega}' f_n\| + \alpha_0 \|\tilde{\phi}_{n,\omega} g_n\|^2 + \beta_0 \|\tilde{\phi}_{n,\omega} f_n\|^2. \end{aligned}$$

Similarly, spelling out $I\left(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}\right)$ and using $\tilde{\phi}_{n,\omega}^2 + \tilde{\psi}_{n,\omega}^2 \equiv 1$, we get

$$\begin{aligned} & I\left(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}\right) + I\left(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}\right) \\ & \leq I(f_n, g_n) + \|\tilde{\phi}'_{n,\omega} g_n\|^2 + 2\|\tilde{\phi}_{n,\omega} g'_n\| \|\tilde{\phi}'_{n,\omega} g_n\| + \|\tilde{\phi}'_{n,\omega} f_n\|^2 \\ & \quad + 2\|\tilde{\phi}_{n,\omega} f'_n\| \|\tilde{\phi}'_{n,\omega} f_n\| + \|\tilde{\psi}'_{n,\omega} g_n\|^2 + 2\|\tilde{\psi}_{n,\omega} g'_n\| \|\tilde{\psi}'_{n,\omega} g_n\| \\ & \quad + \|\tilde{\psi}'_{n,\omega} f_n\|^2 + 2\|\tilde{\psi}_{n,\omega} f'_n\| \|\tilde{\psi}'_{n,\omega} f_n\|. \end{aligned}$$

Next, using

$$|\tilde{\phi}'_{n,\omega}|_\infty, |\tilde{\psi}'_{n,\omega}|_\infty \sim O(1/\omega) = O(\epsilon),$$

and, since $(f_n, g_n) \in X$, that

$$\|f_n\|, \|f'_n\|, \|g_n\|, \|g'_n\| \leq C,$$

for some C independent of n , part (i) follows. For part (ii), we compute

$$\int_{\mathbb{R}} \left(u_{n,\omega}^{(1)}\right)^2 \eta_{n,\omega}^{(1)} + \int_{\mathbb{R}} \left(u_{n,\omega}^{(2)}\right)^2 \eta_{n,\omega}^{(2)} = \int_{\mathbb{R}} \left[f_n^2 g_n + \left(\tilde{\phi}_{n,\omega}^3 + \tilde{\psi}_{n,\omega}^3 - 1\right) f_n^2 g_n \right].$$

Upon using $\tilde{\phi}_{n,\omega} \equiv 1, \tilde{\psi}_{n,\omega} \equiv 0$ for $|x - y_n| \leq \omega$ and $\tilde{\phi}_{n,\omega} \equiv 0, \tilde{\psi}_{n,\omega} \equiv 1$ for $|x - y_n| \geq 2\omega$, along with the fact that $|\tilde{\phi}_{n,\omega}^3 + \tilde{\psi}_{n,\omega}^3 - 1| \leq 2$, we get

$$\begin{aligned} & \left| \int_{\mathbb{R}} \left(\tilde{\phi}_{n,\omega}^3 + \tilde{\psi}_{n,\omega}^3 - 1\right) f_n^2 g_n \right| = \left| \int_{\omega \leq |x - y_n| \leq 2\omega} \left(\tilde{\phi}_{n,\omega}^3 + \tilde{\psi}_{n,\omega}^3 - 1\right) f_n^2 g_n \right| \\ & \leq 2|g_n|_\infty \left(\int_{\omega \leq |x - y_n| \leq 2\omega} \rho_n \right) \leq C\epsilon, \end{aligned}$$

for some C independent of ω and n , and part (ii) is established. \square

Proposition 2.5. *The case $\gamma \in (0, \lambda)$ cannot occur.*

Proof. Suppose the case $\gamma \in (0, \lambda)$ does occur. Consider a minimizing sequence (f_n, g_n) for M_λ . Then, by Lemma 2.4, given $\epsilon > 0$, there exist sequences $(u_{n,\omega}^{(i)}, \eta_{n,\omega}^{(i)})$, $i = 1, 2$ such that

$$I(f_n, g_n) = I\left(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}\right) + I\left(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}\right) + O(\epsilon).$$

$$\int_{\mathbb{R}} f_n^2 g_n = \int_{\mathbb{R}} \left(u_{n,\omega}^{(1)}\right)^2 \eta_{n,\omega}^{(1)} + \int_{\mathbb{R}} \left(u_{n,\omega}^{(2)}\right)^2 \eta_{n,\omega}^{(2)} + O(\epsilon).$$

Then, since $\{(f_n, g_n)\}$ is bounded uniformly in X , so are $(u_{n,\omega}^{(i)}, \eta_{n,\omega}^{(i)})$, $i = 1, 2$. Hence, by passing to subsequences but retaining the same notation, the following limits are well-defined:

$$\sigma_1(\epsilon, \omega) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left(u_{n,\omega}^{(1)}\right)^2 \eta_{n,\omega}^{(1)} \quad \text{and} \quad \sigma_2(\epsilon, \omega) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left(u_{n,\omega}^{(2)}\right)^2 \eta_{n,\omega}^{(2)}.$$

Next, choose a sequence $\epsilon_j \rightarrow 0$, and then a sequence $\{\omega_j\}$ depending only on $\{\epsilon_j\}$ so that $\omega_j \rightarrow \infty$, and finally, define

$$\lim_{j \rightarrow \infty} \sigma_1(\epsilon_j, \omega_j) = \sigma_1 \quad \text{and} \quad \lim_{j \rightarrow \infty} \sigma_2(\epsilon_j, \omega_j) = \sigma_2.$$

Since, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n^2 g_n = \lambda$, we must have $\sigma_1 + \sigma_2 = \lambda$. Without loss of generality, suppose $\sigma_2 \geq \sigma_1$ as it is only a matter of interchanging σ_1 and σ_2 . The following three cases arise:

Case 1: $\sigma_1, \sigma_2 \in (0, \lambda)$.

Taking the limit $n \rightarrow \infty$, and using Proposition 2.3,

$$I(f_n, g_n) = I\left(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}\right) + I\left(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}\right) + O(\epsilon_j)$$

yields

$$\begin{aligned} M_\lambda &= M_{\sigma_1(\epsilon_j, \omega_j)} + M_{\sigma_2(\epsilon_j, \omega_j)} + O(\epsilon_j) \\ &= \left[\left(\frac{\sigma_1(\epsilon_j, \omega_j)}{\lambda}\right)^{2/3} + \left(\frac{\sigma_2(\epsilon_j, \omega_j)}{\lambda}\right)^{2/3} \right] M_\lambda + O(\epsilon_j). \end{aligned}$$

Next, taking the limit $j \rightarrow \infty$, we have

$$M_\lambda = \left[\left(\frac{\sigma_1}{\lambda}\right)^{2/3} + \left(\frac{\sigma_2}{\lambda}\right)^{2/3} \right] M_\lambda > M_\lambda,$$

which is a contradiction.

Case 2: $\sigma_1 = 0$ and $\sigma_2 = \lambda$.

$$\begin{aligned}
& I(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}) \\
&= \int_{\mathbb{R}} \left[(u_{n,\omega}^{(1)'})^2 + (\eta_{n,\omega}^{(1)'})^2 + \alpha_0 (u_{n,\omega}^{(1)})^2 + \beta_0 (\eta_{n,\omega}^{(1)})^2 \right] \\
&\geq \min\{1, \alpha_0, \beta_0\} \int_{\mathbb{R}} \left[(u_{n,\omega}^{(1)'})^2 + (\eta_{n,\omega}^{(1)'})^2 + (u_{n,\omega}^{(1)})^2 + (\eta_{n,\omega}^{(1)})^2 \right] \\
&= C \int_{|x-y_n| \leq 2\omega} [(f'_n(x))^2 + (g'_n(x))^2 + f_n^2(x) + g_n^2(x)] dx + O(\epsilon_j) \\
&\geq C\gamma + O(\epsilon_j).
\end{aligned}$$

Taking the limit $n \rightarrow \infty$,

$$\begin{aligned}
I(f_n, g_n) &= I(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}) + I(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}) + O(\epsilon_j) \\
&\geq C\gamma + I(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}) + O(\epsilon_j)
\end{aligned}$$

yields

$$M_\lambda \geq C\gamma + M_{\sigma_2(\epsilon_j, \omega_j)} + O(\epsilon_j).$$

Upon taking the limit $j \rightarrow \infty$,

$$M_\lambda \geq C\gamma + M_\lambda > M_\lambda,$$

which is a contradiction.

Case 3: $\sigma_1 < 0$ and $\sigma_2 = \lambda - \sigma_1 > \lambda$.

Taking the limit $n \rightarrow \infty$, and using Proposition 2.3,

$$\begin{aligned}
I(f_n, g_n) &= I(u_{n,\omega}^{(1)}, \eta_{n,\omega}^{(1)}) + I(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}) + O(\epsilon_j) \\
&\geq I(u_{n,\omega}^{(2)}, \eta_{n,\omega}^{(2)}) + O(\epsilon_j)
\end{aligned}$$

yields

$$\begin{aligned}
M_\lambda &\geq M_{\sigma_2(\epsilon_j, \omega_j)} + O(\epsilon_j) \\
&= \left(\frac{\sigma_2(\epsilon_j, \omega_j)}{\lambda} \right)^{2/3} M_\lambda + O(\epsilon_j).
\end{aligned}$$

Finally, taking the limit $j \rightarrow \infty$,

$$M_\lambda \geq \left(\frac{\sigma_2}{\lambda} \right)^{2/3} M_\lambda > M_\lambda,$$

which is a contradiction. \square

III. Proof of Theorem 1.1. As we ruled out both vanishing and dichotomy, Lions' concentration compactness lemma guarantees that the sequence $\{\rho_n\}$ is tight up to translation. Thus, there exists a sequence of real numbers $\{y_n\}$ such that given any $\varepsilon > 0$, we can find $\omega = \omega(\varepsilon) \geq \frac{1}{\varepsilon}$ such that for all $n \in \mathbb{N}$,

$$\int_{|x-y_n| \leq \omega} \rho_n(x) dx \geq \gamma - \varepsilon, \quad \int_{|x-y_n| \geq \omega} \rho_n(x) dx \leq \varepsilon,$$

and

$$(2.2) \quad \left| \int_{|x-y_n| \geq \omega} f_n^2(x) g_n(x) dx \right| \leq C \|(f_n, g_n)\|_X \int_{|x-y_n| \geq \omega} \rho_n(x) dx = O(\varepsilon).$$

Denote $\tilde{f}_n = f_n(\cdot + y_n)$ and $\tilde{g}_n = g_n(\cdot + y_n)$. Then, by (2.2),

$$\begin{aligned} \left| \int_{-\omega}^{\omega} \tilde{f}_n^2 \tilde{g}_n - \lambda \right| &= \left| \int_{|x-y_n| \geq \omega} f_n^2(x) g_n(x) dx - \int_{\mathbb{R}} f_n^2(x) g_n(x) dx \right| \\ &= \left| \int_{|x-y_n| \leq \omega} f_n^2(x) g_n(x) dx \right| = O(\varepsilon). \end{aligned}$$

Thus,

$$(2.3) \quad \lambda - O(\varepsilon) \leq \int_{-\omega}^{\omega} \tilde{f}_n^2 \tilde{g}_n \leq \lambda + O(\varepsilon).$$

Since $\{(\tilde{f}_n, \tilde{g}_n)\}$ is uniformly bounded in X , by Banach-Alaoglu's theorem, there exists a subsequence, again labeled $\{(\tilde{f}_n, \tilde{g}_n)\}$ for notational convenience, that converges weakly in X , say, to (ϕ, ψ) . Using the Cauchy-Schwarz inequality and the compact embedding of $H^1([-\omega, \omega])$ in $L^2([-\omega, \omega])$, we have

$$\begin{aligned} \int_{-\omega}^{\omega} |\tilde{f}_n^2 \tilde{g}_n - \phi^2 \psi| &= \int_{-\omega}^{\omega} |(\tilde{f}_n + \phi) \tilde{g}_n (\tilde{f}_n - \phi) + \phi^2 (\tilde{g}_n - \psi)| \\ &\leq \|\tilde{f}_n + \phi\|_{\infty} \cdot \|\tilde{g}_n\| \cdot \|\tilde{f}_n - \phi\|_{L^2(-\omega, \omega)} + \|\phi\|^2 \cdot \|\tilde{g}_n - \psi\|_{L^2(-\omega, \omega)} \\ &\leq C \left(\|\tilde{f}_n - \phi\|_{L^2(-\omega, \omega)} + \|\tilde{g}_n - \psi\|_{L^2(-\omega, \omega)} \right) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence, there exists a large enough N such that for all $n \geq N$, (2.3) yields

$$\lambda - O(\varepsilon) \leq \int_{-\omega}^{\omega} \phi^2 \psi \leq \lambda + O(\varepsilon).$$

Now, choosing $\varepsilon_j = \frac{1}{j}, j \in \mathbb{N}$ and determining $\omega_j = \omega_j(\varepsilon_j) \geq \frac{1}{\varepsilon_j} = j$ and letting $j \rightarrow \infty$, we obtain

$$\int_{\mathbb{R}} \phi^2 \psi = \lambda.$$

Moreover, since I is weak lower semi-continuous and invariant under translation,

$$M_\lambda = \lim_{n \rightarrow \infty} I(\tilde{f}_n, \tilde{g}_n) \geq I(\phi, \psi) \geq M_\lambda.$$

This means $(\phi, \psi) \in \mathcal{S}_\lambda$. Since (ϕ, ψ) is a solution of the variational problem of minimizing $I(f, g)$ subject to the constraint $\int_{\mathbb{R}} f^2 g = \lambda$, it must satisfy the corresponding Euler-Lagrange equation. That is, there exists some Lagrange multiplier $\kappa \in \mathbb{R}$ such that

$$(2.4) \quad \begin{cases} -\phi'' + \alpha_0 \phi = \kappa \phi \psi, \\ -\psi'' + \beta_0 \psi = \frac{\kappa}{2} \phi^2. \end{cases}$$

Multiplying the first and second equations by ϕ and ψ respectively and integrating over \mathbb{R} yields

$$\begin{cases} \int_{\mathbb{R}} (\phi')^2 + \alpha_0 \phi^2 = \kappa \int_{\mathbb{R}} \phi^2 \psi, \\ \int_{\mathbb{R}} (\psi')^2 + \beta_0 \psi^2 = \frac{\kappa}{2} \int_{\mathbb{R}} \phi^2 \psi. \end{cases}$$

Adding the two equations together and using the fact that $(\phi, \psi) \in \mathcal{S}_\lambda$, we get

$$\kappa = \frac{2M_\lambda}{3\lambda} > 0.$$

Clearly, $(\kappa\phi, \kappa\psi)$ solves the system (1.3), or, equivalently,

$$W(x, t) = \kappa e^{i\sigma t} \phi(x), \quad V(x, t) = \kappa e^{2i\sigma t} \psi(x)$$

are solitary wave solutions to the system (1.1). Finally, we establish the smoothness of ϕ and ψ using a standard bootstrap argument. The system (1.3) can be rewritten as

$$(2.5) \quad \begin{cases} \phi = K_{\alpha_0} \star \phi \psi \\ \psi = \frac{1}{2} K_{\beta_0} \star \phi^2, \end{cases}$$

where given any $s > 0$, the kernel K_s is defined via its Fourier transform as

$$q(\zeta) = \hat{K}_s(\zeta) = \frac{1}{s + \zeta^2}.$$

Since $H^s(\mathbb{R}^n)$ is an algebra if $s > n/2$, we have $\phi\psi, \phi^2 \in H^1(\mathbb{R})$. Since the convolution operation takes H^s to H^{s+2} for any $s \geq 0$, (2.5) implies that $\phi, \psi \in H^3$. Again, using the fact that $\phi\psi, \phi^2 \in H^3$, we obtain $\phi, \psi \in H^5$. Continuation of this induction argument leads to $\phi, \psi \in H^\infty$. Exponential decay of ϕ and ψ at infinity follows from another standard technique. Using Theorem 8.1.1 in [6], there exists a $\delta > 0$ such that $e^{\delta|\cdot|}\phi, e^{\delta|\cdot|}\psi \in L^\infty$. Thus,

$$|\phi(x)| \leq \left| e^{\delta|\cdot|}\phi \right|_\infty e^{-\delta|x|} \rightarrow 0, \quad |\psi(x)| \leq \left| e^{\delta|\cdot|}\psi \right|_\infty e^{-\delta|x|} \rightarrow 0,$$

as $|x| \rightarrow \infty$. \square

In the course of the proof of Theorem 1.1 above, we have also managed to establish the following result:

Proposition 2.6. *If $\lambda > 0$, the Lagrange multiplier $\kappa \in \mathbb{R}$ associated to the variational problem of minimizing $I(f, g)$ subject to the constraint $\int_{\mathbb{R}} f^2 g = \lambda$ is positive.*

Acknowledgments The author is grateful to JCHS Faculty Development Grant Program for a grant that partially supported this project.

REFERENCES

- [1] J. ALBERT. Concentration compactness and the stability of solitary-wave solutions to non-local equations. Applied analysis (Baton Rouge, LA, 1996), 1–29, Contemp. Math., vol. **221**. Providence, RI, Amer. Math. Soc., 1999.
- [2] J. ALBERT, S. BHATTARAI. Existence and stability of a two-parameter family of solitary waves for an NLS-KdV system. *Adv. Differential Equations* **18**, 11–12 (2013) 1129–1164.
- [3] J. ANGULO PAVA, F. LINARES. Periodic pulses of coupled nonlinear Schrödinger equations in optics. *Indiana Univ. Math. J.* **56**, 2 (2007), 847–877.
- [4] S. BHATTARAI. Solitary waves and a stability analysis for an equation of short and long dispersive waves. *Nonlinear Anal.* **75**, 18 (2012), 6506–6519.
- [5] S. BHATTARAI. Stability of normalized solitary waves for three coupled nonlinear Schrödinger equations. *Discr. Cont. Dyn. Syst.* **36**, 4 (2016), 1789–1811.

- [6] T. CAZENAVE. Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics, vol. **10**. Providence, RI, American Mathematical Society, 2003.
- [7] M. CHEN, N. NGUYEN, S.-M. SUN. Existence of travelling-wave solutions to Boussinesq systems. *Differential Integral Equations* **24**, 9–10 (2011) 895–908.
- [8] P. FERRO, S. TRILLO. Periodical waves, domain walls, and modulational instability in dispersive quadratic nonlinear media. *Phys. Rev. E* **51**, 5 (1995), 4994.
- [9] H. HAJAIEJ. Orbital stability of standing waves of some ℓ -coupled nonlinear Schrödinger equations. *Commun. Contemp. Math.* **14**, 6 (2012), 1250039, 10 pp.
- [10] S. LEVANDOSKY. A stability analysis of fifth-order water wave models. *Phys. D* **125**, 3–4 (1999) 222–240.
- [11] P. L. LIONS. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**, 2 (1984) 109–145.
- [12] C. R. MENYUK, R. SCHIEK, L. TORNER. Solitary waves due to $\chi^{(2)} : \chi^{(2)}$ cascading. *J. Opt. Soc. Amer. B* **11**, 12 (1994), 2434–2443.
- [13] S. SILWAL. Existence of bound states for $(N + 1)$ -coupled long-wave-short-wave interaction equations. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **23**, 5 (2016) 365–377.
- [14] A. C. YEW, A. R. CHAMPNEYS, P. J. MCKENNA. Multiple solitary waves due to second-harmonic generation in quadratic media. *J. Nonlinear Sci.* **9**, 1 (1999), 33–52.
- [15] A. C. YEW. Multipulses of nonlinearly coupled Schrödinger equations. *J. Differential Equations* **173**, 1 (2001), 92–137.

Jefferson College of Health Sciences
101 Elm Ave SE, Roanoke
VA 24013, USA
e-mail: sdsilwal@jchs.edu

Received February 22, 2017