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SELECTION PRINCIPLES IN $(3,1,\rho)$-D-METRIZABLE SPACES AND $(3,2,\rho)$-D-METRIZABLE SPACES

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Abstract. In this paper we introduce and investigate M, R and H-boundedness properties in $(3,j,\rho)-D$-metrizable spaces, $j \in \{1,2\}$, related to the classical covering properties of Menger, Rothberger and Hurewicz.

0. Introduction. Selection Principles Theory is a field of mathematics having a rich history going back to papers of Borel, Menger, Hurewicz, Rothberger. A systematic investigation in this area rapidly increased and attracted a big number of mathematicians in the last two-three decades. This theory has deep connections with many branches of mathematics such as Set Theory, General Topology, Topological Game Theory, Uniform Structures, Function Spaces and Hyperspaces, Dimension Theory. In [14], Kočinac introduced selection principles in uniform spaces and proved that these selection principles are different from selection principles in topological spaces.

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Key words: $(3,1,\rho)$-D-metrizable spaces, $(3,2,\rho)$-D-metrizable spaces, M-bounded space, R-bounded space, H-bounded space, D-precompact space, σ-D-precompact space, D-*pre*-Lindelof space, D-bounded space.

The geometric properties, their axiomatic classification and generalizations of metric spaces have been considered in lot of papers. Some of them are: K. Menger [17], P. S. Alexandroff and V. V. Niemytskii [1], Z. Mamuzić [16], S. Gähler [10], S. Nedev, M. Choban [19, 20, 21], R. Kopperman [11], B.C. Dhage, Z. Mustafa, B. Sims [4, 18]. The notion of (n, m, ρ)-metric, $n > m$, generalizing the usual notion of pseudometric (the case $n = 2$, $m = 1$), and the notion of $(n + 1)$-metric (as in [17] and [10]) was introduced in [5]. Connections between some of the topologies induced by $(3, 1, \rho)$-metric and topologies induced by pseudo-o-metric, o-metric and symmetric (as in [20]), are given in [6]. Other characterizations of $(3, j, \rho)$-metrizable topological spaces, $j \in \{1, 2\}$, are given in [2, 3, 7, 8, 9].

We start the investigation of selection principles in $(3, j, \rho)$-D-metrizable spaces, $j \in \{1, 2\}$, in manner as it was done with selection principles theory for uniform spaces, topological spaces and fuzzy metric spaces (as in [12, 13, 14, 15]). Our goal is to investigate the similarities and differences between selection properties of these spaces and the selection properties of already known spaces.

1. Preliminaries. We give the basic definitions about $(3, j, \rho)$-D-metrizable spaces as in [2], and the classical Menger, Rothberger and Hurewicz covering properties as in [13].

Let $M \neq \emptyset$ and let $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x, y, z)\alpha(u, v, w) \Leftrightarrow (u, v, w) = \pi(x, y, z),$$

(π stands for permutation).

Let $d : M^{(3)} \to [0, +\infty)$. We consider the following conditions for such a map:

(**M0**) $d(x, x, x) = 0$, for any $x \in M$;

(**M1**) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$, for any $x, y, z, a \in M$;

(**M2**) $d(x, y, z) \leq d(x, a, b) + d(y, a, b) + d(z, a, b)$, for any $x, y, z, a, b \in M$;

(**Ms**) $d(x, x, y) = d(x, y, y)$, for any $x, y \in M$.

Let ρ be a subset of $M^{(3)}$. We consider the following conditions for such a set:

(**E0**) $(x, x, x) \in \rho$, for any $x \in M$;

(**E1**) (x,y,a), (x,a,z), $(a,y,z) \in \rho \Rightarrow (x,y,z) \in \rho$, for any $x,y,z,a \in M$;

(**E2**) (x,a,b), (y,a,b), $(z,a,b) \in \rho \Rightarrow (x,y,z) \in \rho$, for any $x,y,z,a,b \in M$.

Definition 1.1. *i) A subset ρ of $M^{(3)}$ satisfying $(E0)$ and $(E1)$, is called a $(3,1)$-equivalence.*

ii) A subset ρ of $M^{(3)}$ satisfying $(E0)$ and $(E2)$ is called a $(3,2)$-equivalence.

iii) A subset ρ of $M^{(3)}$ satisfying $(E0)$, $(E1)$ and $(E2)$ is called a 3-equivalence on M.

The following statement follows directly from the definition.

Theorem 1.1. *Let $\rho_d = \{(x,y,z) | (x,y,z) \in M^{(3)}, d(x,y,z) = 0\}$. If d satisfies $(M0)$ and $(M1)$, then $\rho = \rho_d$ is a $(3,1)$-equivalence on M. If d satisfies $(M0)$ and $(M2)$, then $\rho = \rho_d$ is a $(3,2)$-equivalence on M.*

Definition 1.2. *i) If d satisfies $(M0)$ and (Mj) we say that d is a $(3,j,\rho)$-metric on M.*

ii) If d satisfies $(M0)$, (Mj) and (Ms) we say that d is a $(3,j,\rho)$-symmetric on M.

iii) If $\rho = \Delta = \{(x,x,x) | x \in M\}$, then we write $(3,j)$ instead of $(3,j,\Delta)$.

Let $x \in M$ and $\varepsilon > 0$. We define an ε-ball with center x and radius ε by:

$$B(x,x,\varepsilon) = \{y | y \in M, d(x,x,y) < \varepsilon\}.$$

For any $A \subseteq M$, the set $B(A,\varepsilon) = \bigcup\{B(a,a,\varepsilon) | a \in A\}$ is called ε-neighborhood of A. We denote by $\tau(D,d)$ the topology on M generated by all ε-balls $B(x,x,\varepsilon)$.

Definition 1.3. *We say that a topological space (M,τ) is $(3,j,\rho)$-D-metrizable if there is $(3,j,\rho)$-metric d such that $\tau = \tau(D,d)$.*

Remark 1.1. Every ε-neighborhood of A is an open set.

Definition 1.4. *We say that a topological space (M,τ) has the:*

i) Menger;

ii) Rothberger;

iii) Hurewicz
covering property if for each sequence of open covers $\{\mathcal{U}_n\}_{n=1}^{\infty}$ *on* M *there is a sequence*

i) $\{\mathcal{V}_n\}_{n=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$, $\mathcal{V}_n$ *is a finite subset of* $\mathcal{U}_n$ *and* $M = \bigcup_{n=1}^{\infty} \bigcup \mathcal{V}_n$;

ii) $\{U_n\}_{n=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ *and* $M = \bigcup_{n=1}^{\infty} U_n$;

iii) $\{\mathcal{W}_n\}_{n=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$, $\mathcal{W}_n$ *is finite subset of* $\mathcal{U}_n$ *and each* $x \in M$ *belongs to* $\bigcup \mathcal{W}_n$ *for all but finitely many* $n \in \mathbb{N}$.

Remark 1.2. For a family $\mathcal{U}$ of subsets of M we denote by $\bigcup \mathcal{U}$ the set $\bigcup \{U | U \in \mathcal{U}\}$.

Next we introduce and investigate some boundedness properties in $(3, j, \rho)$-D-metrizable spaces, $j \in \{1, 2\}$, related to the classical covering properties of Menger, Rothberger and Hurewicz.

From now on (M, τ) will be a $(3, j, \rho)$-metrizable space, $j \in \{1, 2\}$, via $(3, j, \rho)$-metric d.

Definition 1.5. *We say that the space* (M, τ) *is*
i) M-bounded;
ii) R-bounded;
iii) H-bounded
if for each sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ *of real positive numbers there is a sequence*

i) $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* M *such that* $M = \bigcup_{n=1}^{\infty} \bigcup_{a \in A_n} B(a, a, \varepsilon_n)$

ii) $\{x_n\}_{n=1}^{\infty}$ *of elements of* M *such that* $M = \bigcup_{n=1}^{\infty} B(x_n, x_n, \varepsilon_n)$;

iii) $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* M *such that for each* $x \in M$ *there exists* $n_0 \in \mathbb{N}$ *such that* $x \in \bigcup_{a \in A_n} B(a, a, \varepsilon_n)$ *for every* $n \geq n_0$.

We give generalizations for the usual notions of compact, σ-compact and Lindelöf topological spaces concerning $(3,j,\rho)$-metrizable spaces, $j \in \{1,2\}$.

Definition 1.6. *i) A subset $X \subseteq M$ is called D-precompact if for every $\varepsilon > 0$ there exists a finite subset $A \subseteq X$ such that $X = \bigcup_{a\in A} B(a,a,\varepsilon)$.*

ii) The space (M,τ) is called σ-D-precompact if it can be represented as a countable union of D-precompact subsets.

iii) The space (M,τ) is called D-pre-Lindelöf if for every $\varepsilon > 0$ there exists countable subset $A \subseteq M$ such that $M = \bigcup_{a\in A} B(a,a,\varepsilon)$.

iv) The space (M,τ) is called D-bounded if for every $n \in \mathbb{N}$ there exist $x_1, x_2, \ldots, x_n \in M$ such that $M = \bigcup_{i=1}^{n} B(x_i, x_i, 1/n)$.

2. Selection principles.

Proposition 2.1. *If (M,τ) has the Menger covering property, then (M,τ) is M-bounded.*

Proof. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then $\mathcal{U}_n = \{B(x,x,\varepsilon_n) | x \in M\}$ is a sequence of open coverings of M. Menger property implies existence of sequence $\{A_n^0\}_{n=1}^{\infty}$ of finite sets such that $A_n^0 \subseteq \mathcal{U}_n$ and $M = \bigcup_{n=1}^{\infty} \bigcup A_n^0$. Hence, $M = \bigcup_{n=1}^{\infty} \bigcup_{a\in A_n} B(a,a,\varepsilon_n)$, where

$$A_n = \{a \in M | B(a,a,\varepsilon_n) \subseteq A_n^0\}. \qquad \square$$

Proposition 2.2. *If (M,τ) has the Rothberger covering property, then (M,τ) is R-bounded.*

Proof. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then $\mathcal{U}_n = \{B(x,x,\varepsilon_n) | x \in M\}$ is a sequence of open coverings of M. Rothberger property implies existence of a sequence $\{U_n\}_{n=1}^{\infty}$ such that $U_n \in \mathcal{U}_n$ and $M = \bigcup_{n=1}^{\infty} U_n$. Hence, there exists a sequence $\{x_n\}_{n=1}^{\infty} \in M$ such that $M = \bigcup_{n=1}^{\infty} B(x_n,x_n,\varepsilon_n)$. $\square$

Proposition 2.3. *If (M,τ) has the Hurewicz covering property, then (M,τ) is H-bounded.*

Proof. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then $\mathcal{U}_n = \{B(x,x,\varepsilon_n)|x \in M\}$ is a sequence of open coverings of M. Hurewicz property implies existence of sequence $\{A_n^0\}_{n=1}^{\infty}$ of finite sets satisfying $A_n^0 \subseteq \mathcal{U}_n$ and $x \in M$ belongs to $\bigcup A_n^0$ for all but finitely many $n \in \mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that $x \in \bigcup A_n^0$ for $n \geq n_0$, i.e., $x \in \bigcup\limits_{a \in A_n} B(a,a,\varepsilon_n)$ for $n \geq n_0$, where $A_n = \{a \in M|B(a,a,\varepsilon_n) \in A_n^0\}$. $\square$

Corollary 2.1. *If (M,τ) is σ-compact, then (M,τ) is M-bounded.*

Proof. We will prove that (M,τ) has the Menger covering property. Let $\{\mathcal{U}_n\}_{n=1}^{\infty}$ be a sequence of open coverings of M. There exists a sequence $\{O_n\}_{n=1}^{\infty}$ of compact subsets of M satisfying $M = \bigcup\limits_{n=1}^{\infty} O_n$. Since O_n is compact, there is a finite set $\mathcal{V}_n$ such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $O_n \subseteq \bigcup \mathcal{V}_n$. We obtain, $M = \bigcup\limits_{n=1}^{\infty} \bigcup \mathcal{V}_n$, i.e., (M,τ) has the Menger covering property. From the Proposition 2.1. it follows that (M,τ) is M-bounded. $\square$

Corollary 2.2. *(M,τ) is compact $\Rightarrow$ (M,τ) is σ-compact $\Rightarrow$ (M,τ) has Menger covering property $\Rightarrow$ (M,τ) is M-bounded.*

Proposition 2.4. *If (M,τ) is D-precompact, then (M,τ) is H-bounded.*

Proof. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M satisfying $M = \bigcup\limits_{a \in A_n} B(a,a,\varepsilon_n)$. Hence, for every $x \in M$ there exists $n_0 \in \mathbb{N}$ such that $x \in \bigcup\limits_{a \in A_n} B(a,a,\varepsilon_n)$ for every $n \geq n_0$. $\square$

Proposition 2.5. *If (M,τ) is H-bounded, then (M,τ) is M-bounded.*

Proof. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M satisfying: for every $x \in M$ there exists $n_0 \in \mathbb{N}$ such that $x \in \bigcup\limits_{a \in A_n} B(a,a,\varepsilon_n)$ for every $n \geq n_0$. Obviously

$x \in \bigcup_{n=1}^{\infty} \bigcup_{a \in A_n} B(a,a,\varepsilon_n)$, i.e., $M = \bigcup_{n=1}^{\infty} \bigcup_{a \in A_n} B(a,a,\varepsilon_n)$. $\square$

Proposition 2.6. *If (M,τ) is M-bounded, then (M,τ) is D-pre-Lindelöf.*

Proof. Let $\varepsilon > 0$ and $\varepsilon_n = \varepsilon$ for every $n \in \mathbb{N}$. There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M satisfying $M = \bigcup_{a \in A_n} B(a,a,\varepsilon_n) = \bigcup_{a \in A} B(a,a,\varepsilon)$, where $A = \bigcup_{n=1}^{\infty} A_n$. Since the set A is countable, one obtains the claim. $\square$

Proposition 2.7. *If (M,τ) is R-bounded, then (M,τ) is M-bounded.*

Proof. For a sequence of positive real numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \in M$ such that $M = \bigcup_{n=1}^{\infty} B(x_n,x_n,\varepsilon_n)$. Choosing $A_n = \{x_n\}$ for every $n \in \mathbb{N}$ one obtains $M = \bigcup_{n=1}^{\infty} \bigcup_{a \in A_n} B(a,a,\varepsilon_n)$. $\square$

Proposition 2.8. *A space (M,τ) is D-bounded iff it is D-precompact.*

Proof. For arbitrary $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since (M,τ) is D-bounded there are $x_1, x_2, \ldots, x_n \in M$ such that $M = \bigcup_{i=1}^{n} B(x_i,x_i,1/n)$. Then for $A = \{x_1, x_2, \ldots, x_n\}$ one obtains $M = \bigcup_{a \in A} B(a,a,1/n) = \bigcup_{a \in A} B(a,a,\varepsilon)$. For the other implication let $n \in \mathbb{N}$ and $\varepsilon = 1/n$. There exists a finite set $A = \{x_1, x_2, \ldots, x_n\}$ satisfying $M = \bigcup_{a \in A} B(a,a,\varepsilon) = \bigcup_{i=1}^{n} B(x_i,x_i,1/n)$. $\square$

In the next theorem we prove the existence of M, R and H-bounded spaces.

Theorem 2.1. *Let (M,D) be a metric space that has Menger (Rothberger) [Hurewitz] covering property and let d_D be the 3-symmetric on M defined by:*

$$d_D(x,y,z) = \frac{D(x,y) + D(x,z) + D(y,z)}{2}.$$

Then, the 3-D-symmetrizable space (M,τ), via the 3-symmetric d_D is M-bounded (R-bounded) [H-bounded].

P r o o f. Obviously $B(x,x,\varepsilon) = T(x,\varepsilon)$, where $T(x,\varepsilon) = \{y|D(x,y) < \varepsilon\}$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers and $\mathcal{U}_n = \{T(x,\varepsilon_n)|x \in M\}$. If (M,D) has the Menger covering property, then there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subset of M such that for every $n \in \mathbb{N}$, $A_n \subseteq \mathcal{U}_n$ and $M = \bigcup\limits_{n=1}^{\infty}\bigcup\limits_{a\in A_n} T(a,\varepsilon_n) = \bigcup\limits_{n=1}^{\infty}\bigcup\limits_{a\in A_n} B(a,a,\varepsilon_n)$. Hence, (M,τ) is M-bounded. If (M,D) has the Rothberger covering property, then there exists a sequence $\{U_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $M = \bigcup\limits_{n=1}^{\infty} U_n$. Since $U_n \in \mathcal{U}_n$ there exists $x_n \in M$ such that $U_n = T(x_n,\varepsilon_n)$, hence there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in M such that $M = \bigcup\limits_{n=1}^{\infty} T(x_n,\varepsilon_n) = \bigcup\limits_{n=1}^{\infty} B(x_n,x_n,\varepsilon_n)$. The later means that, (M,τ) is R-bounded. If (M,D) has Hurewitz covering property, then there exists a sequence of finite sets $\{\mathcal{W}_n\}_{n=1}^{\infty}$ such that $\mathcal{W}_n \subseteq \mathcal{U}_n$ and $x \in M$ is an element in $\bigcup \mathcal{W}_n$ for all but finitely $n \in \mathbb{N}$. Choosing $A_n = \{a|T(a,\varepsilon_n) \in \mathcal{W}_n\}$ one obtains $x \in \bigcup\limits_{a\in A_n} T(a,\varepsilon_n) = \bigcup\limits_{a\in A_n} B(a,a,\varepsilon_n)$ for all but finitely $n \in \mathbb{N}$ which implies the desired claim. $\square$

Proposition 2.9. *If (M,τ) has the Hurewicz covering property, then (M,τ) is σ-D-precompact.*

P r o o f. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that $0 < \varepsilon_{n+1} < \varepsilon_n$ for $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ when $n \to \infty$. For such a sequence we define $\mathcal{U}_n = \{B(x,x,\varepsilon_n)|x \in M\}$. For every $n \in \mathbb{N}$ there exists a finite set $V_n \subseteq \mathcal{U}_n$ satisfying $x \in M$ is an element of $\bigcup V_n$ for all but finitely many $n \in \mathbb{N}$. So there exists $k \in \mathbb{N}$, such that $x \in \bigcup V_n$ for $n \geq k$. Let $M_n = \bigcap\limits_{m\geq n}\bigcup V_m$ for every $n \in \mathbb{N}$. Obviously $M_{n_1} \subseteq M_{n_2}$ for $n_1 \leq n_2$. Because $x \in \bigcup V_n$ for every $n \geq k$, it follows that $x \in M_n$ for every $n \geq k$, hence $M \subseteq \bigcup\limits_{n=1}^{\infty} M_n$. We will show that M_n is D-precompact for every $n \in \mathbb{N}$. Let $\varepsilon > 0$. Choose $m > n$ such that $\varepsilon_m < \varepsilon$ and let $V'_m = \{B(a,a,\varepsilon)|B(a,a,\varepsilon_m) \in V_m\}$. Then every element in V'_m is an open set and V'_m is a finite covering of M_n. $\square$

Proposition 2.10. *For a space (M,τ) the following conditions are equivalent:*

1) *For every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M such that every finite $E \subseteq M$ is contained in some $B(A_n, \varepsilon_n)$;*

2) *For every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M and monotonically increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that every finite $E \subseteq M$ is contained in some $\bigcup\{B(A_i, \varepsilon_i) | n_k \leq i < n_{k+1}\}$.*

Proof. The implication 1) $\Rightarrow$ 2) is trivial. 2) $\Rightarrow$ 1). Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. For $n \in \mathbb{N}$ let $\delta_n = min\{\varepsilon_i | i \leq n\}$. There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M and monotonically increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that every finite $E \subseteq M$ is contained in $\bigcup\{B(A_i, \delta_i) | n_k \leq i < n_{k+1}\}$ for some $k \in \mathbb{N}$. We define finite subset of M as follows:

$$A'_n = \begin{cases} \bigcup\limits_{i<n_1} A_i, & n < n_1 \\ \bigcup\limits_{n_k \leq i < n_{k+1}} A_i, & n_k \leq i < n_{k+1} \end{cases}.$$

Let F be a finite subset of M. There exists $k \in \mathbb{N}$ such that $F \subseteq \bigcup\{B(A_i, \delta_i) | n_k \leq i < n_{k+1}\}$. For every $n \in \mathbb{N}$, $n_k \leq n < n_{k+1}$, we obtain $A'_n = \bigcup\limits_{n_k \leq i < n_{k+1}} A_i$. For every $x \in F$ there exists $j \in \mathbb{N}$, $n_k \leq j < n_{k+1}$ and $y \in A_j$ such that $x \in B(y, y, \delta_j)$. From $B(y, y, \delta_j) \subseteq B(y, y, \varepsilon_j)$ and $y \in A'_n$, one easily obtains $x \in B(A'_j, \varepsilon_j)$, hence $F \subseteq B(A'_j, \varepsilon_j)$. $\square$

Proposition 2.11. *For a space (M, τ) the following conditions are equivalent:*

1) *For every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M such that every finite $E \subseteq M$ is contained in $B(A_n, \varepsilon_n)$ for all but finitely many $n \in \mathbb{N}$;*

2) *For every sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M and monotonically increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that every finite $E \subseteq M$ is contained in $\bigcup\{B(A_i, \varepsilon_i) | n_k \leq i < n_{k+1}\}$ for all but finitely many $n \in \mathbb{N}$.*

P r o o f. The implication 1) $\Rightarrow$ 2) is trivial. We will prove 2) $\Rightarrow$ 1). Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. For $n \in \mathbb{N}$ let $\delta_n = \min\{\varepsilon_i | i \leq n\}$. There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of M and monotonically increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that every finite $E \subseteq M$ is contained in $\bigcup\{B(A_i, \delta_i) | n_k \leq i < n_{k+1}\}$ for all but finitely many $k \in \mathbb{N}$. We define finite subset of M as follows:

$$A'_n = \begin{cases} \bigcup\limits_{i<n_1} A_i, & n < n_1 \\ \bigcup\limits_{n_k \leq i < n_{k+1}} A_i, & n_k \leq i < n_{k+1} \end{cases}.$$

Let F be a finite subset of M. Then $F \subseteq \bigcup\{B(A_i, \delta_i) | n_k \leq i < n_{k+1}\}$ starting from some $k \in \mathbb{N}$. Let $s_n = \min\{n_m | n_m > n\}$. Then, we obtain $\delta_{s_n} \leq \delta_n \leq \varepsilon_n$. The sequence $\{s_n\}_{n=1}^{\infty}$ is monotonically increasing and subsequence of $\{n_k\}_{k=1}^{\infty}$. There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $s_n \geq n_k$. Also, there exists $p \geq k$ such that $s_n = n_p$. Then $n_p \leq s_n < n_{p+1}$ and $\varepsilon_n \geq \delta_{s_n} = \delta_{n_p} \geq \delta_i \geq \delta_{n_{p+1}}$. Hence, $B(A'_{s_n}, \varepsilon_n) = \bigcup\limits_{a \in A'_{s_n}} B(a, a, \varepsilon_n) = \bigcup\limits_{n_p \leq i < n_{p+1}} \bigcup\limits_{a \in A_i} B(a, a, \varepsilon_n) = \bigcup\limits_{n_p \leq i < n_{p+1}} B(A_i, \varepsilon_n) \supseteq \bigcup\limits_{n_p \leq i < n_{p+1}} B(A_i, \delta_i) \supseteq F$. This proves the desired claim. $\square$

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