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**HOMOLOGICAL DIMENSION AND DIMENSIONAL  
FULL-VALUEDNESS\***

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*Communicated by V. Gutev*

**ABSTRACT.** There are different definitions of homological dimension of metric compacta involving either Čech homology or exact (Steenrod) homology. In this paper we investigate the relation between these homological dimensions with respect to different groups. It is shown that all homological dimensions of a metric compactum  $X$  with respect to any field coincide provided  $X$  is homologically locally connected with respect to the singular homology up to dimension  $n = \dim X$  (br.,  $X$  is  $lc^n$ ). We also prove that any two-dimensional  $lc^2$  metric compactum  $X$  satisfies the equality  $\dim(X \times Y) = \dim X + \dim Y$  for any metric compactum  $Y$  (i.e.,  $X$  is dimensionally full-valued). This improves the well known result of Kodama [6, Theorem 8] that every two-dimensional  $ANR$  is dimensionally full-valued. Actually, the condition  $X$  to be  $lc^2$  can be weakened to the existence at every point  $x \in X$  of a neighborhood  $V$  of  $x$  such that the inclusion homomorphism  $H_k(\overline{V}; \mathbb{S}^1) \rightarrow H_k(X; \mathbb{S}^1)$  is trivial for all  $k = 1, 2$ .

**1. Introduction.** Everywhere below by a space we mean a metric compactum and by a group an abelian group.

The homological dimension  $d_G X$  of a space  $X$  with respect to a given group  $G$  was introduced by Alexandroff [1] in terms of Vietoris homology:  $d_G X$

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of a space  $X$  is the largest integer  $n$  such that there exists a closed set  $\Phi \subset X$  carrying an  $(n - 1)$ -dimensional cycle on  $\Phi$  which is not-homologous to zero in  $\Phi$ , but is homologous to zero in  $X$ . According to Lefschetz [8, Theorem 26.1], the Vietoris homology is isomorphic to Čech homology  $H_*(X; G)$ , where  $G$  is considered as a discrete group. So,  $d_G X$  can be defined as the largest integer  $n$  such that there exists a closed set  $\Phi \subset X$  and a non-trivial element  $\gamma \in H_{n-1}(\Phi; G)$  with  $i_{\Phi, X}^{n-1}(\gamma) = 0$ , where  $i_{\Phi, X}^{n-1} : H_{n-1}(\Phi; G) \rightarrow H_{n-1}(X; G)$  is the homomorphism generated by the inclusion  $\Phi \hookrightarrow X$ .

Because Čech homology is not exact, some authors prefer to define homological dimension involving exact homology groups. In particular, Sklyarenko and his students are using the following definition (see [11], [10], [3]): the homological dimension of  $X$ , denoted by  $h \dim_G X$ , is the largest integer  $n$  such that there exists a closed subset  $\Phi \subset X$  and a non-trivial element of  $\widehat{H}_n(X, \Phi; G)$ . Here  $\widehat{H}_*$  is the exact homology introduced by Sklyarenko in [11], which is isomorphic to the Steenrod homology [13] for metric compacta.

We always have  $d_G X \leq \dim X$ , and if  $0 < \dim X < \infty$ , then  $d_{\mathbb{Q}_1} X = d_{\mathbb{S}^1} X = \dim X$ , where  $\mathbb{S}^1$  is the circle group and  $\mathbb{Q}_1$  is the group of rational elements of  $\mathbb{S}^1$ , see [1].

In Section 2 we investigate the relations between  $d_G$  and  $h \dim_G$ . For example, we show that if  $\dim X = n$  and  $X$  is homologically locally connected with respect to the singular homology up to dimension  $n$  (br.,  $X$  is  $lc^n$ ), then  $d_G = h \dim_G X$  for any field  $G$  (see Corollary 2.3). We apply our results from Section 2 to establish in Section 3 that every two-dimensional compactum  $X$  is dimensionally full-valued if  $X$  is  $lc^2$  (Corollary 3.3). The last result improves a theorem of Kodama [6, Theorem 8].<sup>1</sup> Actually, the condition in Corollary 3.3  $X$  to be  $lc^2$  can be weakened (see Theorem 3.2) to the existence at every point  $x \in X$  of a neighborhood  $V$  of  $x$  such that the inclusion homomorphism  $H_k(\overline{V}; \mathbb{S}^1) \rightarrow H_k(X; \mathbb{S}^1)$  is trivial for all  $k = 1, 2$ .

**2. Some relations between  $d_G$  and  $h \dim_G$ .** It was noted that, in the class of metric compacta, the exact homology  $\widehat{H}_*$  is isomorphic to Steenrod homology, where  $G$  is any module over a commutative ring with unity. Moreover, for every module  $G$  and a compact pair  $(X, A)$  there exists a natural transformation  $T_{X,A} : \widehat{H}_*(X, A; G) \rightarrow H_*(X, A; G)$  between the exact and Čech homologies such that  $T_{X,A}^n : \widehat{H}_n(X, A; G) \rightarrow H_n(X, A; G)$  is a surjective homomorphism for each  $n$ , see [11] (if  $A$  is the empty set, we denote  $T_{X,\emptyset}^k$  by  $T_X^n$ ). By [11, The-

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<sup>1</sup>According to the referee, we don't know any example of a two-dimensional  $lc^2$ -compactum which is not an ANR

orem 4], this homomorphism is an isomorphism in the following situations: (i)  $\dim X = n$ : (ii)  $G$  admits a compact topology or  $G$  is a vector space over a field; (iii) both the Čech cohomology group  $H^n(X, A; \mathbb{Z})$  and  $G$  are finitely generated modules having finite numbers of relations.

Let  $hd_G X$  be the largest integer  $n$  such that there exists a closed set  $\Phi \subset X$  and a non-trivial  $\gamma \in \widehat{H}_{n-1}(\Phi; G)$  with  $\widehat{i}_{\Phi, X}^{n-1}(\gamma) = 0$ , where  $\widehat{i}_{\Phi, X}^{n-1} : \widehat{H}_{n-1}(\Phi; G) \rightarrow \widehat{H}_{n-1}(X; G)$  is the inclusion homomorphism. Clearly,  $hd_G X$  is the exact homology analogue of  $d_G X$ .

**Proposition 2.1.** *For any group  $G$  we have the following inequalities  $d_G X \leq h \dim_G X \leq \dim X$  and  $hd_G X \leq h \dim_G X$ .*

*Proof.* Suppose  $d_G X = n$ . Then  $n \geq 1$  and there is a closed set  $\Phi \subset X$  and a non-trivial  $\gamma \in H_{n-1}(\Phi; G)$  such that  $i_{\Phi, X}^{n-1}(\gamma) = 0$ . This implies  $H_n(X, \Phi; G) \neq 0$ , see [14, Proposition 4.6]. Consequently,  $\widehat{H}_n(X, \Phi; G) \neq 0$  (recall that  $T_{X, \Phi}^n : \widehat{H}_n(X, \Phi; G) \rightarrow H_n(X, \Phi; G)$  is surjective). Therefore,  $d_G X \leq h \dim_G X$ . The inequality  $h \dim_G X \leq \dim X$  follows from the fact that all groups  $\widehat{H}_k(X, \Phi; G)$  are trivial for  $k > \dim X$ .

For every closed  $\Phi \subset X$  and every  $n \geq 1$  we have the exact sequence

$$\rightarrow \widehat{H}_n(X, \Phi; G) \rightarrow \widehat{H}_{n-1}(\Phi; G) \rightarrow \widehat{H}_{n-1}(X; G) \rightarrow \cdots,$$

which yields the inequality  $hd_G X \leq h \dim_G X$ .  $\square$

Recall that a space  $X$  is *homologically locally connected in dimension  $n$*  (br.,  $n - lc$ ) if for every  $x \in X$  and a neighborhood  $U$  of  $x$  in  $X$  there exists a neighborhood  $V \subset U$  of  $x$  such that the homomorphism  $\widetilde{i}_{V, U}^n : \widetilde{H}_n(V; \mathbb{Z}) \rightarrow \widetilde{H}_n(U; \mathbb{Z})$  is trivial, where  $\widetilde{H}_*(\cdot; \cdot)$  denotes the singular homology groups. The above definition has two variations: (i) if the group  $\mathbb{Z}$  is replaced by a group  $G$ , we say that  $X$  is  *$n - lc$  with respect to  $G$* ; (ii) if every  $x \in X$  has a neighborhood  $V$  such that the homomorphism  $\widetilde{i}_{V, X}^n : \widetilde{H}_n(V; \mathbb{Z}) \rightarrow \widetilde{H}_n(X; \mathbb{Z})$  is trivial, we say that  $X$  is *semi- $n - lc$* . Using the Universal Coefficient Theorem for singular homology, one can show that  $X \in n - lc$  with respect to any group  $G$  provided  $X \in k - lc$  for every  $k \in \{n, n - 1\}$ . We say that  $X$  is *homologically locally connected up to dimension  $n$*  (br.,  $X \in lc^n$ ) provided  $X$  is  $k - lc$  for all  $k \leq n$ .

According to [9, Theorem 1] the following is true: If  $(X, A)$  is a pair of paracompact spaces with both  $X$  and  $A$  being  $lc^n$  and semi- $(n + 1) - lc$ , then there exists a natural transformation  $M_{X, A}$  between the singular and the Čech homologies of  $(X, A)$  such that for each group  $G$  the homomorphisms  $M_{X, A}^k : \widetilde{H}_k(X, A; G) \rightarrow H_k(X, A; G)$ ,  $k \leq n + 1$ , are isomorphisms. There exists also a similar connection between the singular homology and the exact homology, see

[11, Proposition 9]: Let  $(X, A)$  be a pair of locally compact metric spaces with both  $X$  and  $A$  being  $lc^n$ . Then there is a natural transformation  $S_{X,A}$  between the singular and the exact homologies with compact supports  $\widehat{H}_k^c(X, A; G)$  such that  $S_{X,A}^k : \widetilde{H}_k(X, A; G) \rightarrow \widehat{H}_k^c(X, A; G)$  is an isomorphism for each  $k \leq n - 1$  and it is surjective for  $k = n$ .

Therefore, combining the above results we obtain that if  $(X, A)$  is a compact metric pair such that both  $X$  and  $A$  are  $lc^n$ , then the homology groups  $\widetilde{H}_k(X, A; G)$ ,  $\widehat{H}_k(X, A; G)$  and  $H_k(X, A; G)$  are naturally isomorphic for each  $k \leq n - 1$  and each  $G$ .

**Proposition 2.2.** *If  $G$  is any group and  $X$  is  $lc^n$  with  $n = h \dim_G X$ , then  $d_G X \leq hd_G X = h \dim_G X$ .*

*Proof.* According to Proposition 2.1, all we need to show is the equality  $hd_G X = n$ . This is true if  $n = 1$  because always  $1 \leq hd_G X \leq h \dim_G X$ . So, let  $n \geq 2$ . By [10, Corollary 2], there is a point  $x \in X$  such that the module  $H_n^x = \varinjlim_{x \in U} \widehat{H}_n(X, X \setminus U; G)$  is non-trivial and  $n$  is the maximal integer

with this property. Therefore,  $\widehat{H}_n(X, X \setminus U; G) \neq 0$  for all sufficiently small neighborhoods  $U$  of  $x$ . Because  $X$  is  $lc^n$ , it is  $p - lc$  with respect to the group  $G$  for all  $p \in \{n - 1, n\}$ . Hence, there exists an open neighborhood  $V$  of  $x$  such that the inclusion homomorphism  $\widetilde{i}_{V,X}^p : \widetilde{H}_p(V; G) \rightarrow \widetilde{H}_p(X; G)$  is trivial for  $p = n$  and  $p = n - 1$ . According to the mentioned above natural transformation between the singular and the exact homology with compact supports, we have the following commutative diagrams

$$\begin{array}{ccc} \widetilde{H}_p(V; G) & \xrightarrow{\widetilde{i}_{V,X}^p} & \widetilde{H}_p(X; G) \\ \downarrow S_V^p & & \downarrow S_X^p \\ \widehat{H}_p^c(V; G) & \xrightarrow{\widehat{i}_{V,X}^p} & \widehat{H}_p(X; G) \end{array}$$

such that both homomorphisms  $S_V^p$  and  $S_X^p$  are isomorphisms for  $p = n - 1$  and surjective for  $p = n$ . This implies that the homomorphism  $\widehat{i}_{V,X}^p$  is trivial for all  $p \in \{n - 1, n\}$ . Consequently, if  $F \subset V$  is any compact set and  $p \in \{n - 1, n\}$ , then the homomorphism  $\widehat{i}_{F,X}^p : \widehat{H}_p(F; G) \rightarrow \widehat{H}_p(X; G)$  is also trivial.

Now we choose a neighborhood  $W$  of  $x$  with  $\overline{W} \subset V$  and  $\widehat{H}_n(X, X \setminus W; G) \neq 0$ . Then, by the excision axiom,  $\widehat{H}_n(X, X \setminus W; G)$  is isomorphic to

$\widehat{H}_n(\overline{W}, \text{bd } \overline{W}; G)$ . Consider the exact sequence

$$\rightarrow \widehat{H}_n(\overline{W}; G) \rightarrow \widehat{H}_n(\overline{W}, \text{bd } \overline{W}; G) \rightarrow \widehat{H}_{n-1}(\text{bd } \overline{W}; G) \rightarrow \cdots$$

We claim that  $L = \partial(\widehat{H}_n(\overline{W}, \text{bd } \overline{W}; G)) \neq 0$ , where  $\partial$  denotes the boundary homomorphism  $\partial : \widehat{H}_n(\overline{W}, \text{bd } \overline{W}; G) \rightarrow \widehat{H}_{n-1}(\text{bd } \overline{W}; G)$ . Indeed, otherwise the exactness of the above sequence yield  $\widehat{H}_n(\overline{W}; G) \neq 0$ . This means that  $hd_G X \geq n + 1$  (recall that, according to the choice of  $V$ , the homomorphism  $\widehat{i}_{\overline{W}, X}^n : \widehat{H}_n(\overline{W}; G) \rightarrow \widehat{H}_n(X; G)$  should be trivial), a contradiction.

Therefore,  $L$  is a non-trivial subgroup of  $\widehat{H}_{n-1}(\text{bd } \overline{W}; G)$ . Finally, the exactness of the above sequence implies  $\widehat{i}_{\text{bd } \overline{W}, \overline{W}}^{n-1}(L) = 0$ . Hence,  $\widehat{i}_{\text{bd } \overline{W}, X}^{n-1}(L)$  is also trivial, which yields  $hd_G X = n$ .  $\square$

**Corollary 2.3.** *Let  $X$  be  $lc^n$  with  $n = h \dim_G X$ . Then  $d_G X = hd_G X = h \dim_G X$  provided  $G$  has the following property: the homomorphisms  $T_{\text{bd } \overline{U}}^{n-1} : \widehat{H}_{n-1}(\text{bd } \overline{U}; G) \rightarrow H_{n-1}(\text{bd } \overline{U}; G)$  are isomorphisms for all open sets  $U \subset X$  (in particular, this is true for any field  $G$ ).*

**Proof.** We need to show the equality  $d_G X = hd_G X$ . It follows from the proof of Proposition 2.2 that there exists a point  $x \in X$  and its neighborhood  $U$  such that  $\widehat{H}_{n-1}(\text{bd } \overline{U}; G) \neq 0$  and the homomorphism  $\widehat{i}_{\text{bd } \overline{U}, X}^{n-1}$  is trivial. Obviously, we have the commutative diagram

$$\begin{array}{ccc} \widehat{H}_{n-1}(\text{bd } \overline{U}; G) & \xrightarrow{\widehat{i}_{\text{bd } \overline{U}, X}^{n-1}} & \widehat{H}_{n-1}(X; G) \\ \downarrow T_{\text{bd } \overline{U}}^{n-1} & & \downarrow T_X^{n-1} \\ H_{n-1}(\text{bd } \overline{U}; G) & \xrightarrow{i_{\text{bd } \overline{U}, X}^{n-1}} & H_{n-1}(X; G) \end{array}$$

such that  $T_{\text{bd } \overline{U}}^{n-1}$  is an isomorphism. Therefore,  $H_{n-1}(\text{bd } \overline{U}; G) \neq 0$  and  $i_{\text{bd } \overline{U}, X}^{n-1}(H_{n-1}(\text{bd } \overline{U}; G)) = 0$ . Hence,  $d_G X = n$ .  $\square$

Recall that the cohomological dimension  $\dim_G X$  is the largest integer  $m$  such that there exists a closed set  $A \subset X$  such that Čech cohomology group  $H^m(X, A; G)$  is non-trivial. It is well known that  $\dim_G X \leq n$  iff every map  $f : A \rightarrow K(G, n)$  can be extended to a map  $\tilde{f} : X \rightarrow K(G, n)$ , where  $K(G, n)$  is the Eilenberg-MacLane space of type  $(G, n)$ , see [12]. We also say that a finite-dimensional metric compactum  $X$  is *dimensionally full-valued* if  $\dim X \times Y = \dim X + \dim Y$  for all metric compacta  $Y$ , or equivalently (see [7, Theorem 11]),  $\dim_G X = \dim_{\mathbb{Z}} X$  for any abelian group  $G$ .

**Corollary 2.4.** *If  $X$  is dimensionally full-valued and  $X$  is  $lc^n$  with  $n = \dim X$ , then  $d_G X = h d_G X = h \dim_G X = \dim_G X = \dim X$  for any field  $G$ .*

*Proof.* By [3],  $h \dim_G X = \dim_G X$ . On the other hand,  $\dim_G X = \dim X$  because  $X$  is dimensionally full-valued. Then, Corollary 2.3 completes the proof.  $\square$

One of the main problems concerning homogeneous finite-dimensional *ANR* compacta is whether any such space is dimensionally full-valued. According to [15, Theorem 1.1], any homogeneous *ANR* compactum satisfies the hypotheses of next proposition. So, this proposition provides some information about such spaces which are not dimensionally full-valued.

**Proposition 2.5.** *Let  $X$  be  $lc^{n-1}$  with  $\dim X = n$  such that each  $x \in X$  has a local base  $\mathcal{B}_x$  with the following property:  $H^{n-1}(\text{bd } \overline{U}; \mathbb{Z})$  is finitely generated for each  $U \in \mathcal{B}_x$ . Then  $d_{\mathbb{Z}} X = h \dim_{\mathbb{Z}} X = n - 1$  provided  $X$  is not dimensionally full-valued.*

*Proof.* Since  $n - 1 \leq h \dim_{\mathbb{Z}} X \leq n$  and  $X$  is not dimensionally full valued,  $h \dim_{\mathbb{Z}} X = n - 1$  (see [3, p.364]). So, by Proposition 2.2 and the arguments from the proof of Proposition 2.2, we have  $d_{\mathbb{Z}} X \leq n - 1$  and there is a point  $x \in X$  and its neighborhood  $U$  such that  $\widehat{H}_{n-2}(\text{bd } \overline{U}; \mathbb{Z}) \neq 0$  and the homomorphism  $\widehat{i}_{\text{bd } \overline{U}, X}^{n-2}$  is trivial. Since  $H^{n-1}(\text{bd } \overline{U}; \mathbb{Z})$  are finitely generated, Theorem 4(4) from [11] implies that  $T_{\text{bd } \overline{U}}^{n-2} : \widehat{H}_{n-2}(\text{bd } \overline{U}; \mathbb{Z}) \rightarrow H_{n-2}(\text{bd } \overline{U}; \mathbb{Z})$  is an isomorphism. Hence,  $H_{n-2}(\text{bd } \overline{U}; \mathbb{Z}) \neq 0$  and triviality of  $\widehat{i}_{\text{bd } \overline{U}, X}^{n-2}$  yields triviality of  $i_{\text{bd } \overline{U}, X}^{n-2}$ . Therefore,  $d_{\mathbb{Z}} X = n - 1$ .  $\square$

The last statement in this section is the following analogue of Corollary 2.8 from [14].

**Proposition 2.6.** *Suppose  $X$  is a compact space and  $d_G X = n$ . Then there exists a point  $x \in X$  and a local base  $\mathcal{B}_x$  at  $x$  such that  $\text{bd } U$  contains a closed set  $C_U \subset X$  with  $H_{n-1}(C_U; G) \neq 0$  for any  $U \in \mathcal{B}_x$ .*

*Proof.* Since  $d_G X = n$ , there is a closed set  $\Phi \subset X$  such that  $i_{\Phi, X}^{n-1}(\gamma) = 0$  for some non-trivial  $\gamma \in H_{n-1}(\Phi; G)$ . By [14, Lemma 6], there exists a closed set  $K \subset X$  containing  $\Phi$  such that  $i_{\Phi, K}^{n-1}(\gamma) = 0$  but  $i_{\Phi, F}^{n-1}(\gamma) \neq 0$  for all proper closed subsets  $F$  of  $K$  containing  $\Phi$ . Choose  $x \in K \setminus \Phi$  and let  $\mathcal{B}_x$  consist of all open neighborhoods  $U$  of  $x$  with  $\overline{U} \cap \Phi = \emptyset$ . Denote  $A = K \setminus U$ ,  $B = \overline{U} \cap K$  and  $C_U = \text{bd}_K B$ . Then  $K = A \cup B$  and  $A \cap B = C_U$ . Since  $A$  is a proper subset of  $K$  and contains  $\Phi$ ,  $\gamma_U = i_{\Phi, A}^{n-1}(\gamma)$  is a non-trivial element of  $H_{n-1}(A; G)$ . Consider

the following, in general not exact, sequence

$$H_{n-1}(C_U; G) \xrightarrow{\psi} H_{n-1}(A; G) \oplus H_{n-1}(B; G) \xrightarrow{\phi} H_{n-1}(K; G),$$

where  $\psi(\theta) = (i_{C_U, A}^{n-1}(\theta), i_{C_U, B}^{n-1}(\theta))$  and  $\phi((\theta_1, \theta_2)) = i_{A, K}^{n-1}(\theta_1) - i_{B, K}^{n-1}(\theta_2)$ . Since  $i_{A, K}^{n-1}(\gamma_U) = i_{\Phi, K}^{n-1}(\gamma) = 0$ ,  $\phi((\gamma_U, 0)) = 0$ .

Take a family  $\{\omega_\alpha\}$  of finite open covers of  $K$  such that the homology groups  $H_{n-1}(C_U; G)$ ,  $H_{n-1}(A; G)$ ,  $H_{n-1}(B; G)$  and  $H_{n-1}(K; G)$  are limit of the inverse systems  $\{H_n(N_\alpha^{C_U}; G), \pi_{\beta, \alpha}^*\}$ ,  $\{H_{n-1}(N_\alpha^A; G), \pi_{\beta, \alpha}^*\}$ ,  $\{H_{n-1}(N_\alpha^B; G), \pi_{\beta, \alpha}^*\}$  and  $\{H_{n-1}(N_\alpha^K; G), \pi_{\beta, \alpha}^*\}$ . Here,  $N_\alpha^F$ ,  $F \subset K$ , denotes the nerve of  $\omega_\alpha$  restricted on  $F$ , and  $\pi_{\beta, \alpha} : N_\beta^F \rightarrow N_\alpha^F$  are the corresponding simplicial maps with  $\beta$  being a refinement of  $\alpha$ . Because  $\gamma_U \neq 0$ , there is  $\alpha_0$  with  $\gamma_{\alpha_0} = \pi_{\alpha_0}^*(\gamma_U) \neq 0$ , where  $\pi_\alpha : A \rightarrow N_\alpha^A$  is the natural map. Since  $K$  is a metric space, we can suppose that each cover  $\omega_\alpha$  has the following property: if  $\{V_j : j = 1, \dots, k\} \subset \omega_\alpha$  such that  $\bigcap_{j=1}^k V_j$  meets both  $A$  and  $B$ , then it also meet  $C_U$  (this can be done by

considering first a finite open family  $\omega'$  in  $K$ , which covers  $C_U$  and satisfies the following condition: for any  $\Omega \subset \omega'$  we have  $\cap \Omega \neq \emptyset$  if and only if  $\cap \Omega$  meets  $C_U$ ; then add to  $\omega'$  open subsets of  $K$  disjoint from  $C_U$  to obtain a cover of  $K$ ). The advantage of this type of covers is that the intersection of the nerves  $N_\alpha^A$  and  $N_\alpha^B$  (considered as sub-complexes of  $N_\alpha^K$ ) is the nerve  $N_\alpha^{C_U}$  and  $N_\alpha^A \cup N_\alpha^B = N_\alpha^K$  for all  $\alpha$ .

Then, we have the Mayer-Vietoris exact sequence (the coefficient group  $G$  is suppressed)

$$H_{n-1}(N_{\alpha_0}^{C_U}) \xrightarrow{\psi_{\alpha_0}} H_{n-1}(N_{\alpha_0}^A) \oplus H_{n-1}(N_{\alpha_0}^B) \xrightarrow{\phi_{\alpha_0}} H_{n-1}(N_{\alpha_0}^K) \cdots$$

Obviously,  $\phi((\gamma_U, 0)) = 0$  implies  $\phi_{\alpha_0}((\gamma_{\alpha_0}, 0)) = 0$ , and the exactness of this sequence yields  $H_{n-1}(N_{\alpha_0}^{C_U}; G) \neq 0$ . Therefore,  $H_{n-1}(C_U; G) \neq 0$ .  $\square$

**3. Dimensionally full-valued compacta.** In this section we are going to improve the result of Kodama [6, Theorem 8] that every two-dimensional  $ANR$ -compactum is dimensionally full-valued. We need the following statement:

**Lemma 3.1.** *Suppose  $G$  is a torsion free group and  $X$  is compact such that  $h \dim_G X = \dim X = n$ . If the groups  $\widehat{H}_n(X, F; G)$  and  $H_n(X, F; G)$  are isomorphic for each closed set  $F \subset X$ , then  $X$  is dimensionally full-valued.*

*Proof.* Since  $h \dim_G X = n$ , there exists a closed set  $\Phi \subset X$  with  $\widehat{H}_n(X, \Phi; G) \neq 0$ . So,  $H_n(X, \Phi; G) \neq 0$ , and according to [14, Proposition 4.5],



$H_n(X, \Phi; \mathbb{Z})$  is also non-trivial. Finally, by [5, Corollary 1],  $X$  is dimensionally full-valued.  $\square$

**Theorem 3.2.** *Suppose  $X$  is a two-dimensional compactum satisfying the following condition: for any  $x \in X$  there exists a neighborhood  $V$  of  $x$  such that the homomorphism  $i_{\bar{V}, X}^k : H_k(\bar{V}; \mathbb{S}^1) \rightarrow H_k(X; \mathbb{S}^1)$  is trivial for  $k = 1, 2$ . Then  $X$  is dimensionally full-valued.*

*Proof.* Because  $\mathbb{S}^1$  is a compact group, the exact homology  $\widehat{H}_*(\cdot; \mathbb{S}^1)$  is naturally isomorphic to Čech homology  $H_*(\cdot; \mathbb{S}^1)$ . So,  $d_{\mathbb{S}^1} X = hd_{\mathbb{S}^1} X$ . Moreover, according to [1],  $d_{\mathbb{S}^1} X = h \dim_{\mathbb{S}^1} X = 2$ . The last equality implies the existence of a point  $x \in X$  such that  $H_2^x = \varinjlim_{x \in U} \widehat{H}_2(X, X \setminus U; \mathbb{S}^1)$  is non-trivial, see [10,

Corollary 2]. Thus,  $\widehat{H}_2(\bar{U}, \text{bd } \bar{U}; \mathbb{S}^1)$ , being isomorphic to  $\widehat{H}_2(X, X \setminus U; \mathbb{S}^1)$ , is not trivial for all sufficiently small neighborhoods  $U$  of  $x$ . On the other hand, there is a neighborhood  $V$  of  $x$  such that the homomorphisms  $\widehat{i}_{\bar{V}, X}^k : \widehat{H}_k(\bar{V}; \mathbb{S}^1) \rightarrow \widehat{H}_k(X; \mathbb{S}^1)$ ,  $k = 1, 2$ , are trivial. Consequently, the homomorphisms  $\widehat{i}_{\bar{U}, X}^k$ ,  $k = 1, 2$ , are also trivial for all neighborhoods  $U$  of  $x$  with  $\bar{U} \subset V$ . Hence,  $\widehat{H}_2(\bar{U}; \mathbb{S}^1) = 0$  provided  $\bar{U} \subset V$  (otherwise we would have  $hd_{\mathbb{S}^1} X > 2$ ).

Therefore, for any  $U$  with  $\bar{U} \subset V$  we have the exact sequence

$$0 \rightarrow \widehat{H}_2(\bar{U}, \text{bd } \bar{U}; \mathbb{S}^1) \xrightarrow{\partial} \widehat{H}_1(\text{bd } \bar{U}; \mathbb{S}^1) \rightarrow \widehat{H}_1(\bar{U}; \mathbb{S}^1) \dots$$

Since  $\widehat{H}_2(\bar{U}, \text{bd } \bar{U}; \mathbb{S}^1) \neq 0$ ,  $\widehat{H}_1(\text{bd } \bar{U}; \mathbb{S}^1) \neq 0$ .

So, for all small neighborhoods  $U$  of  $x$  the groups  $\widehat{H}_2(\bar{U}, \text{bd } \bar{U}; \mathbb{S}^1)$  and  $\widehat{H}_1(\text{bd } \bar{U}; \mathbb{S}^1)$  are non-trivial, while the homomorphisms  $\widehat{i}_{\bar{U}, X}^1 : \widehat{H}_1(\bar{U}; \mathbb{S}^1) \rightarrow \widehat{H}_1(X; \mathbb{S}^1)$  are trivial. This implies that the homomorphisms  $\widehat{i}_{\text{bd } \bar{U}, X}^1 : \widehat{H}_1(\text{bd } \bar{U}; \mathbb{S}^1) \rightarrow \widehat{H}_1(X; \mathbb{S}^1)$  are also trivial. Since  $\dim X = 2$ ,  $H^3(X; \mathbb{Z}) = H^3(\bar{U}, \text{bd } \bar{U}; \mathbb{Z}) = H^3(\bar{U}; \mathbb{Z}) = 0$ . So, by the Universal Coefficient Theorem (see [11, Theorem 3]), we have the isomorphisms

$$\widehat{H}_2(\bar{U}, \text{bd } \bar{U}; \mathbb{S}^1) \cong \text{Hom}(H^2(\bar{U}, \text{bd } \bar{U}; \mathbb{Z}), \mathbb{S}^1),$$

$$\widehat{H}_2(\bar{U}; \mathbb{S}^1) \cong \text{Hom}(H^2(\bar{U}; \mathbb{Z}), \mathbb{S}^1) \text{ and } \widehat{H}_2(X; \mathbb{S}^1) \cong \text{Hom}(H^2(X; \mathbb{Z}), \mathbb{S}^1).$$

Similarly,  $\dim \text{bd } \bar{U} \leq 1$  yields  $H^2(\text{bd } \bar{U}; \mathbb{Z}) = 0$ . So,

$$\widehat{H}_1(\text{bd } \bar{U}; \mathbb{S}^1) \cong \text{Hom}(H^1(\text{bd } \bar{U}; \mathbb{Z}), \mathbb{S}^1).$$

Thus,  $H^2(\bar{U}, \text{bd } \bar{U}; \mathbb{Z}) \neq 0$ ,  $H^1(\text{bd } \bar{U}; \mathbb{Z}) \neq 0$  and the triviality of the homomorphisms  $\widehat{i}_{\text{bd } \bar{U}, X}^1$  yields the triviality of the inclusion homomorphisms  $j_{X \text{bd } \bar{U}}^1 :$

$H^1(X; \mathbb{Z}) \rightarrow H^1(\text{bd } \overline{U}; \mathbb{Z})$ . Indeed, by [4], the group  $H_1(X; \mathbb{S}^1)$  is the character group of  $H^1(X; \mathbb{Z})$ . So,  $\text{Hom}(H^1(X; \mathbb{Z}), \mathbb{S}^1)$  is isomorphic to  $H_1(X; \mathbb{S}^1)$ . Similarly,  $\text{Hom}(H^1(\text{bd } \overline{U}; \mathbb{Z}), \mathbb{S}^1)$  is isomorphic to  $H_1(\text{bd } \overline{U}; \mathbb{S}^1)$ . Hence, identifying the Čech homology groups with the corresponding exact homology groups, we obtain that the inclusion homomorphism  $\widehat{i}_{\text{bd } \overline{U}, X}^1$  is equivalent to a homomorphism from  $\text{Hom}(H^1(\text{bd } \overline{U}; \mathbb{Z}), \mathbb{S}^1)$  into  $\text{Hom}(H^1(X; \mathbb{Z}), \mathbb{S}^1)$  assigning to each character  $\varphi \in \text{Hom}(H^1(\text{bd } \overline{U}; \mathbb{Z}), \mathbb{S}^1)$  the composition  $\varphi \circ j_{X \text{bd } \overline{U}}^1$ . Thus,  $j_{X \text{bd } \overline{U}}^1$  is trivial.

On the other hand, it is well known that the simplicial one-dimensional cohomology groups with integer coefficients are free, so any non-trivial one-dimensional Čech cohomology group  $H^1(\cdot; \mathbb{Z})$  is torsion free, see for example [2, Theorem 2.5]. In particular,  $H^1(\text{bd } \overline{U}; \mathbb{Z})$  is torsion free. It follows from the exact sequence

$$\dots \rightarrow H^1(X; \mathbb{Z}) \xrightarrow{j_{X \text{bd } \overline{U}}^1} H^1(\text{bd } \overline{U}; \mathbb{Z}) \xrightarrow{\partial_X} H^2(X, \text{bd } \overline{U}; \mathbb{Z}) \rightarrow \dots$$

that  $\partial_X$  is an injective homomorphism. Hence,  $H^2(X, \text{bd } \overline{U}; \mathbb{Z})$  contains elements of infinite order. This implies  $H^2(X, \text{bd } \overline{U}; \mathbb{Q}) \neq 0$ . So,  $\dim_{\mathbb{Q}} X = 2$ . On the other hand, by [3, p.364],  $\dim_{\mathbb{Q}} X = h \dim_{\mathbb{Q}} X$ . Finally, Lemma 3.1 yields  $X$  is dimensionally full-valued.  $\square$

**Corollary 3.3.** *Every two-dimensional  $lc^2$ -compactum is dimensionally full-valued.*

*Proof.* We already observed the  $lc^2$ -property implies that the following condition for any group  $G$  and open sets  $V \subset X$ :

- the groups  $\widetilde{H}_k(V, G)$  and  $\widetilde{H}_k(X, G)$  are isomorphic to  $H_k(V, G)$  and  $H_k(X, G)$ , respectively, for all  $k \leq 2$  (see [9, Theorem 1]);

Because  $X$  is  $lc^2$ , any point  $x \in X$  has a neighborhood  $V$  such that the inclusion homomorphisms  $\widetilde{i}_{V, X}^k : \widetilde{H}_k(V; G) \rightarrow \widetilde{H}_k(X; G)$  are trivial for  $k = 1, 2$ . So, we can apply Theorem 3.2.  $\square$

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