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TAUBERIAN THEOREMS FOR THE MEAN OF LEBESGUE-STIELTJES INTEGRALS

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Abstract. Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is locally integrable with respect to a Radon measure μ on $[a, \infty)$. The mean of $s(x)$ with respect to μ is defined to be

$$\tau(t) = \frac{1}{F(t)} \int\limits_a^t s(x)\mu(dx),$$

where $F(x) = \mu(a, x]$. A scallar l is called the statistical limit of $s(x)$ as $x \to \infty$ if for every $\varepsilon > 0$,

$$\lim_{b\to\infty} \frac{1}{b-a} |\{x \in (a, b) : |s(x) - l| > \varepsilon\}| = 0.$$

This is denoted by $\operatorname*{st-lim}_{x\to\infty} s(x) = l$. The following Tauberian theorems are proved under mild assymptotic conditions on $F(t)$ and assuming that $s(x)$ is slowly decreasing with respect to $F(t)$.

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1. If $\lim_{t\to\infty} \tau(t) = l$, then $\lim_{x\to\infty} s(x) = l$.
2. If $\operatorname{st-lim}_{x\to\infty} s(x) = l$, then $\lim_{x\to\infty} s(x) = l$.
3. If $\operatorname{st-lim}_{t\to\infty} \tau(t) = l$, then $\lim_{x\to\infty} s(x) = l$.

This work extends results obtained by F. Móricz and Z. Németh in [3] and [4] for the case $F(t) = \log(t)$.

1. Introduction.

Definition 1.1. *A number $l \in \mathbb{R}$, is called the statistical limit of $s(x) : [a,\infty) \mapsto \mathbb{R}$ at infinity if for any $\varepsilon > 0$,*

$$\lim_{b\to\infty} \frac{1}{b-a} |\{x \in (a,b) : |s(x) - l| > \varepsilon\}| = 0, \tag{1.1}$$

where $|A|$ is the Lebesgue measure of the set A. We write this as:

$$\operatorname{st-lim}_{x\to\infty} s(x) = l.$$

It is easy to work out a relationship between the ordinary limit and the statistical limit. We omit the easy proof. The converse is not true in general.

Proposition 1.1. *If $\lim_{x\to\infty} s(x) = l$, then $\operatorname{st-lim}_{x\to\infty} s(x) = l$.*

The converse of Proposition 1.1 is not true in general.

Definition 1.2. *Let μ be a Radon measure on $[a,\infty)$ with $\mu[a,\infty) = \infty$, and let*

$$F(t) := \mu(a,t]$$

be its cumulative distribution function. For any $s(x) : [a,\infty) \mapsto \mathbb{R}$, locally integrable with respect to μ, define the mean of $s(x)$ with respect to μ to be

$$\tau(t) = \frac{1}{F(t)} \int_a^t s(x)\mu(dx), \quad \textit{for } t > a. \tag{1.2}$$

The following proposition and its proof are standard.

Proposition 1.2. *If $\lim_{x\to\infty} s(x) = l$, then $\lim_{t\to\infty} \tau(t) = l$.*

The converse of Proposition 1.2 is not always true, unless one imposes additional conditions on the function $s(x)$. Such conditions are called *Tauberian conditions*, after the work of Tauber [1]. Sufficient conditions for the converse are given in Theorem 2.1, which is the first main result of this work. We extend the slowly decreasing condition proposed by F. Móricz in [3] to fit the definition of the mean of $s(x)$ with respect to μ.

Definition 1.3. *A function $s(x) : [a,\infty) \mapsto \mathbb{R}$ is said to be slowly decreasing with respect to $F(t)$, if*

$$\lim_{\lambda\to 1^+} \liminf_{x\to\infty} \inf_{F(x)\leqslant F(t)\leqslant \lambda F(x)} (s(t)-s(x)) \geqslant 0.$$

We keep the term 'slowly decreasing' even though it is somewhat misleading since every increasing function $s(x)$ satisfies Definition 1.3. More appropriate description would be to say that $s(x)$ is not quickly decreasing. We do not use this definition directly but instead the equivalent characterization given below. We include the proof for completeness.

Proposition 1.3. *A function $s(x) : [a,\infty) \mapsto \mathbb{R}$ is slowly decreasing with respect to $F(t)$ if and only if for every $\varepsilon > 0$, there exist $\lambda_0 > 1$, such that for all $\lambda \in (1, \lambda_0)$, there exists an $x_0 > a$, such that*

$$s(t) - s(x) > -\varepsilon, \tag{1.3}$$

whenever x and t satisfy $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant \lambda F(x)$.

Proof. Let

$$a(\lambda) := \liminf_{x\to\infty} \inf_{F(x)\leqslant F(t)\leqslant \lambda F(x)} (s(t)-s(x)).$$

It is easy to see that $a(\lambda)$ is a non-increasing function on $(1,\infty)$. Indeed, for any $1 < \lambda_1 < \lambda_2$ and a fixed x, we have

$$\{t : F(x) \leqslant F(t) \leqslant \lambda_1 F(x)\} \subseteq \{t : F(x) \leqslant F(t) \leqslant \lambda_2 F(x)\},$$

implying that

$$\inf_{F(x)\leqslant F(t)\leqslant \lambda_1 F(x)} (s(t)-s(x)) \geqslant \inf_{F(x)\leqslant F(t)\leqslant \lambda_2 F(x)} (s(t)-s(x)).$$

Therefore,

$$a(\lambda_1) = \liminf_{x\to\infty} \inf_{F(x)\leqslant F(t)\leqslant \lambda_1 F(x)} (s(t)-s(x))$$

$$\geqslant \liminf_{x\to\infty} \inf_{F(x)\leqslant F(t)\leqslant\lambda_2 F(x)} (s(t)-s(x)) = a(\lambda_2),$$

which shows that $a(\lambda)$ is decreasing.

Next, $s(x)$ is slowly decreasing if and only if

$$\lim_{\lambda\to 1^+} a(\lambda) \geqslant 0.$$

This means, for every $\varepsilon > 0$, there exists $\lambda_0 > 1$, such that for all $\lambda \in (1, \lambda_0)$,

$$a(\lambda) \geqslant -\frac{\varepsilon}{2}.$$

For each fixed $\lambda > 1$, define

$$h(\lambda, x) := \inf_{F(x)\leqslant F(t)\leqslant\lambda F(x)} (s(t)-s(x)),$$

and

$$\bar{h}(\lambda, x) := \inf\{h(\lambda, y) : y \in [x, \infty)\}.$$

Then, by the definition of limit infimum, we have

$$a(\lambda) = \lim_{x\to\infty} \bar{h}(\lambda, x)$$

for all $\lambda > 1$. Thus, $s(x)$ is slowly decreasing if and only if for every $\varepsilon > 0$, there exists $\lambda_0 > 1$, such that for all $\lambda \in (1, \lambda_0)$, there is x_0, such that

$$\bar{h}(\lambda, x) \geqslant -\frac{\varepsilon}{2} \tag{1.4}$$

holds for all $x \geq x_0$. But (1.4) is equivalent to $h(\lambda, x) \geqslant -\varepsilon/2$ and the result follows from here. □

2. Main results. Recall that the function $F : [a, \infty) \mapsto \mathbb{R}$ is non-decreasing, right-continuous, and satisfies $F(a) = 0$ and

$$\lim_{x\to\infty} F(x) = \infty.$$

It is a standard practice to denote by $F(x-)$ the left limit of F at x. The main results require that we impose one or both of the following two conditions on $F(t)$.

F.1) $\lim_{x\to\infty} F(x)/F(x-) = 1.$

F.2) $\limsup_{x\to\infty} F(\lambda x)/F(x) \le \lambda$ for all $\lambda \geqslant 1$ sufficiently close to 1.

These conditions are met for broad classes of functions. It is clear that if $F(t)$ is a continuous function, then F.1) holds. Condition F.1) is equivalent to $\lim_{x\to\infty} F(x-)/F(x) = 1$ or to

$$\lim_{x\to\infty} \frac{F(x) - F(x-)}{F(x)} = 0.$$

The last limit says that the jumps of the function $F(x)$ increase at a slower rate than $F(x)$. In particular, if $F(t)$ has bounded jumps, then F.1) holds.

Proposition 2.1. *Condition* F.2) *holds, whenever* $F(x)$ *is concave.*

P r o o f. The concavity of F implies that for any $x > a$, there is a number d_x, called *subgradient of* F *at* x, such that

$$F(y) \le F(x) + d_x(y - x) \text{ for all } y \ge a.$$

The number d_x may not be unique, but any choice $x \mapsto d_x$ gives a non-increasing, function, see [5]. The fact that $F(t)$ is non-decreasing implies that $d_x \geqslant 0$. Thus, for any $\lambda \geqslant 1$, one has

$$\begin{aligned} \limsup_{x\to\infty} \frac{F(\lambda x)}{F(x)} &\leqslant \limsup_{x\to\infty} \frac{F(x) + d_x(\lambda-1)x}{F(x)} = \limsup_{x\to\infty} \Big(1 + \frac{d_x(\lambda-1)x}{F(x)}\Big) \\ &\leqslant \limsup_{x\to\infty} \Big(1 + \frac{d_x(\lambda-1)x}{F(a) + d_x(x-a)}\Big) = \limsup_{x\to\infty} \Big(1 + \frac{(\lambda-1)x}{x-a}\Big) \\ &= \lambda, \end{aligned}$$

where we used that $F(a) = 0$. □

Examples of functions that satisfy the two conditions are t^r for $r \in (0, 1]$, $\log(t)$, and $\log(\log(t))$. A function $F(t)$ does not need to be continuous to satisfy the two conditions, see Subsection 3.4 for such an example.

Proposition 2.2. *Suppose* $s(x)$ *is slowly decreasing with respect to* $F(t)$*. If* $F(t)$ *satisfies condition* F.2)*, then* $s(x)$ *is slowly decreasing with respect to* t*.*

P r o o f. Fix $\epsilon > 0$. Since $s(x)$ is slowly decreasing with respect to $F(t)$, there exists $\lambda_0 > 1$ such that for any $\lambda \in (1, \lambda_0)$ there is an x_0, such that

$$s(t) - s(x) > -\varepsilon, \tag{2.1}$$

whenever $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant \lambda F(x)$.

Fix $\lambda \in (1, \lambda_0)$ and let $\eta > 0$ be such that $\lambda + \eta \in (1, \lambda_0)$. Choose x_0 so that (2.1) holds, whenever $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant (\lambda+\eta)F(x)$. Choose x_0 larger, if necessary, so that, by condition F.2) we have $F(\lambda x) \leqslant (\lambda+\eta)F(x)$ for all $x \geqslant x_0$.

Now, if x and t satisfy $x_0 \leqslant x \leqslant t \leqslant \lambda x$, then $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant F(\lambda x) \leqslant (\lambda+\eta)F(x)$, so (2.1) holds. That is, $s(x)$ is slowly decreasing with respect to t. □

The next theorem gives a sufficient condition for the converse of Proposition 1.2.

Theorem 2.1. *Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is locally integrable with respect to μ and slowly decreasing with respect to $F(t)$. If $F(t)$ satisfies conditions* F.1) *and*

$$\lim_{t\to\infty} \tau(t) = l,$$

then

$$\lim_{x\to\infty} s(x) = l.$$

The next theorem gives a sufficient condition for the converse of Proposition 1.1.

Theorem 2.2. *Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is slowly decreasing with respect to t. If*

$$\operatorname*{st-lim}_{x\to\infty} s(x) = l,$$

then

$$\lim_{x\to\infty} s(x) = l.$$

Combining Theorem 2.2 with Proposition 2.2 gives the following corollary, which extends Theorem 1 from [4]. (Recall, that $F(t) = \log(t)$ satisfies condition F.2).)

Corollary 2.1. *Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is slowly decreasing with respect to $F(t)$. If $F(t)$ satisfies conditions* F.2) *and*

$$\operatorname*{st-lim}_{x\to\infty} s(x) = l,$$

then

$$\lim_{x\to\infty} s(x) = l.$$

Theorem 2.3. *Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is locally integrable with respect to μ and slowly decreasing with respect to $F(t)$. If $F(t)$ satisfies conditions* F.1) *and* F.2)*, then $\tau(t)$ is slowly decreasing with respect to $F(t)$.*

In the last theorem, one may also conclude that $\tau(t)$ is slowly decreasing with respect to t, invoking Proposition 2.2.

Corollary 2.2. *Suppose $s(x) : [a, \infty) \mapsto \mathbb{R}$ is locally integrable with respect to μ and slowly decreasing with respect to $F(t)$. If $F(t)$ satisfies conditions* F.1) *and* F.2) *and*

$$\operatorname*{st-lim}_{t\to\infty} \tau(t) = l,$$

then

$$\lim_{x\to\infty} s(x) = l.$$

P r o o f. By Theorem 2.3, $\tau(t)$ is slowly decreasing with respect to $F(t)$, and hence with respect to t, by Proposition 2.2. Thus, by Theorem 2.2, we have $\lim_{t\to\infty} \tau(t) = l$. Finally, Theorem 2.1 implies $\lim_{x\to\infty} s(x) = l$. □

3. Examples.

3.1.* $(C, 1)$ *summability. This is the case when $F(t) = t$ for $t \geqslant 0$. Then we have

$$\tau(t) = \frac{1}{t} \int_0^t s(x)dx.$$

Since $F(t)$ satisfies conditions F.1) and F.2), the three theorems apply.

3.2.* $(L, 1)$ *summability. Summability $(L, 1)$ is the case when $F(t) = \log(t)$ for $t \geqslant 1$. Then we have

$$\tau(t) = \frac{1}{\log(t)} \int_1^t \frac{s(x)}{x} dx.$$

Since $F(t)$ is continuous and concave, conditions F.1) and F.2) hold. The three theorems apply.

This particular case of Theorem 2.1 is given in [3, Corollary 1] and in this particular case Theorem 2.2 and Corollary 2.2 are given in [4, Theorems 1 and 3].

3.3. $(L,2)$ summability. Summability $(L,2)$ is the case when $F(t) = \log(\log(t))$ for $t \geqslant e$. Then, we have

$$\tau(t) = \frac{1}{\log(\log(t))} \int_e^t \frac{s(x)}{x \log x} dx.$$

Since $F(t)$ is continuous and concave, conditions F.1) and F.2) hold. The three theorems apply.

3.4. $(L,1)$ summability of numerical sequences. Consider a numerical sequence $\{s_n\}_{n=1}^{\infty}$ and the function $s(x) : [1,\infty) \to \mathbb{R}$ defined by $s(x) := s_{[x]}$. For $t \geq 1$, let

$$F(t) = \sum_{k=1}^{[t]} \frac{1}{k},$$

then

$$\tau(t) = \frac{1}{F(t)} \sum_{k=1}^{[t]} \frac{s(k)}{k}.$$

Function $F(t)$ satisfies condition F.1), since its jumps are all less than or equal to 1. So Theorem 2.1 holds. This case was considered in [3, Corollary 3]. Using the fact that

$$\log([t]+1) \leqslant F(t) \leqslant \log([t]) + 1$$

one can show that condition F.2) also holds. Thus, all three theorems apply.

3.5. Stolz' theorem. The classical Stolz' theorem is a discrete version of the L'Hospital's rule. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $\{b_n\}_{n=1}^{\infty}$ is strictly increasing and converging to infinity. If

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l,$$

where $l \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

This theorem is a special case of Proposition 1.2 if we set $a_0 = b_0 := 0$, define

$$s(x) = \frac{a_{[x]} - a_{[x]-1}}{b_{[x]} - b_{[x]-1}}, \quad \text{for } x \geq 1$$

and for $t \geq 0$, define

$$F(t) = \mu(0,t] = b_{[t]}.$$

Indeed,

$$\tau(t) = \frac{1}{b_{[t]}} \int_1^t s(x)\mu(dx) = \frac{1}{b_{[t]}} \sum_{k=1}^{[t]} \frac{a_k - a_{k-1}}{b_k - b_{k-1}} (b_k - b_{k-1}) = \frac{a_{[t]}}{b_{[t]}}.$$

Condition F.1) is satisfied if and only if

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1,$$

while for F.2), we have the following sufficient criteria.

Lemma 3.1. *The function $F(t) = b_{[t]}$ satisfies condition* F.2), *whenever for any integers $k \geqslant m > 0$, we have*

$$\limsup_{s\to\infty} \frac{b_{sk+k-1}}{b_{sm}} \leq \frac{k}{m}.$$

P r o o f. Observe that it suffices to prove the inequality

$$\limsup_{n\to\infty} \frac{b_{[\lambda n]}}{b_{[n]}} \leqslant \lambda$$

only for rational numbers $\lambda \geqslant 1$. (One can approximate an irrational λ with rationals from above and use the fact that $\{b_n\}$ is increasing sequence.)

So let $\lambda := k/m$ for some integers $k \geqslant m > 0$. Let $n = ms + l$, where $l \in \{0, 1, \dots, m-1\}$. We have

$$\limsup_{n\to\infty} \frac{b_{[kn/m]}}{b_{[n]}} = \limsup_{s\to\infty} \frac{b_{[ks+kl/m]}}{b_{[ms+l]}} \leqslant \limsup_{s\to\infty} \frac{b_{[ks+k(m-1)/m]}}{b_{[ms]}} \leqslant \limsup_{s\to\infty} \frac{b_{ks+k-1}}{b_{ms}}.$$

This concludes the proof. □

In fact, we can say a little bit more.

Corollary 3.1. *If the function $F(t) = b_{[t]}$ satisfies condition* F.1), *then it satisfies condition* F.2), *whenever for any integers $k \geqslant m > 0$, we have*

$$\limsup_{s\to\infty} \frac{b_{sk}}{b_{sm}} \leqslant \frac{k}{m}.$$

Theorem 2.1, gives a converse of the Stolz' theorem if $s(x)$ is slowly decreasing and $F(t)$ satisfies condition F.1). The slow decrease condition translates into: for every $\varepsilon > 0$, there is a $\lambda > 1$ and N, such that

$$\frac{a_m - a_{m-1}}{b_m - b_{m-1}} - \frac{a_n - a_{n-1}}{b_n - b_{n-1}} > -\varepsilon,$$

holds, whenever $b_N \leqslant b_n \leqslant b_m \leqslant \lambda b_n$.

3.6. L'Hospital's theorem. Let $f(x)$ and $g(x)$ be differentiable functions on $[a, \infty)$. The classical rule of L'Hospital states that if $\lim_{x\to\infty} g(x) = \infty$, $g'(x) \neq 0$, and

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l.$$

Without loss of generality, assume that $f(a) = g(a) = 0$. If $g(x)$ is non-decreasing, then this theorem is a special case of Proposition 1.2. Indeed, define

$$s(x) = \frac{f'(x)}{g'(x)}, \quad \text{for } x \geq a$$

and for $t \geq 0$, define

$$F(t) = \mu(0, t] = g(t),$$

then

$$\tau(t) = \frac{1}{g(t)} \int_a^t s(x)\mu(dx) = \frac{1}{g(t)} \int_a^t f'(t)dt = \frac{f(t)}{g(t)}.$$

Condition F.1) is satisfied since $F(t)$ is continuous. Theorem 2.1, gives a converse of the L'Hospital theorem: if $f'(x)/g'(x)$ is slowly decreasing with respect to $g(x)$, if $g(x)$ is non-decreasing, $\lim_{x\to\infty} g(x) = \infty$, $g'(x) \neq 0$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l, \tag{3.1}$$

then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l. \tag{3.2}$$

Recall that if if $f'(x)/g'(x)$ is increasing, then it is slowly decreasing with respect to any $F(t)$. Thus, we get two particular cases.

Suppose that $f(x)$ is non-decreasing and convex, while $g(x)$ is non-decreasing and concave. If $\lim_{x\to\infty} g(x) = \infty$, $g'(x) \neq 0$, and if (3.1) holds, then (3.2) holds.

Suppose that $f(x)$ is non-increasing and convex, while $g(x)$ is non-decreasing and convex. If $\lim_{x\to\infty} g(x) = \infty$, $g'(x) \neq 0$, and if (3.1) holds, then (3.2) holds.

3.7. Slowly decreasing does not imply convergent. This example, exhibits a continuous function $s(x)$ that is slowing decreasing with respect to $F(x)$ but has no limit as x approaches infinity. Suppose $F(x) = x$ and $s : [1, \infty) \to \mathbb{R}$ be defined by

$$s(x) = \begin{cases} \dfrac{x}{2^{2n}} - 1 & \text{if } x \in (2^{2n}, 2^{2n+1}], \\ -\dfrac{x}{2^{2n+1}} + 2 & \text{if } x \in (2^{2n+1}, 2^{2n+2}]. \end{cases}$$

Fix $\varepsilon \in (0, 1)$, let $\lambda_0 = 1 + \varepsilon/2$ and $x_0 > 1$. Fix $\lambda \in (1, \lambda_0)$ and let x, t satisfy $x_0 < x < t \leqslant \lambda x$, then $t \in (x, 3x/2)$.

If $x \in (2^{2n}, 2^{2n+1}]$ and $t \in (2^{2n}, 2^{2n+1}]$, then since $s(x)$ is increasing in $(2^{2n}, 2^{2n+1}]$, we have $s(t) - s(x) > 0 > -\varepsilon$.

If $x \in (2^{2n}, 2^{2n+1}]$ and $t \in (2^{2n+1}, 2^{2n+2}]$, then

$$\begin{aligned} s(t) - s(x) &= -\frac{t}{2^{2n+1}} - \frac{x}{2^{2n}} + 3 \geqslant -\frac{1}{2^{2n}}\Big(\frac{\lambda}{2} + 1\Big)x + 3 \geqslant -2\Big(\frac{\lambda}{2} + 1\Big) + 3 \\ &= 1 - \lambda = -\frac{\varepsilon}{2} > -\varepsilon. \end{aligned}$$

If $x \in (2^{2n+1}, 2^{2n+2}]$ and $t \in (2^{2n+1}, 2^{2n+2}]$, then

$$s(t) - s(x) = -\frac{1}{2^{2n+1}}(t - x) \geqslant -\frac{1}{2^{2n+1}}(\lambda x - x) > -\frac{x}{2^{2n+1}}\frac{\varepsilon}{2} \geqslant -\varepsilon.$$

If $x \in (2^{2n+1}, 2^{2n+2}]$ and $t \in (2^{2n+2}, 2^{2n+3}]$, then since $s(x)$ is increasing over the latter interval, we have

$$s(t) - s(x) > s(2^{2n+2}) - s(x) \geqslant -\varepsilon.$$

So, $s(x)$ is slowing decreasing. However, it is clear that every point in $[0, 1]$ is a limit point of $s(x)$ at infinity.

4. Proofs of the main results.

Proof of Theorem 2.1. Fix $\varepsilon > 0$. The proof consists of two analogous parts.

First, choose $\lambda > 1$ so that Proposition 1.3 holds. By Lemma 5.1, part (2) for any $\gamma \in (1, \lambda)$ there exists x_0 such that for all $x > x_0$, there exists $t > x$ satisfying

$$\gamma F(x) \leqslant F(t) \leqslant \lambda F(x).$$

For $\lambda_x := F(t)/F(x)$, we have $1 < \gamma \leqslant \lambda_x \leqslant \lambda$ and $F(t) = \lambda_x F(x)$. Using Lemma 5.2, part (1) (with $\gamma := \lambda_x$, $t := x$, and $t^* := t$), we have

$$\limsup_{x\to\infty}(s(x) - \tau(x)) \leqslant \limsup_{x\to\infty}\Big\{\frac{\lambda_x}{\lambda_x - 1}(\tau(t) - \tau(x))\Big\} + \limsup_{x\to\infty}\Big\{\frac{-1}{(\lambda_x - 1)F(x)}\int_x^t [s(u) - s(x)]\mu(du)\Big\}.$$

Since

$$1 < \frac{\lambda_x}{\lambda_x - 1} \leqslant \frac{\gamma}{\gamma - 1} \quad\text{and}\quad \lim_{x\to\infty}(\tau(t) - \tau(x)) = 0$$

(recall that $t > x$ depends on x), we have

$$\limsup_{x\to\infty}\Big\{\frac{\lambda_x}{\lambda_x - 1}(\tau(t) - \tau(x))\Big\} = 0.$$

Focusing on the integral, when $u \in (x, t]$, we have

$$F(x) \leqslant F(u) \leqslant F(t) = \lambda_x F(x) \leqslant \lambda F(x). \tag{4.1}$$

Hence, by Proposition 1.3, we have $s(u) - s(x) > -\varepsilon$. So,

$$\begin{aligned}\limsup_{x\to\infty}(s(x) - \tau(x)) &\leqslant \limsup_{x\to\infty}\Big\{\frac{\varepsilon}{(\lambda_x - 1)F(x)}\int_x^t \mu(du)\Big\} \\ &= \limsup_{x\to\infty}\Big\{\frac{\varepsilon}{(\lambda_x - 1)F(x)}[F(t) - F(x)]\Big\} \\ &= \varepsilon,\end{aligned}$$

where we used the equality in (4.1).

Second, choose $0 < \lambda < 1$ so that Proposition 1.3 holds with $1/\lambda$. By Lemma 5.1, part (4) for any $\gamma \in (\lambda, 1)$ there exists x_0 such that for all $x > x_0$, there exists $t < x$ satisfying

$$\lambda F(x) \leqslant F(t) \leqslant \gamma F(x) \tag{4.2}$$

Set $\lambda_x = F(t)/F(x)$, then we have $\lambda \leqslant \lambda_x \leqslant \gamma < 1$ and $F(t) = \lambda_x F(x)$. Using Lemma 5.2 part (2) (with $\gamma := \lambda_x$, $t := x$, and $t^* := t$), we have

$$\begin{aligned}\liminf_{x\to\infty}(s(x)-\tau(x)) \geqslant \liminf_{x\to\infty}\Big\{\frac{\lambda_x}{1-\lambda_x}(\tau(x)-\tau(t))\Big\}\\ + \liminf_{x\to\infty}\Big\{\frac{1}{(1-\lambda_x)F(x)}\int_t^x [s(x)-s(u)]\mu(du)\Big\}.\end{aligned}$$

Since

$$0 < \frac{\lambda_x}{1-\lambda_x} \leqslant \frac{\gamma}{1-\gamma} \text{ and } \lim_{x\to\infty}(\tau(x)-\tau(t)) = 0$$

(recall that $t < x$ depends on x, but since $F(x)$ approaches infinity as x does, inequality (4.2) shows that t approaches infinity in that case as well), we have

$$\liminf_{x\to\infty}\Big\{\frac{\lambda_x}{1-\lambda_x}(\tau(x)-\tau(t))\Big\} = 0.$$

Considering the integral, when $u \in (t, x]$, we have

$$\lambda F(x) \leqslant \lambda_x F(x) = F(t) \leqslant F(u) \leqslant F(x) \leqslant (1/\lambda)F(u). \tag{4.3}$$

So, by Proposition 1.3, we have $s(x) - s(u) > -\varepsilon$. Thus,

$$\begin{aligned}\liminf_{t\to\infty}(s(x)-\tau(x)) &\geqslant \liminf_{x\to\infty}\Big\{\frac{-\varepsilon}{(1-\lambda_x)F(x)}\int_t^x \mu(du)\Big\}\\ &= \liminf_{x\to\infty}\Big\{\frac{-\varepsilon}{(1-\lambda_x)F(x)}[F(x)-F(t)]\Big\}\\ &= -\varepsilon,\end{aligned}$$

where we used the equality in (4.3).

Both parts of the proof, together show that

$$\lim_{x\to\infty}(s(x)-\tau(x)) = 0$$

and the result follows. □

Proof of Theorem 2.2. Fix $\varepsilon > 0$. Choose $\lambda > 1$ and $x_0 > a$ so that Proposition 1.3 holds for $F(t) = t$.

Define inductively an increasing sequence $\{b_n\}_{n=1}^{\infty}$ as follows. By the definition of statistical limit we can find a b_1 such that

$$|s(b_1) - l| \leqslant \varepsilon.$$

Suppose, $b_1, \dots, b_n$ have been chosen. Select, b_{n+1} according to the following two cases.

Case 1. If

$$|s(t) - l| \leqslant \varepsilon \text{ for some } t \in (\sqrt{\lambda} b_n, \lambda b_n],$$

then let b_{n+1} be that t. (It does not matter which one if there is a choice.)

Case 2. Otherwise, we have

$$|s(t) - l| > \varepsilon \text{ for every } t \in (\sqrt{\lambda} b_n, \lambda b_n] \tag{4.4}$$

and by the definition of statistical limit we can find a $b_{n+1} > \lambda b_n$ for which

$$|s(b_{n+1}) - l| \leqslant \varepsilon$$

holds.

By construction, we have

$$|s(b_n) - l| \leqslant \varepsilon \text{ for all } n = 1, 2, \dots \tag{4.5}$$

Since $b_{n+1} \geqslant \sqrt{\lambda} b_n$ for all n, and $\lambda > 1$, we have that the sequence $\{b_n\}_{n=1}^{\infty}$ increases to infinity.

Suppose that in the constriction of the sequence, case 2 has been applied infinitely many times. That is, (4.4) holds for infinitely many n. Then

$$\begin{aligned}\frac{1}{\lambda b_n - a} |\{t \in (a, \lambda b_n) : \, |s(t) - l| > \varepsilon\}| > \frac{\lambda b_n - \sqrt{\lambda} b_n}{\lambda b_n - a} &= \frac{\lambda - \sqrt{\lambda}}{\lambda - a/b_n} \\ &> \frac{1}{2} \frac{\lambda - \sqrt{\lambda}}{\lambda} > 0\end{aligned}$$

is a contradiction with the fact that the statistical limit of $s(x)$ is l. So, there is an N, such that for all $n \geqslant N$, we have

$$b_{n+1} \in (\sqrt{\lambda} b_n, \lambda b_n].$$

Next, for any $t \in (b_n, b_{n+1}]$, $n \geqslant N$, we have

$$b_n \leqslant t \leqslant b_{n+1} \leqslant \lambda b_n \leqslant \lambda t.$$

Thus, by Proposition 1.3, $b_n \leqslant t \leqslant \lambda b_n$ implies $s(t) - s(b_n) > -\varepsilon$, which together with (4.5) gives

$$s(t) - l \geqslant s(t) - s(b_n) + s(b_n) - l \geqslant -2\varepsilon.$$

Similarly, by Proposition 1.3, $t \leqslant b_{n+1} \leqslant \lambda t$ implies $s(b_{n+1}) - s(t) > -\varepsilon$, which together with (4.5) gives

$$s(t) - l \geqslant s(t) - s(b_{n+1}) + s(b_{n+1}) - l \leqslant 2\varepsilon.$$

So, $|s(t) - l| \leqslant 2\varepsilon$ for every $t \in \bigcup_{n=N}^{\infty} (b_n, b_{n+1}]$, and using the fact that $\{b_n\}_{n=1}^{\infty}$ increases to infinity, concludes the proof. $\square$

P r o o f o f T h e o r e m 2.3. We prove that if $s(x)$ is slowly decreasing, then $\tau(t)$ is also slowly decreasing. For any $x_0 \leqslant x \leqslant t$, one estimates

$$\begin{aligned}
\tau(t) - \tau(x) &= \frac{1}{F(t)} \int_a^t s(u)\mu(du) - \frac{1}{F(x)} \int_a^x s(u)\mu(du) \\
&= -\Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) \int_a^x s(u)\mu(du) + \frac{1}{F(t)} \int_x^t s(u)\mu(du) \\
&\quad + \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) \int_a^x s(x)\mu(du) - \frac{1}{F(t)} \int_x^t s(x)\mu(du) \\
&= \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) \int_a^x \big(s(x) - s(u)\big)\mu(du) \\
&\quad + \frac{1}{F(t)} \int_x^t \big(s(u) - s(x)\big)\mu(du) \\
&= \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big)\Big(\int_a^{x_0} + \int_{x_0}^x\Big)\big(s(x) - s(u)\big)\mu(du) \\
&\quad + \frac{1}{F(t)} \int_x^t \big(s(u) - s(x)\big)\mu(du) \\
&= \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) s(x)F(x_0) \\
&\quad - \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) \int_a^{x_0} s(u)\mu(du) \\
&\quad + \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) \int_{x_0}^x \big(s(x) - s(u)\big)\mu(du) \\
&\quad + \frac{1}{F(t)} \int_x^t \big(s(u) - s(x)\big)\mu(du) \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We consider each one of the expressions J_i, $i = 1, 2, 3, 4$, separately.

Considering J_1, we have

$$J_1 = \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big) x F(x_0) \frac{s(x)}{x}.$$

so, by Lemma 5.3, keeping in mind that $x \le t$, we have

$$\liminf_{x\to\infty} J_1 \geqslant 0.$$

Considering J_2, we have

$$\begin{aligned} |J_2| &= \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big)\Big|\int_a^{x_0} s(u)\mu(du)\Big| \\ &\leqslant \Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big)\int_a^{x_0} |s(u)|\mu(du). \end{aligned}$$

Since the integral does not depend on x and t, we see that

$$\lim_{x\to\infty} J_2 = 0.$$

Considering J_3, fix $\varepsilon > 0$ and let $\lambda_0 > 1$ and x_0 be such that Proposition 1.3 holds. Decrease $\lambda_0 > 1$, if necessary, so that Lemma 5.6 holds. That is, for any $\lambda \in (1, \lambda_0)$ any $\gamma \in (1/\lambda, 1)$ and any $\theta \in (\gamma, 1)$, there is an x_0 such that for any x and t satisfying

$$\lambda F(x_0) < F(x) \le F(t) \le \lambda F(x), \tag{4.6}$$

we have

$$\begin{aligned} J_3 &\geqslant -\Big(\frac{1}{F(x)} - \frac{1}{F(t)}\Big)F(x)\Big(-\frac{2\gamma}{\log(\theta)}\log(\lambda) - \frac{2\gamma}{\log(\theta)} + 1 - \frac{1}{\lambda}\Big)\varepsilon \\ &= -\Big(1 - \frac{F(x)}{F(t)}\Big)\Big(-\frac{2\gamma}{\log(\theta)}\log(\lambda) - \frac{2\gamma}{\log(\theta)} + 1 - \frac{1}{\lambda}\Big)\varepsilon \\ &\geqslant -\Big(1 - \frac{1}{\lambda}\Big)\Big(-\frac{2\gamma}{\log(\theta)}\log(\lambda) - \frac{2\gamma}{\log(\theta)} + 1 - \frac{1}{\lambda}\Big)\varepsilon. \\ &= \frac{\lambda - 1}{\log(\theta)}\frac{2\gamma}{\lambda}(\log(\lambda) + 1)\varepsilon - \Big(1 - \frac{1}{\lambda}\Big)^2\varepsilon. \end{aligned} \tag{4.7}$$

Now, for any $\lambda \in (1, \lambda_0)$ let

$$\gamma := \frac{1 + 1/\lambda}{2} = \frac{\lambda + 1}{2\lambda} \text{ and } \theta := \frac{1 + \gamma}{2} = \frac{3\lambda + 1}{4\lambda}.$$

It is trivial to check that $\gamma \in (1/\lambda, 1)$ and $\theta \in (\gamma, 1)$, so there is an x_0 such that (4.7) holds, whenever x and t satisfy (4.6). Next, it is simple calculus verification that the following inequality holds

$$\theta = \frac{3\lambda + 1}{4\lambda} \leqslant \Big(\frac{9}{10}\Big)^{\lambda - 1} \text{ for all } \lambda \in [1, 2].$$

Thus, $(\lambda - 1)/\log(\theta) \geqslant 1/\log(0.9)$ for all $\lambda \in [1, 2]$. Substituting the expressions for γ and θ into (4.7), we obtain

$$\begin{aligned} J_3 &\geqslant \frac{1}{\log(0.9)} \frac{\lambda + 1}{\lambda^2} (\log(\lambda) + 1)\varepsilon - \Big(1 - \frac{1}{\lambda}\Big)^2 \varepsilon \geqslant -\Big(- \frac{2}{\log(0.9)} + 1\Big)\varepsilon \\ &\geqslant -20\varepsilon, \end{aligned}$$

using the fact that $(\lambda + 1)(\log(\lambda) + 1)/\lambda^2$ is a decreasing function in $\lambda \geqslant 1$ and so it achieves its maximum at $\lambda = 1$.

To summarise, we showed that for every $\lambda \in (1, \lambda_0)$, there is a x_0 such that for any x, t satisfying (4.6), we have

$$J_3 \geqslant -20\varepsilon.$$

Considering J_4, let $\varepsilon > 0$, $\lambda_0 > 1$, and x_0 be chosen as in the previous case and let x and t satisfy $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant \lambda F(x)$. For any $u \in (x, t]$, we have

$$F(x_0) \leqslant F(x) \leqslant F(u) \leqslant F(t) \leqslant \lambda F(x)$$

so Proposition 1.3 implies $s(u) - s(x) > -\varepsilon$. Hence,

$$\begin{aligned} J_4 &= \frac{1}{F(t)} \int_x^t \big(s(u) - s(x)\big)\mu(du) \geqslant -\frac{\varepsilon}{F(t)} \int_x^t \mu(du) = -\frac{\varepsilon}{F(t)}\big(F(t) - F(x)\big) \\ &= -\Big(1 - \frac{F(x)}{F(t)}\Big)\varepsilon \geqslant -\Big(1 - \frac{1}{\lambda}\Big)\varepsilon > -\varepsilon. \end{aligned}$$

In conclusion, for all $x \geqslant x_0$, we have

$$\tau(t) - \tau(x) = J_1 + J_2 + J_3 + J_4 > -23\varepsilon,$$

whenever $F(x_0) \leqslant F(x) \leqslant F(t) \leqslant \lambda F(x)$. This concludes the proof. $\square$

5. Appendix: Auxiliary results.

Lemma 5.1. *Under the assumption* $\lim_{x\to\infty} F(x) = \infty$*, the following conditions are equivalent.*

1. $\lim_{x\to\infty} F(x)/F(x-) = 1.$

2. *For all* $\lambda > 1$ *sufficiently close to one and any* $\gamma \in (1, \lambda)$*, there exists* x_0*, such that for every* $x > x_0$*, there exists* $t > x$ *satisfying*

$$\gamma F(x) \leqslant F(t) \leqslant \lambda F(x).$$

3. *For all* $\lambda > 1$ *sufficiently close to one, there exists* $\gamma \in (1, \lambda]$ *and* x_0*, such that for every* $x > x_0$*, there exists* $t > x$ *satisfying*

$$\gamma F(x) \leqslant F(t) \leqslant \lambda F(x).$$

4. *For all* $\lambda < 1$ *sufficiently close to one and any* $\gamma \in (\lambda, 1)$*, there exists* x_0*, such that for every* $x > x_0$*, there exists* $t < x$ *satisfying*

$$\lambda F(x) \leqslant F(t) \leqslant \gamma F(x).$$

5. *For all* $\lambda < 1$ *sufficiently close to one, there exists* $\gamma \in [\lambda, 1)$ *and* x_0*, such that for every* $x > x_0$*, there exists* $t < x$ *satisfying*

$$\lambda F(x) \leqslant F(t) \leqslant \gamma F(x).$$

Proof. (1) $\Rightarrow$ (2). Fix $\lambda > 1$, any $\gamma \in (1, \lambda)$, and let $\varepsilon := \lambda/\gamma - 1$. Since (1) is equivalent to

$$\lim_{x\to\infty} \frac{F(x) - F(x-)}{F(x-)} = 0,$$

there exists x_1, such that for all $x > x_1$, we have $F(x) - F(x-) \leqslant \varepsilon F(x-)$. Let x_0 be so large that for all $x > x_0$, we have

$$t := \sup\{y : F(y) \leqslant \gamma F(x)\} > x_1.$$

(This is possible, since $\lim_{x\to\infty} F(x) = \infty$.) On the one hand, using the right-continuity of F at t, we obtain $F(t) \geqslant \gamma F(x) > F(x)$. This implies that $t > x$, since F is non-decreasing. On the other hand, we have $F(t-) \leqslant \gamma F(x)$. Since $t > x_1$, we have $F(t) - F(t-) \leqslant \varepsilon F(t-)$. Hence,

$$F(t) \leqslant F(t-) + \varepsilon F(t-) \leqslant (1+\varepsilon)\gamma F(x) = \lambda F(x).$$

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Suppose that for all $\lambda > 1$ sufficiently close to one, there exists $\gamma \in (1, \lambda]$ and x_0, such that for every $x > x_0$, there exists $t > x$ satisfying

$$\gamma F(x) \leqslant F(t) \leqslant \lambda F(x).$$

We claim that for every $x > x_0$, there is a $t \geq x$ such that

$$F(t) \leqslant \lambda F(x-). \tag{5.1}$$

Indeed, fix an $x > x_0$. For all large enough n, there is a $t_n > x - 1/n$ such that

$$\gamma F(x - 1/n) \leqslant F(t_n) \leqslant \lambda F(x - 1/n) \leqslant \lambda F(x-). \tag{5.2}$$

Since $\lim_{x\to\infty} F(x) = \infty$, the sequence $\{t_n\}$ is bounded. Hence, it has an accumulation point, call it t^*. Without loss of generality, or else choose a subsequence, assume $\{t_n\}$ converges to t^*. Clearly, $t^* \geq x$. Again, by choosing a subsequence, we may assume that $\{t_n\}$ is a monotone sequence.

If $\{t_n\}$ is a decreasing sequence, then by the right-continuity of F, we obtain

$$\gamma F(x-) \leqslant F(t^*) \leqslant \lambda F(x-)$$

and, taking $t := t^*$, we are done.

If $\{t_n\}$ is an increasing sequence, then taking the limit in all sides of (5.2), we obtain

$$\gamma F(x-) \leqslant F(t^*-) \leqslant \lambda F(x-).$$

In this case, we must have $t^* > x$, or else we reach a contradiction with $\gamma > 1$. Now, simply take $t \in (x, t^*)$ to obtain

$$F(t) \leqslant F(t^*-) \leqslant \lambda F(x-)$$

concluding the proof of the claim.

Thus, for every $x > x_0$, there is a $t \geq x$ such that (5.1) holds, implying that

$$F(x) - F(x-) \leqslant F(t) - F(x-) \leqslant (\lambda - 1)F(x-).$$

Therefore,

$$0 \leqslant \liminf_{x\to\infty} \frac{F(x) - F(x-)}{F(x-)} \leqslant \limsup_{x\to\infty} \frac{F(x) - F(x-)}{F(x-)} \leqslant \lambda - 1.$$

Letting λ approach one, one obtains (1).

(1) $\Rightarrow$ (4). Fix $\lambda \in (0, 1)$, any $\gamma \in (\lambda, 1)$, and let $\varepsilon = \gamma - \lambda$. Since (1) is equivalent to

$$\lim_{x\to\infty} \frac{F(x) - F(x-)}{F(x)} = 0,$$

there exists x_1, such that for all $x > x_1$, we have $F(x) - F(x-) \leqslant \varepsilon F(x)$. Let x_0 be so large that for all $x > x_0$, we have

$$t := \sup\{y : F(y) \leqslant \lambda F(x)\} > x_1.$$

(This is possible, since $\lim_{x\to\infty} F(x) = \infty$.) Then, we have $F(t) \geqslant \lambda F(x)$ and $F(t-) \leqslant \lambda F(x) < F(x)$. The latter implies that $x \geqslant t$ and hence $F(t) \leqslant F(x)$. Combining everything, leads to

$$F(t) - F(t-) \leqslant \varepsilon F(t) \leqslant (\gamma - \lambda)F(x).$$

Thus,

$$F(t) \leqslant \gamma F(x) + F(t-) - \lambda F(x) \leqslant \gamma F(x) < F(x),$$

and so $t < x$.

(4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (1). For all $\lambda < 1$ sufficiently close to one, there exists $\gamma \in [\lambda, 1)$ and x_0, such that for every $x > x_0$, there exists $t < x$ satisfying

$$\lambda F(x) \leqslant F(t) \leqslant \gamma F(x).$$

Since $F(t) \leqslant F(x-)$, we obtain

$$F(x) - F(x-) \leqslant F(x) - F(t) \leqslant (1 - \lambda)F(x).$$

Dividing by $F(x)$ leads to

$$0 \leqslant \liminf_{x\to\infty} \frac{F(x) - F(x-)}{F(x)} \leqslant \limsup_{x\to\infty} \frac{F(x) - F(x-)}{F(x)} \leqslant 1 - \lambda.$$

Letting λ approach one, we get $\lim_{x\to\infty} (F(x) - F(x-))/F(x) = 0$, which is easily seen to be equivalent to (1). $\square$

Lemma 5.2. *Let $s(x)$ and $\tau(t)$ be as in Definition* 1.2.

1. *Suppose for $t > a$ and $\gamma > 1$, there exists t^* with the property $F(t^*) = \gamma F(t)$. Then, we have*

$$(5.3)\quad s(t) - \tau(t) = \frac{\gamma}{\gamma - 1}(\tau(t^*) - \tau(t)) - \frac{1}{(\gamma - 1)F(t)} \int_t^{t^*} [s(u) - s(t)]\mu(du).$$

2. *Suppose for $t > a$ and $0 < \gamma < 1$, there exists t^* with the property $F(t^*) = \gamma F(t)$. Then, we have*

$$(5.4)\quad s(t) - \tau(t) = \frac{\gamma}{1 - \gamma}(\tau(t) - \tau(t^*)) + \frac{1}{(1 - \gamma)F(t)} \int_{t^*}^{t} [s(t) - s(u)]\mu(du).$$

P r o o f. We prove the first part only since the proof of the second one is similar.

Note that $t > a$, $\gamma > 1$ and $F(t^*) = \gamma F(t)$, imply $t^* > t$. Then, we have

$$\begin{aligned}
\frac{\gamma}{\gamma - 1}(\tau(t^*) - \tau(t)) &= \frac{\gamma}{\gamma - 1}\Big\{\frac{1}{F(t^*)} \int_a^{t^*} s(u)\mu(du) - \frac{1}{F(t)} \int_a^{t} s(u)\mu(du)\Big\} \\
&= \frac{\gamma}{\gamma - 1}\Big\{\frac{1}{\gamma F(t)} \int_a^{t^*} s(u)\mu(du) - \frac{\gamma}{\gamma F(t)} \int_a^{t} s(u)\mu(du)\Big\} \\
&= \frac{1}{\gamma - 1}\Big\{\frac{1 - \gamma}{F(t)} \int_a^{t} s(u)\mu(du) + \frac{1}{F(t)} \int_t^{t^*} s(u)\mu(du)\Big\} \\
&= -\tau(t) + \frac{1}{(\gamma - 1)F(t)} \int_t^{t^*} s(u)\mu(du) \\
&= -\tau(t) + \frac{1}{(\gamma - 1)F(t)} \int_t^{t^*} [s(u) - s(t)]\mu(du) \\
&\quad + \frac{s(t)}{(\gamma - 1)F(t)} \int_t^{t^*} \mu(du) \\
&= s(t) - \tau(t) + \frac{1}{(\gamma - 1)F(t)} \int_t^{t^*} [s(u) - s(t)]\mu(du).
\end{aligned}$$

So, Equation (5.3) holds. □

Lemma 5.3. *Suppose $s(x)$ is slowly decreasing with respect to $F(t)$. If $F(t)$ satisfies conditions* F.1) *and* F.2)*, then*

$$\liminf_{x \to \infty} \frac{s(x)}{x} \geqslant 0.$$

P r o o f. Let $\varepsilon := 1$ and fix $\lambda_0 > 1$ and $x_0 > a$ whose existence is guaranteed by Proposition 1.3. Decrease $\lambda_0 > 1$, if necessary, so that condition F.2) holds for all $\lambda \in (1, \lambda_0)$. Decrease $\lambda_0 > 1$ even further, if necessary, so that Lemma 5.1, part (2), holds for every $\lambda \in (1, \lambda_0)$ and every $\gamma \in (1, \lambda)$. Fix such a λ and γ. Increase x_0 so that the conclusion of Lemma 5.1, part (2), holds. Fix an $\eta \in (0, \lambda - \gamma)$ and increase x_0 so that by condition F.2) (applied with $\lambda := \gamma - \eta \in (1, \lambda_0)$) wet have

$$\gamma F(x) = ((\gamma - \eta) + \eta)F(x) > F((\gamma - \eta)x) \tag{5.5}$$

for every $x > x_0$.

We construct an increasing sequence $\{b_n\}$ starting with any $b_0 > x_0$. Suppose $b_0, \dots, b_n$ have been constructed, then by Lemma 5.1, part (2), since $b_n > x_0$, there is a $b_{n+1} > b_n$ such that

$$\gamma F(b_n) \leqslant F(b_{n+1}) \leqslant \lambda F(b_n).$$

Thus we have

$$F(x_0) \leqslant F(b_n) \leqslant F(b_{n+1}) \leqslant \lambda F(b_n) \text{ for all } n = 0, 1, 2, \dots$$

and Proposition 1.3 guarantees that

$$\begin{aligned} s(b_n) - s(b_0) = (s(b_n) - s(b_{n-1})) + (s(b_{n-1}) - s(b_{n-2})) \\ + \dots + (s(b_1) - s(b_0)) \geqslant -n. \end{aligned}$$

So,

$$\frac{s(b_n)}{b_n} \geqslant \frac{s(b_0)}{b_n} - \frac{n}{b_n}.$$

Observing that $F(b_n) \geqslant \gamma F(b_{n-1}) \geqslant \dots \geqslant \gamma^n F(b_0)$, we have

$$\begin{aligned} F(b_n) \geqslant \gamma^n F(b_0) &= \gamma^{n-1}\big(\gamma F(b_0)\big) > \gamma^{n-1} F((\gamma - \eta) b_0) \\ &= \gamma^{n-2}\big(\gamma F((\gamma - \eta) b_0)\big) \\ &> \gamma^{n-2} F((\gamma - \eta)^2 b_0) > \dots > F((\gamma - \eta)^n b_0). \end{aligned}$$

This shows that $b_n \geqslant (\gamma - \eta)^n b_0$ and in particular that the sequence $\{b_n\}$ approaches infinity exponentially. Hence, we can estimate

$$\lim_{n\to\infty} \frac{n}{b_n} \leqslant \lim_{n\to\infty} \frac{n}{(\gamma - \eta)^n b_0} = 0.$$

So,

$$\liminf_{n\to\infty} \frac{s(b_n)}{b_n} \geqslant \lim_{n\to\infty} \left(\frac{s(b_0)}{b_n} - \frac{n}{b_n} \right) = 0.$$

Suppose $\{x_n\}$ is an increasing sequence, approaching infinity, such that

$$\lim_{n\to\infty} \frac{s(x_n)}{x_n} = L.$$

We show that $L \geqslant 0$ by constructing appropriate subsequences of $\{b_n\}$ and $\{x_n\}$, inductively, as follows.

Let p_1 be the index such that $b_{p_1} < x_1 \leqslant b_{p_1+1}$, set $n_1 = 1$. Assume that indices $p_1, \dots, p_{k-1}$ and $n_1, \dots, n_{k-1}$ have been chosen. Let n_k be the first index such that $x_{n_k} \notin (b_{p_{k-1}}, b_{p_{k-1}+1}]$ and let p_k be the index such that $b_{p_k} < x_{n_k} \leqslant b_{p_k+1}$.

For the so-chosen subsequences $\{b_{p_k}\}$ and $\{x_{n_k}\}$, we have

$$F(b_{p_k}) \leqslant F(x_{n_k}) \leqslant F(b_{p_k+1}) \leqslant \lambda F(b_{p_k}).$$

Thus, $s(x_{n_k}) - s(b_{p_k}) \geqslant -1$ and

$$\frac{s(x_{n_k})}{x_{n_k}} \geqslant \frac{s(b_{p_k})}{x_{n_k}} - \frac{1}{x_{n_k}} \geqslant \frac{s(b_0) - p_k}{x_{n_k}} - \frac{1}{x_{n_k}} > \frac{s(b_0)}{x_{n_k}} - \frac{p_k}{b_{p_k}} - \frac{1}{x_{n_k}}.$$

Since $b_{p_k} \geqslant (\gamma - \eta)^{p_k} b_0$, we have

$$0 \leqslant \frac{p_k}{b_{p_k}} \leqslant \frac{p_k}{(\gamma - \eta)^{p_k} b_0},$$

where the last ratio converges to zero as k approaches infinity. So,

$$L = \lim_{k\to\infty} \frac{s(x_{n_k})}{x_{n_k}} \geqslant \lim_{k\to\infty} \left(\frac{s(x_0)}{x_{n_k}} - \frac{p_k}{b_{p_k}} - \frac{1}{x_{n_k}} \right) = 0.$$

This concludes the proof of the lemma. □

Lemma 5.4. *Suppose $s(x)$ is slowly decreasing with respect to $F(x)$ and suppose $F(x)$ satisfies condition* F.1). *Then, for every $\varepsilon > 0$, there is a $\gamma_0 < 1$, such that for any $\gamma \in (\gamma_0, 1)$ and any $\theta \in (\gamma, 1)$ there exists an x_0, such that*

$$s(t) - s(x) \geqslant \frac{2\varepsilon}{\log(\theta)} \log \left(\frac{F(t)}{F(x)} \right) \tag{5.6}$$

holds, whenever x and t satisfy $F(x_0) \leqslant F(x) < \gamma F(t)$.

P r o o f. Fix $\varepsilon > 0$. Let $\gamma_0 < 1$ be such that Proposition 1.3 holds (with $\lambda_0 := 1/\gamma_0$). Increase $\gamma_0 < 1$, if necessary, so that Lemma 5.1, part (4) holds (with $\lambda := \gamma_0$). Note that, by increasing γ_0, Proposition 1.3 continues to hold.

Now, choose any $\gamma \in (\gamma_0, 1)$ and any $\theta \in (\gamma, 1)$ and choose x_0 large enough so that both the condition in Proposition 1.3 (with $\lambda := 1/\gamma$) and the condition in Lemma 5.1, part (4), (with $\lambda := \gamma$ and $\gamma := \theta$) hold.

Fix $x \geqslant x_0$ and t satisfying

$$F(x_0) \leqslant F(x) < \gamma F(t) \tag{5.7}$$

and note that $x_0 \leqslant x < t$. We want to show that (5.6) holds.

Define a decreasing sequence $\{t_n\}_{n=0}^{\infty}$ inductively as follows. Let $t_0 := t$ and suppose $t_n < \cdots < t_0 = t$ have been defined. Lemma 5.1, part (4) (applied with $\lambda := \gamma$, $\gamma := \theta$, $x := t_n$) says that *if* $x_0 < t_n$, then there exists a $t_{n+1} < t_n$, such that

$$\gamma F(t_n) \leqslant F(t_{n+1}) \leqslant \theta F(t_n). \tag{5.8}$$

Since, $F(t_{n+1}) \leqslant \theta^{n+1} F(t_0)$ and $\theta < 1$, there is an index m such that

$$t_{m+1} \leqslant x < t_m.$$

That is, $n := m$ is the largest index for which (5.8) is guaranteed to hold. We have

$$F(x) \leqslant F(t_m) \leqslant (1/\gamma)F(t_{m+1}) \leqslant (1/\gamma)F(x), \text{ and} \tag{5.9}$$

$$F(t_{n+1}) \leqslant F(t_n) \leqslant (1/\gamma)F(t_{n+1}) \text{ for all } n = 0, 1, \ldots, m. \tag{5.10}$$

Proposition 1.3, together with (5.9), implies

$$s(t_m) - s(x) \geqslant -\varepsilon,$$

while Proposition 1.3, together with (5.10), implies

$$s(t_n) - s(t_{n+1}) \geqslant -\varepsilon \text{ for all } n = 0, 1, \ldots, m.$$

Hence,

$$s(t) - s(x) = \sum_{n=0}^{m-1} (s(t_n) - s(t_{n+1})) + s(t_m) - s(x) \geqslant -(m+1)\varepsilon. \tag{5.11}$$

Next, since $F(x) \leqslant F(t_m) \leqslant \theta^m F(t_0) = \theta^m F(t)$, one obtains

$$m \leqslant -\frac{1}{\log(\theta)} \log\Big(\frac{F(t)}{F(x)}\Big).$$

Inequality (5.7) implies that $\log(1/\gamma) < \log(F(t)/F(x))$, and so

$$1 < \frac{1}{\log(1/\gamma)} \log\Big(\frac{F(t)}{F(x)}\Big) < -\frac{1}{\log(\theta)} \log\Big(\frac{F(t)}{F(x)}\Big),$$

where $\gamma < \theta < 1$ was also used. Therefore, we have

$$m + 1 < -\frac{2}{\log(\theta)} \log\Big(\frac{F(t)}{F(x)}\Big).$$

Substituting in (5.11), completes the proof of (5.6). $\square$

For the next lemma we need some technical preparations about the Lebegue-Stieltjes integral. For the right-continuous, increasing function $F(x)$, define its *generalized inverse* by

$$F^{-1}(y) := \sup\{x : F(x) < y\}$$

for all $y \in (0, \infty)$. If $F(x)$ is the cumulative distribution function of a probability measure, then $F^{-1}(y)$ is also known as the *quantile function.* It is not difficult to show that

$$F(F^{-1}(y)) \geq y \text{ for all } y \in (0, \infty). \tag{5.12}$$

The following lemma is proved in [2, Lemma 2.2].

Lemma 5.5. *Any non-decreasing, right-continuous functions $F(x)$ and $G(x)$ satisfy*

$$\int_a^b G(x) dF(x) = \int_{F(a)}^{F(b)} G(F^{-1}(y)) dy.$$

Lemma 5.6. *Suppose $s(x)$ is slowly decreasing with respect to $F(x)$ and suppose $F(x)$ satisfies condition* F.1)*. Then, for every $\varepsilon > 0$, there is a $\lambda_0 > 1$, such that for any $\lambda \in (1, \lambda_0)$ any $\gamma \in (1/\lambda, 1)$ and any $\theta \in (\gamma, 1)$, there is an x_0, such that*

$$\frac{1}{F(t)} \int_{x_0}^{t} [s(t) - s(x)] \mu(dx) \geqslant -\Big(-\frac{2\gamma}{\log(\theta)} \log(\lambda) - \frac{2\gamma}{\log(\theta)} + 1 - \frac{1}{\lambda}\Big) \varepsilon, \tag{5.13}$$

whenever $F(t) > \lambda F(x_0)$.

P r o o f. Fix $\varepsilon > 0$. Let $\gamma_0 < 1$ be such that Lemma 5.4 holds. Choose, $\lambda_0 > 1$ such that Proposition 1.3 holds. Decrease $\lambda_0 > 1$, if necessary, so that $\gamma_0 < 1/\lambda_0$; and Lemma 5.1, part (4) holds (with $\lambda := 1/\lambda_0$). Note that by decreasing $\lambda_0 > 1$ Proposition 1.3 continues to hold.

Now fix any $\lambda \in (1, \lambda_0)$, any $\gamma \in (1/\lambda, 1)$, and any $\theta \in (\gamma, 1)$. Note that Proposition 1.3 continues to hold for the chosen λ. Since $\gamma_0 < 1/\lambda_0 < 1/\lambda < \gamma < 1$, we have that $\gamma \in (\gamma_0, 1)$, so Lemma 5.4 holds for the chosen γ and θ. Finally, since $1/\lambda \in (1/\lambda_0, 1)$ and $\gamma \in (1/\lambda, 1)$, Lemma 5.1, part (4) with $1/\lambda < 1$ and γ.

Thus, one can choose x_0 large enough, so that all three results hold for that x_0 and their respective parameters.

Next, fix t satisfying

$$F(t) > \lambda F(x_0)$$

and note that $t > x_0$.

By Lemma 5.1, part (4), there exists $t^* < t$, such that

$$F(x_0) < (1/\lambda)F(t) \leqslant F(t^*) \leqslant \gamma F(t). \tag{5.14}$$

This implies $x_0 < t^* < t$, and we have

$$\int_{x_0}^{t} [s(t) - s(x)]\mu(dx) = \Big(\int_{x_0}^{t^*} + \int_{t^*}^{t} \Big) [s(t) - s(x)]\mu(dx) =: I_1 + I_2.$$

Considering I_2, for any $x \in [t^*, t]$, we have

$$(1/\lambda)F(t) \leqslant F(t^*) \leqslant F(x) \leqslant F(t) \leqslant \lambda F(t^*) \leqslant \lambda F(x).$$

Thus, by Proposition 1.3, we have $s(t) - s(x) \geqslant -\varepsilon$ and so,

$$\frac{1}{\varepsilon} I_2 \geqslant -\int_{t^*}^{t} \mu(dx) = -F(t) + F(t^*) \geqslant -F(t)\Big(1 + \frac{1}{\lambda}\Big). \tag{5.15}$$

Considering I_1, for any $x \in [x_0, t^*]$, we have

$$F(x_0) \leqslant F(x) \leqslant F(t^*) \leqslant \gamma F(t),$$

where in the last inequality we used (5.14). By Lemma 5.4, and using the definition $\mu(dx) = dF(x)$, we obtain

$$\frac{1}{\varepsilon} I_1 \geqslant \frac{2}{\log(\theta)} \int_{x_0}^{t^*} \log \Big(\frac{F(t)}{F(x)} \Big) dF(x)$$

$$= \frac{2}{\log(\theta)} \log(F(t)) \int_{x_0}^{t^*} \mu(dx) - \frac{2}{\log(\theta)} \int_{x_0}^{t^*} \log(F(x)) dF(x)$$
$$= \frac{2}{\log(\theta)} \log(F(t))[F(t^*) - F(x_0)] - \frac{2}{\log(\theta)} \int_{x_0}^{t^*} \log(F(x)) dF(x)$$

Consider the last integral separately. By Lemma 5.5 and inequality (5.12), we have

$$\begin{aligned}\int_{x_0}^{t^*} \log(F(x)) dF(x) &= \int_{F(x_0)}^{F(t^*)} \log(F(F^{-1}(y))) dy \geqslant \int_{F(x_0)}^{F(t^*)} \log(y) dy \\ &= F(t^*) \log(F(t^*)) - F(x_0) \log(F(x_0)) - \int_{F(x_0)}^{F(t^*)} 1 dy \\ &= F(t^*) \log(F(t^*)) - F(x_0) \log(F(x_0)) - F(t^*) + F(x_0).\end{aligned}$$

Using (5.14), we continue

$$\begin{aligned}&\int_{x_0}^{t^*} \log(F(x)) dF(x) \\ &\qquad \geqslant F(t^*) \log(F(t)/\lambda) - F(x_0) \log(F(x_0)) - \gamma F(t) + F(x_0) \\ &\qquad = F(t^*) \log(F(t)) - F(t^*) \log(\lambda) - F(x_0) \log(F(x_0)) \\ &\qquad\quad - \gamma F(t) + F(x_0) \\ &\qquad \geqslant F(t^*) \log(F(t)) - \gamma F(t) \log(\lambda) - F(x_0) \log(F(x_0)) \\ &\qquad\quad - \gamma F(t) + F(x_0).\end{aligned}$$

Putting everything together, we obtain the following bound. We use the fact that $\log(\theta) < 0$ and (5.14).

$$\begin{aligned}\frac{1}{\varepsilon} I_1 &\geqslant \frac{2}{\log(\theta)} \big(\gamma F(t) \log(\lambda) + F(x_0) \log(F(x_0)) \\ &\qquad\qquad + \gamma F(t) - F(x_0) - F(x_0) \log(F(t))\big) \\ &\geqslant \frac{2}{\log(\theta)} \big(\gamma F(t) \log(\lambda) + F(x_0) \log(F(x_0)) \\ &\qquad\qquad + \gamma F(t) - F(x_0) - F(x_0) \log(\lambda F(x_0))\big) \\ &\geqslant \frac{2}{\log(\theta)} \big(\gamma F(t) \log(\lambda) + \gamma F(t) - F(x_0) - F(x_0) \log(\lambda)\big) \\ &\geqslant -F(t) \Big(- \frac{2\gamma}{\log(\theta)} \log(\lambda) - \frac{2\gamma}{\log(\theta)} \Big).\end{aligned}$$

Combining with (5.15) one obtains (5.13). □

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