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PRIMITIVE DECOMPOSITION OF ELEMENTS OF THE FREE METABELIAN LIE ALGEBRA OF RANK TWO

Ela Aydın

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Abstract. We give a primitive decomposition of the elements of the free metabelian Lie algebra of rank 2 over a field of characteristic 0 and determine the primitive length of the elements.

1. Introduction. Let L be the free metabelian Lie algebra freely generated by a set $\{x, y\}$ over a field K of characteristic zero. We denote by L' the commutator subalgebra $[L, L]$ of L. An element u in L is primitive if it can be extended to a free generating set of L. There are several studies about primitive elements of free metabelian Lie algebras. In [2] Chirkov and Shevelin found a necessary condition for an element z of L to belong to the ideal $\langle y\rangle_{\mathrm{id}}$ generated by an element y of L. They proved that if z is primitive and $z \in \langle y\rangle_{\mathrm{id}}$ then the elements z and y are conjugate by means of an inner automorphism of L. In 2003 Drensky and Yu [4] investigated the primitivity of an element of the free

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metabelian Lie algebra under certain conditions. We refer to [5] and [6] for a background on free metabelian Lie algebras, their automorphisms and primitive elements.

Our purpose in this paper is to give a presentation of the elements of L as a sum of primitive elements. The result is in the spirit of a result by Bardakov, Shpilrain, and Tolstykh [1] who determined the primitive width of free groups.

2. Preliminaries. Let L be the free metabelian Lie algebra on the free generators x and y. It is known that the commutator ideal L' of L has a basis consisting of all elements

$$[y,x]\mathrm{ad}^p x\mathrm{ad}^q y = [[\ldots[[y,x],\underbrace{x],\ldots],x}_{p \text{ times}}],\underbrace{y],\ldots],y}_{q \text{ times}}], \quad p,q \geq 0,$$

and is generated by the commutator $[y,x] = y\mathrm{ad}x$. In what follows we fix a field K of characteristic zero. We define an action of the polynomial algebra $K[t,u]$ on L' by

$$t^r u^s * [y,x]\mathrm{ad}^p x\mathrm{ad}^q y = [y,x]\mathrm{ad}^{p+r} x\mathrm{ad}^{q+s} y$$

and in this way give to L' the structure of a free $K[t,u]$-module generated by $[y,x]$. Hence all elements of the commutator ideal L' are of the form

$$w = a(t,u)[y,x], \text{ where } a(t,u) \in K[t,u].$$

By the theorem of Cohn [3], the automorphisms of the free Lie algebra of any rank are tame. In particular, for the free Lie algebra with two generators all automorphisms are linear. The automorphisms of the metabelian Lie algebra L are described by Shmelkin [7]. Recall that the inner automorphism $e^{\mathrm{ad}v}, v \in L'$, acts by

$$e^{\mathrm{ad}v}(w) = w + [w,v], \quad w \in L.$$

Proposition 1 ([7]). *Every automorphism of L is a composition of a linear automorphism and an inner automorphism. In particular, if $v = c(t,u)[y,x]$, then we have the equalities*

$$\begin{aligned} e^{\mathrm{ad}v}(w) &= e^{\mathrm{ad}c(t,u)[y,x]}(w) \\ &= w + c(t,u)[[y,x],w], \end{aligned}$$

$$\begin{aligned} e^{\mathrm{ad}v}(x) &= e^{\mathrm{ad}c(t,u)[y,x]}(x) \\ &= x + c(t,u)[[y,x],x] = x + tc(t,u)[y,x], \end{aligned}$$

and

$$\begin{aligned} e^{\mathrm{ad}v}(y) &= e^{\mathrm{ad}c(t,u)[y,x]}(y) \\ &= x + c(t,u)[[y,x],y] = x + uc(t,u)[y,x]. \end{aligned}$$

Corollary 2. (i) *The primitive elements of L are of the form*

$$f = \alpha x + \beta y + (\alpha t + \beta u)a(t,u)[y,x], \quad (\alpha,\beta) \neq (0,0), a(t,u) \in K[t,u].$$

(ii) *If* $b(0,0) \neq 0$, *then the element*

$$f = \alpha x + \beta y + b(t,u)[y,x] \in L, \quad b(t,u) \in K[t,u],$$

without any restrictions on $\alpha, \beta \in K$, is not a sum of primitive elements.

Proof. (i) By Proposition 1 every automorphism φ of L is of the form $\theta \circ e^{\mathrm{ad} v}$, where $v = c(t,u)[y,x]$, $c(t,u) \in K[t,u]$, and the linear automorphism θ acts on x and y by

$$\theta(x) = \alpha x + \beta y, \quad \theta(y) = \gamma x + \delta y, \quad \alpha,\beta,\gamma,\delta \in K, \alpha\delta - \beta\gamma \neq 0.$$

Hence

$$\begin{aligned}
\varphi(x) &= \theta(e^{\mathrm{ad} v}(x)) = \theta(x + tc(t,u)[y,x]) \\
&= (\alpha x + \beta y) + (\alpha t + \beta u)c(\alpha t + \beta u, \gamma t + \delta u)[\gamma x + \delta y, \alpha x + \beta y] \\
&= (\alpha x + \beta y) + (\alpha t + \beta u)c(\alpha t + \beta u, \gamma t + \delta u)(\alpha\delta - \beta\gamma)[y,x] \\
&= (\alpha x + \beta y) + b(t,u)[y,x]
\end{aligned}$$

and $b(t,u) = (\alpha t + \beta u)c(\alpha t + \beta u, \gamma t + \delta u)(\alpha\delta - \beta\gamma)$ is divisible by $\alpha t + \beta u$.

(ii) Since for any primitive element $f = \alpha x + \beta y + b(t,u)[y,x]$ of L the polynomial $b(t,u)$ has a linear factor and $b(0,0) = 0$, we obtain that the condition $b(0,0) = 0$ holds also for all sums of primitive elements. □

Definition 3. *The primitive length $|g|_{pr}$ of a nonzero element $g \in L$ is the smallest number m such that g can be presented as a sum of m different primitive elements. We define $|g|_{pr} = \infty$ if g cannot be presented as a sum of primitive elements.*

3. Decomposition of elements of L as a sum of primitive elements. The following theorem is the main result of our paper.

Theorem 4. *Let f be a nonzero element of L,*

$$f = \alpha x + \beta y + b(t,u)[y,x], \quad \alpha,\beta \in K, b(t,u) \in K[t,u].$$

Then

$$|f|_{\mathrm{pr}} = \begin{cases} \infty, & \text{if } b(0,0) \neq 0, \\ 1, & \text{if } (\alpha,\beta) \neq (0,0) \text{ and } \alpha t + \beta u \text{ divides } b(t,u), \\ 2, & \text{if } (\alpha,\beta) \neq (0,0) \text{ and } \alpha t + \beta u \text{ does not divide } b(t,u), \\ 2, & \text{if } \alpha = \beta = 0, b(0,0) = 0 \text{ and } b(t,u) \text{ has a linear factor}, \\ 3, & \text{if } \alpha = \beta = 0, b(0,0) = 0 \text{ and } b(t,u) \text{ does not have a linear factor}. \end{cases}$$

The equalities $|f|_{\mathrm{pr}} = \infty$ when $b(0,0) \neq 0$ and $|f|_{\mathrm{pr}} = 1$ when $(\alpha, \beta) \neq (0,0)$ and $\alpha t + \beta u$ divides $b(t,u)$ follow from Corollary 2. The proofs of the other three cases are given in a series of lemmas.

Lemma 5. *Let $f = \alpha x + \beta y + b(t,u)[y,x] \in L$, $(\alpha,\beta) \neq (0,0)$, $b(0,0) = 0$, and let $\alpha t + \beta u$ do not divide $b(t,u)$. Then $|f|_{\mathrm{pr}} = 2$.*

P r o o f. By Corollary 2 (i) f is not primitive, i.e., $|f|_{\mathrm{pr}} \geq 2$. Changing linearly the generators x and y of L we may assume that $f = x + b(t,u)[y,x]$ and that t does not divide $b(t,u)$. We divide $b(t,u)$ by $t - u$ and present it in the form

$$\begin{aligned} b(t,u) &= (t-u)^k a_0(u) + (t-u)^{k-1} a_1(u) + \cdots + (t-u) a_{k-1}(u) + a_k(u) \\ &= (t-u) a'(t,u) + a_k(u). \end{aligned}$$

Since $b(0,0) = a_k(0) = 0$ we obtain that $a_k(u) = u a''(u)$ and

$$b(t,u) = (t-u) a'(t,u) + u a''(u).$$

Then

$$f = ((x-y) + (t-u) a'(t,u)[y,x]) + (y + u a''(u)[y,x])$$

and by Corollary 2 (i) both $(x-y) + (t-u)a'(t,u)[y,x]$ and $y + ua''(u)[y,x]$ are primitive elements. Hence $|f|_{\mathrm{pr}} = 2$. □

Lemma 6. *Let $f = b(t,u)[y,x] \in L'$, and let $b(t,u)$ have a linear factor. Then $|f|_{\mathrm{pr}} = 2$.*

P r o o f. Let $\alpha t + \beta u$ divide $b(t,u)$, i.e., $b(t,u) = (\alpha t + \beta u) b_1(t,u)$. Obviously $|f|_{\mathrm{pr}} \geq 2$ because f is not primitive. We can present f as a sum of two primitive elements

$$f = ((\alpha x + \beta y) + (\alpha t + \beta u)(b_1(t,u) + 1))[y,x] + (-(\alpha x + \beta y) - (\alpha t + \beta u)[y,x])$$

and hence $|f|_{\mathrm{pr}} = 2$. □

Lemma 7. *Let $f = b(t,u)[y,x] \in L'$, and let $b(t,u)$ do not have a linear factor. Then $|f|_{\mathrm{pr}} = 3$.*

P r o o f. If $|f|_{\mathrm{pr}} = 2$, then f is a sum of two primitive elements:

$$\begin{aligned} f &= ((\alpha x + \beta y) + (\alpha t + \beta u) a_1(t,u)[y,x]) + ((\gamma x + \delta y) + (\gamma t + \delta u) a_2(t,u)[y,x]) \\ &= (\alpha + \gamma) x + (\beta + \delta) y + ((\alpha t + \beta u) a_1(t,u) + (\gamma t + \delta u) a_2(t,u))[y,x] \\ &= b(t,u)[y,x]. \end{aligned}$$

Hence $\gamma = -\alpha$, $\delta = -\beta$,

$$f = (\alpha t + \beta u)(a_1(t,u) - a_2(t,u))[y,x],$$

and $b(t,u)$ is divisible by $\alpha t + \beta u$ which is a contradiction. In this way $|f|_{\mathrm{pr}} \geq 3$. As in the proof of Lemma 5 we present $b(t,u)$ in the form

$$b(t,u) = (t-u)a'(t,u) + ua''(u) = (t-u)(a'(t,u)+1) - t + u(a''(u)+1).$$

Then f is a sum of three primitive elements:

$$f = ((x-y) + (t-u)(a'(t,u)+1)[y,x]) + (-x - t[y,x]) + (y + u(a''(u)+1)[y,x]),$$

and $|f|_{\mathrm{pr}} = 3$. $\square$

The above lemmas complete the proof of Theorem 4.

Problem 8. *Let L_n be the free metabelian Lie algebra of rank $n > 2$.*

(i) *Is it true that any nonzero element of L_n can be presented as a sum of primitive elements?*

(ii) *If (i) is true, is there an upper bound on the primitive length of the elements of L_n? If such a bound exists, does it depend on n?*

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Department of Mathematics
Çukurova University
01330 Balcalı, Adana, Turkey
e-mail: `eaydin@cu.edu.tr`

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