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CONSTRUCTING SELECTIONS STEPWISE OVER SKELETONS OF NERVES OF COVERS

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Abstract. It is given a simplified and self-contained proof of the classical Michael's finite-dimensional selection theorem. The proof is based on approximate selections constructed stepwise over skeletons of nerves of covers. The method is also applied to simplify the proof of the Schepin–Brodsky's generalisation of this theorem.

1. Introduction. All spaces in this paper are Hausdorff topological spaces. We will use $\Phi : X \rightsquigarrow Y$ to designate that Φ is a map from X to the nonempty subsets of Y, i.e. a *set-valued mapping.* Such a mapping is *lower semi-continuous*, or l.s.c., if the set

$$\Phi^{-1}[U] = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$$

is open in X, for every open $U \subset Y$. Also, let us recall that a map $f : X \to Y$ is a *selection* for $\Phi : X \rightsquigarrow Y$ if $f(x) \in \Phi(x)$, for all $x \in X$.

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Let $n \geq -1$. A family $\mathscr{S}$ of subsets of a space Y is *equi-LC^n* [11] if every neighbourhood U of a point $y \in \bigcup \mathscr{S}$ contains a neighbourhood V of y such that for every $S \in \mathscr{S}$, every continuous map $g : \mathbb{S}^k \to V \cap S$ of the k-sphere $\mathbb{S}^k$, $k \leq n$, can be extended to a continuous map $h : \mathbb{B}^{k+1} \to U \cap S$ of the $(k+1)$-ball $\mathbb{B}^{k+1}$. A space S is called C^n if for every $k \leq n$, every continuous map $g : \mathbb{S}^k \to S$ can be extended to a continuous map $h : \mathbb{B}^{k+1} \to S$. In these terms, a family $\mathscr{S}$ of subsets of Y is equi-LC^{-1} if it consists of nonempty subsets; similarly, each nonempty subset $S \subset Y$ is C^{-1}.

Let $\mathscr{F}(Y)$ be the collection of all nonempty closed subsets of a space Y. The following theorem was proved by Ernest Michael, see [11, Theorem 1.2], and is commonly called the *finite-dimensional selection theorem.*

Theorem 1.1. *Let X be a paracompact space with $\dim(X) \leq n+1$, Y be a completely metrizable space, and $\mathscr{S} \subset \mathscr{F}(Y)$ be an equi-LC^n family such that each $S \in \mathscr{S}$ is C^n. Then each l.s.c. mapping $\Phi : X \to \mathscr{S}$ has a continuous selection.*

The original proof of Theorem 1.1 in [11] takes up most of that paper, and is accomplished in 6 steps. Other proofs of this theorem can be found in the monograph [12], and the book [7]. Actually, in [12] are given two different approaches to obtain the theorem — the one which follows the original Michael's proof, and another one based on filtrations [14]. Other proofs were given by other authors, see e.g. [1] and [9]. However, what all these proofs have in common is that they may somehow discourage the casual reader and make Theorem 1.1 not so accessible to wider audience. The main purpose of this paper is to fill in this gap, and present a simplified and self-contained proof of this theorem.

The paper is organised as follows. The next section contains a brief review of canonical maps and partitions of unity, which is essential for the proper understanding of any of the available proofs of Theorem 1.1. In this regard, let us explicitly remark that these considerations were not made readily available in previous proofs, so they are now included to make the exposition self-contained. The essential preparation for the proof of Theorem 1.1 starts in Section 3, which contains a selection theorem for finite aspherical sequences of lower locally constant mappings (Theorem 3.1). This theorem is similar to a theorem of Uspenskij, see [16, Theorem 1.3], and represents a relaxed version of another theorem proved by the author, see [9, Theorem 3.1]. Section 4 contains several simple constructions of finite aspherical sequences of sets providing the main interface between such sequences of sets and the property of equi-LC^n. Finally, the proof of Theorem 1.1 is accomplished in Section 5. It is based on two constructions which are also present in Michael's proof. The one, Proposition 5.1, relates l.s.c. mappings to lower locally constant mappings; the other — Proposition 5.2, relates selections for lower locally constant mappings to approximate selections for l.s.c. mappings. These constructions are applied together with Theorem 3.1 to deal with two selection properties of l.s.c. equi-LC^n-valued mappings, see Theorems 5.3 and 5.4. The proof of Theorem 1.1 is then obtained as an immediate consequence of these properties.

2. Canonical maps and partitions of unity. The *cozero set*, or the *set-theoretic support*, of a function $\xi : X \to \mathbb{R}$ is the set $\operatorname{coz}(\xi) = \{x \in X : \xi(x) \neq 0\}$. A collection $\xi_a : X \to [0,1]$, $a \in \mathscr{A}$, of continuous functions on a space X is a *partition of unity* if $\sum_{a\in\mathscr{A}} \xi_a(x) = 1$, for each $x \in X$. Here, "$\sum_{a\in\mathscr{A}} \xi_a(x) = 1$" means that only countably many functions ξ_a's do not vanish at x, and the series composed by them is convergent to 1. For a cover $\mathscr{U}$ of a space X, a partition of unity $\{\xi_U : U \in \mathscr{U}\}$ on X is *index-subordinated* to $\mathscr{U}$ if $\operatorname{coz}(\xi_U) \subset U$, for each $U \in \mathscr{U}$, see Remark 2.7. The following theorem is well known, it is a consequence of Urysohn's characterisation of normality [15] and the Lefschetz lemma [10].

Theorem 2.1. *Every locally finite open cover of a normal space has an index-subordinated partition of unity.*

A partition of unity $\{\xi_a : a \in \mathscr{A}\}$ on a space X is called *locally finite* if $\{\operatorname{coz}(\xi_a) : a \in \mathscr{A}\}$ is a locally finite cover of X. Complementary to Theorem 2.1 is the following important property of partitions of unity; it follows from a construction of M. Mather, see [3, Lemma] and [6, Lemma 5.1.8].

Theorem 2.2. *If a cover $\mathscr{U}$ of a space X has an index-subordinated partition of unity, then $\mathscr{U}$ also has an index-subordinated locally finite partition of unity.*

By a *simplicial complex* we mean a collection Σ of nonempty finite subsets of a set S such that $\tau \in \Sigma$, whenever $\emptyset \neq \tau \subset \sigma \in \Sigma$. The set $\bigcup \Sigma$ is the *vertex set* of Σ, while each element of Σ is called a *simplex*. The *k-skeleton* Σ^k of Σ ($k \geq 0$) is the simplicial complex $\Sigma^k = \{\sigma \in \Sigma : \operatorname{Card}(\sigma) \leq k+1\}$, where $\operatorname{Card}(\sigma)$ is the cardinality of σ. In the sequel, for simplicity, we will identify the vertex set of Σ with its 0-skeleton Σ^0. In these terms, a *simplicial map* $g : \Sigma_1 \to \Sigma_2$ is a map $g : \Sigma_1^0 \to \Sigma_2^0$ between the vertices of simplicial complexes Σ_1 and Σ_2 such that $g(\sigma) \in \Sigma_2$, for each $\sigma \in \Sigma_1$. If $g : \Sigma_1 \to \Sigma_2$ is a simplicial map and $g : \Sigma_1^0 \to \Sigma_2^0$ is bijective, then the inverse g^{-1} is also a simplicial map, and we say that g is a *simplicial isomorphism*.

The set Σ_S of all nonempty finite subsets of a set S is a simplicial complex. Another natural example is the *nerve* $\mathscr{N}(\mathscr{U})$ of a cover $\mathscr{U}$ of a set X, which is the subcomplex of $\Sigma_{\mathscr{U}}$ defined by

$$\mathscr{N}(\mathscr{U}) = \left\{\sigma \in \Sigma_{\mathscr{U}} : \bigcap \sigma \neq \emptyset\right\}. \tag{2.1}$$

The k-skeleton of $\mathscr{N}(\mathscr{U})$ is denoted by $\mathscr{N}^k(\mathscr{U})$, and the vertex set $\mathscr{N}^0(\mathscr{U})$ of $\mathscr{N}(\mathscr{U})$ is actually $\mathscr{U}$ because we can always assume that $\emptyset \notin \mathscr{U}$.

For a set $\mathscr{A}$, let $\ell_1(\mathscr{A})$ be the linear space of all functions $y : \mathscr{A} \to \mathbb{R}$ with $\sum_{a\in\mathscr{A}} |y(a)| < \infty$. In fact, $\ell_1(\mathscr{A})$ is a Banach space when equipped with the norm $\|y\|_1 = \sum_{a\in\mathscr{A}} |y(a)|$, but this will play no role in the paper. The vertex set Σ^0 of a simplicial complex Σ is a linearly independent subset of $\ell_1(\Sigma^0)$, where each $v \in \Sigma^0$ is identified with its characteristic function $v : \Sigma^0 \to \{0, 1\}$, namely with the function $v(u) = 0$ for $u \neq v$, and $v(v) = 1$. Then to each $\sigma \in \Sigma$ one can associate the *geometric simplex* $|\sigma| = \operatorname{conv}(\sigma)$, which is the convex hull of σ. Thus, $|\sigma|$ is a *k-dimensional simplex* if and only if $\operatorname{Card}(\sigma) = k + 1$. The set $|\Sigma| = \bigcup_{\sigma\in\Sigma} |\sigma| \subset \ell_1(\Sigma^0)$ is called the *geometric realisation* of Σ. As a topological space, we will consider $|\Sigma|$ endowed with the *Whitehead topology* [18, 19]. In this topology, a subset $U \subset |\Sigma|$ is open if and only if $U \cap |\sigma|$ is open in $|\sigma|$, for every $\sigma \in \Sigma$. Let us explicitly remark that the Whitehead topology on $|\Sigma|$ is not necessarily the subspace topology on $|\Sigma|$ as a subset of the Banach space $\ell_1\left(\Sigma^0\right)$. However, both topologies coincide on each geometric simplex $|\sigma|$, for $\sigma \in \Sigma$.

If $p \in |\sigma|$ for some $\sigma \in \Sigma$, then p is both an element $p \in \ell_1(\Sigma^0)$ and a unique convex combination of the elements of $\sigma \subset \Sigma^0 \subset \ell_1(\Sigma^0)$. Hence, the geometric realisation $|\Sigma|$ is the set of all $p \in \ell_1(\Sigma^0)$ such that

$$p(v) \geq 0,\ v \in \Sigma^0, \quad \text{and} \quad \operatorname{coz}(p) = \left\{v \in \Sigma^0 : p(v) > 0\right\} \in \Sigma. \tag{2.2}$$

Here, $p(v)$ is called the v-th *barycentric* (or *affine*) *coordinate of* $p \in |\Sigma|$, while the simplex $\operatorname{coz}(p) \in \Sigma$ is called the *carrier* of p, and denoted by $\operatorname{car}(p) = \operatorname{coz}(p)$. Since the representation $p = \sum_{v\in\operatorname{car}(p)} p(v) \cdot v$ is unique, the carrier $\operatorname{car}(p)$ is the minimal simplex of Σ with the property that $p \in |\operatorname{car}(p)|$.

To each vertex $v \in \Sigma^0$, we can now associate the function $\alpha_v : |\Sigma| \to [0, 1]$, defined by

$$\alpha_v(p) = p(v), \quad \text{for every } p \in |\Sigma|. \tag{2.3}$$

It is called the v-th *barycentric coordinate function* and is continuous being affine on each simplex $|\sigma|$, for $\sigma \in \Sigma$. The cozero set $\operatorname{coz}(\alpha_v)$ of α_v is called the *open star* of the vertex $v \in \Sigma^0$, and denoted by

$$\operatorname{st}\langle v\rangle = \big\{p \in |\Sigma| : \alpha_v(p) > 0\big\}. \tag{2.4}$$

Clearly, the open star $\operatorname{st}\langle v\rangle$ is open in $|\Sigma|$ because α_v is continuous. The following proposition is an immediate consequence of (2.2), (2.3) and (2.4).

Proposition 2.3. *If Σ is a simplicial complex, then the collection $\big\{\alpha_v : v \in \Sigma^0\big\}$ is a partition of unity on $|\Sigma|$ with $\operatorname{coz}(\alpha_v) = \operatorname{st}\langle v\rangle$, for each $v \in \Sigma^0$.*

We now turn to the other essential concept in this section. For a cover $\mathscr{U}$ of a space X, a continuous map $f : X \to |\mathscr{N}(\mathscr{U})|$ is called *canonical for $\mathscr{U}$* if

$$f^{-1}(\operatorname{st}\langle U\rangle) \subset U, \quad \text{for every } U \in \mathscr{U}. \tag{2.5}$$

Canonical maps are essentially partitions of unity, which are index-subordinated to the corresponding cover of the space.

Theorem 2.4. *A cover $\mathscr{U}$ of a space X has an index-subordinated partition of unity if and only if $\mathscr{U}$ has a canonical map.*

Proof. Let $\mathscr{U}$ be a cover of X and α_U, $U \in \mathscr{U}$, be the barycentric coordinate functions of $|\mathscr{N}(\mathscr{U})|$.

Suppose that $f : X \to |\mathscr{N}(\mathscr{U})|$ is a canonical map for $\mathscr{U}$. Since f is continuous, by Proposition 2.3, $\{\alpha_U \circ f : U \in \mathscr{U}\}$ is a partition of unity on X. By the same proposition and (2.5), we also have that

$$\operatorname{coz}(\alpha_U \circ f) = f^{-1}(\operatorname{coz}(\alpha_U)) = f^{-1}(\operatorname{st}\langle U\rangle) \subset U, \quad U \in \mathscr{U}.$$

Conversely, suppose that $\mathscr{U}$ has an index-subordinated partition of unity. Then by Theorem 2.2, $\mathscr{U}$ also has an index-subordinated locally finite partition of unity $\{\xi_U : U \in \mathscr{U}\}$. For each $x \in X$, let $\sigma_\xi(x) \in \mathscr{N}(\mathscr{U})$ be the simplex determined by the point x and the functions ξ_U, $U \in \mathscr{U}$, namely $\sigma_\xi(x) = \{U \in \mathscr{U} : \xi_U(x) > 0\}$. Next, define a map $f : X \to |\mathscr{N}(\mathscr{U})|$ by

$$f(x) = \sum_{U \in \sigma_\xi(x)} \xi_U(x) \cdot U, \quad x \in X. \tag{2.6}$$

Since $\{\xi_U : U \in \mathscr{U}\}$ is a locally finite partition of unity, each point $p \in X$ has a neighbourhood $V_p \subset X$ such that $\mathscr{U}_p = \{U \in \mathscr{U} : V_p \cap \operatorname{coz}(\xi_U) \neq \emptyset\}$ is a finite set. According to (2.6), this implies that $f(V_p) \subset |\mathscr{N}(\mathscr{U}_p)| \subset \ell_1(\mathscr{U}_p)$. However, $\ell_1(\mathscr{U}_p)$ is now the usual Euclidean space $\mathbb{R}^{\mathscr{U}_p}$ because $\mathscr{U}_p$ is a finite set. For the same reason, $\mathscr{N}(\mathscr{U}_p)$ has finitely many simplices. Therefore, the Whitehead topology on $|\mathscr{N}(\mathscr{U}_p)|$ is the subspace topology on $|\mathscr{N}(\mathscr{U}_p)|$ as a subset of $\mathbb{R}^{\mathscr{U}_p}$. Since each function $\xi_U = \alpha_U \circ f$, $U \in \mathscr{U}_p$, is continuous, so is the restriction $f \restriction V_p$. This shows that f is continuous as well. Finally, let $U \in \mathscr{U}$ and $x \in f^{-1}(\mathrm{st}\langle U\rangle)$. Then $f(x) \in \mathrm{st}\langle U\rangle$ and by (2.4) and (2.6), we get that $\xi_U(x) = \alpha_U(f(x)) > 0$. Accordingly, $U \in \sigma_\xi(x)$ which implies that $x \in U$ because $\mathrm{coz}(\xi_U) \subset U$. Thus, f is canonical for $\mathscr{U}$, see (2.5). □

Canonical maps will be involved in the proof of Theorem 1.1 with two properties, which are briefly discussed below.

For a simplicial complex Σ, as mentioned before, the carrier $\mathrm{car}(p)$ of a point $p \in |\Sigma|$ is the minimal simplex of Σ with $p \in |\,\mathrm{car}(p)|$, see (2.2). According to (2.3) and (2.4), it has the following natural representation

$$\mathrm{car}(p) = \left\{v \in \Sigma^0 : p \in \mathrm{st}\langle v\rangle\right\}. \tag{2.7}$$

For a cover $\mathscr{U}$ of X and $x \in X$, we will associate the simplicial complex

$$\Sigma_{\mathscr{U}}(x) = \left\{\sigma \in \Sigma_{\mathscr{U}} : x \in \bigcap \sigma\right\}. \tag{2.8}$$

According to (2.1), we have that $\Sigma_{\mathscr{U}}(x) \subset \mathscr{N}(\mathscr{U})$, for every $x \in X$. Thus, (2.8) defines a natural set-valued mapping $\Sigma_{\mathscr{U}} : X \rightsquigarrow \mathscr{N}(\mathscr{U})$. To this mapping, we will associate the mapping $|\Sigma_{\mathscr{U}}| : X \rightsquigarrow |\mathscr{N}(\mathscr{U})|$ which assigns to each $x \in X$ the geometric realisation $|\Sigma_{\mathscr{U}}|(x) = |\Sigma_{\mathscr{U}}(x)|$. In terms of this mapping, we have the following selection interpretation of canonical maps which extends an observation of Dowker [4], see Remark 2.9.

Proposition 2.5. *Let $\mathscr{U}$ be a cover of a space X. Then a continuous map $f : X \to |\mathscr{N}(\mathscr{U})|$ is canonical for $\mathscr{U}$ if and only if f is a selection for the mapping $|\Sigma_{\mathscr{U}}| : X \rightsquigarrow |\mathscr{N}(\mathscr{U})|$.*

P r o o f. Let f be a canonical map for $\mathscr{U}$, and $x \in X$. Whenever $U \in \mathrm{car}(f(x))$, it follows from (2.7) that $f(x) \in \mathrm{st}\langle U\rangle$ and therefore, by (2.5), $x \in U$. Thus, by (2.8), $\mathrm{car}(f(x)) \in \Sigma_{\mathscr{U}}(x)$ and we have that $f(x) \in |\,\mathrm{car}(f(x))| \subset |\Sigma_{\mathscr{U}}|(x)$. Conversely, suppose that f is as selection for $|\Sigma_{\mathscr{U}}|$, and $x \in f^{-1}(\mathrm{st}\langle U\rangle)$ for some $U \in \mathscr{U}$. Then by (2.7), $U \in \mathrm{car}(f(x))$ because $f(x) \in \mathrm{st}\langle U\rangle$. Moreover, $f(x) \in |\sigma|$ for some $\sigma \in \Sigma_{\mathscr{U}}(x)$ because $f(x) \in |\Sigma_{\mathscr{U}}|(x)$. Since $\mathrm{car}(f(x))$ is the minimal simplex with this property, we get that $U \in \mathrm{car}(f(x)) \subset \sigma$ and, therefore, $x \in U$. That is, $f^{-1}(\mathrm{st}\langle U\rangle) \subset U$. □

Each simplicial map $g : \Sigma_1 \to \Sigma_2$, between simplicial complexes Σ_1 and Σ_2, can be extended to a continuous map $|g| : |\Sigma_1| \to |\Sigma_2|$ which is affine on each geometric simplex $|\sigma|$, for $\sigma \in \Sigma_1$. This map is simply defined by

$$|g|(p) = \sum_{v \in \operatorname{car}(p)} \alpha_v(p) \cdot g(v), \quad p \in |\Sigma_1|.$$

If a cover $\mathscr{V}$ of X refines another cover $\mathscr{U}$, then there exists a natural simplicial map $r : \mathscr{N}(\mathscr{V}) \to \mathscr{N}(\mathscr{U})$ with $V \subset r(V)$, for each $V \in \mathscr{V}$. Such a map is commonly called a *canonical projection*, or a *refining simplicial map*, or simply a *refining map*. Canonical maps are preserved by refinements in the following sense.

Corollary 2.6. *Let $\mathscr{U}$ and $\mathscr{V}$ be covers of a space X such that $\mathscr{V}$ refines $\mathscr{U}$. If $r : \mathscr{N}(\mathscr{V}) \to \mathscr{N}(\mathscr{U})$ is a refining map and $g : X \to |\mathscr{N}(\mathscr{V})|$ is canonical for $\mathscr{V}$, then the composite map $|r| \circ g : X \to |\mathscr{N}(\mathscr{U})|$ is canonical for $\mathscr{U}$.*

Proof. This follows from Proposition 2.5 and the fact that $r(\Sigma_{\mathscr{V}}(x)) \subset \Sigma_{\mathscr{U}}(x)$, $x \in X$, because $V \subset r(V)$ for every $V \in \mathscr{V}$, see (2.8). □

We conclude this section with several remarks.

Remark 2.7. For a space X, the *support* of a function $\xi : X \to \mathbb{R}$, called also the *topological support*, is the set $\operatorname{supp}(\xi) = \overline{\operatorname{coz}(\xi)}$. In several sources, a partition of unity $\{\xi_U : U \in \mathscr{U}\}$ on a space X is called *index-subordinated* to a cover $\mathscr{U}$ of X if $\operatorname{supp}(\xi_U) \subset U$, for every $U \in \mathscr{U}$; and $\{\xi_U : U \in \mathscr{U}\}$ is called *weakly index-subordinated* to $\mathscr{U}$ if $\operatorname{coz}(\xi_U) \subset U$, for every $U \in \mathscr{U}$, see e.g. [13]. However, these variations in the terminology do not affect the results of this section. Namely, if $\{\eta_U : U \in \mathscr{U}\}$ is a partition of unity on X, then X also has a (locally finite) partition of unity $\{\xi_U : U \in \mathscr{U}\}$ with $\operatorname{supp}(\xi_U) \subset \operatorname{coz}(\eta_U)$, for all $U \in \mathscr{U}$, [13, Proposition 2.7.4]. This property is essentially the construction of M. Mather for proving Theorem 2.2.

Remark 2.8. Canonical maps provide an isomorphism between simplicial complexes and nerves of covers. Namely, if $\mathscr{O}_\Sigma = \left\{ \operatorname{st}\langle v\rangle : v \in \Sigma^0 \right\}$ is the cover of $|\Sigma|$ by the open stars of the vertices of a simplicial complex Σ and $\sigma \subset \Sigma^0$, then $\sigma \in \Sigma$ if and only if $\bigcap_{v \in \sigma} \operatorname{st}\langle v\rangle \neq \emptyset$. That is, $\sigma \in \Sigma$ precisely when $\operatorname{st}\langle\sigma\rangle = \{\operatorname{st}\langle v\rangle : v \in \sigma\} \in \mathscr{N}(\mathscr{O}_\Sigma)$. Hence, $\operatorname{st}\langle\cdot\rangle : \Sigma \to \mathscr{N}(\mathscr{O}_\Sigma)$ is a simplicial isomorphism and the associated map $|\operatorname{st}\langle\cdot\rangle| : |\Sigma| \to |\mathscr{N}(\mathscr{O}_\Sigma)|$ is both a homeomorphism and a canonical map for $\mathscr{O}_\Sigma$.

Remark 2.9. In the case of a point-finite cover $\mathscr{U}$ of X, Proposition 2.5 is reduced to the following selection interpretation of canonical maps given by Dowker [4]. Whenever $x \in X$, let $\sigma(x) = \{U \in \mathscr{U} : x \in U\} \in \mathscr{N}(\mathscr{U})$ be the simplex determined by x. Then a continuous map $f : X \to |\mathscr{N}(\mathscr{U})|$ is canonical for $\mathscr{U}$ if and only if $f(x) \in |\sigma(x)|$, for every $x \in X$. While $\sigma(x)$ is only an element of $\Sigma_{\mathscr{U}}(x)$, we have that $|\sigma(x)| = |\Sigma_{\mathscr{U}}|(x)$ because $\sigma \subset \sigma(x)$, for each $\sigma \in \Sigma_{\mathscr{U}}(x)$.

3. Aspherical sequences of mappings and selections. A mapping $\varphi : X \rightsquigarrow Y$ is *lower locally constant* [9] if the set $\{x \in X : K \subset \varphi(x)\}$ is open in X, for every compact subset $K \subset Y$. This property appeared in a paper of Uspenskij [16]; later on, it was used by some authors (see, for instance, [2, 17]) under the name "strongly l.s.c.", while in papers of other authors strongly l.s.c. was already used for a different property of set-valued mappings (see, for instance, [8]). Every lower locally constant mapping is l.s.c. but the converse fails in general and counterexamples abound. In fact, if we consider a single-valued map $f : X \to Y$ as a set-valued one, then f is l.s.c. if and only if it is continuous, while f will be lower locally constant if and only if it is locally constant. Thus, our terminology provides some natural analogy with the single-valued case.

Let $k \geq 0$. For subsets $S, B \subset Y$, we will write that $S \overset{k}{\hookrightarrow} B$ if every continuous map of the k-sphere in S can be extended to a continuous map of the $(k+1)$-ball in B. Evidently, the relation $S \overset{k}{\hookrightarrow} B$ implies that $S \subset B$. Similarly, for mappings $\varphi, \psi : X \rightsquigarrow Y$, we will write $\varphi \overset{k}{\hookrightarrow} \psi$ to express that $\varphi(x) \overset{k}{\hookrightarrow} \psi(x)$, for every $x \in X$. In these terms, we shall say that a sequence of mappings $\varphi_k : X \rightsquigarrow Y$, $0 \leq k \leq n$, is *aspherical* if $\varphi_k \overset{k}{\hookrightarrow} \varphi_{k+1}$, for every $k < n$. The following theorem will be proved in this section.

Theorem 3.1. *Let X be a paracompact space with $\dim(X) \leq n$, Y be a space, and $\varphi_k : X \rightsquigarrow Y$, $0 \leq k \leq n$, be an aspherical sequence of lower locally constant mappings. Then φ_n has a continuous selection.*

The proof of Theorem 3.1 is based on special skeletal selections motivated by the characterisation of canonical maps in Proposition 2.5. Namely, we shall say that a mapping $\varphi : X \rightsquigarrow Y$ has a *k-skeletal selection*, $k \geq 0$, if there exists and open cover $\mathscr{U}$ of X and a continuous map $u : |\mathscr{N}^k(\mathscr{U})| \to Y$ such that

$$u\left(|\Sigma^k_{\mathscr{U}}(x)|\right) \subset \varphi(x), \quad \text{for every } x \in X. \tag{3.1}$$

Here, $\Sigma^k_{\mathscr{U}}(x)$ is the k-skeleton of the simplicial complex $\Sigma_{\mathscr{U}}(x)$, see (2.8). In fact, just like before, one can consider $\Sigma^k_{\mathscr{U}} : X \rightsquigarrow \mathscr{N}^k(\mathscr{U})$ as a set-valued mapping;

similarly for $|\Sigma^k_{\mathscr{U}}| : X \rightsquigarrow |\mathscr{N}^k(\mathscr{U})|$. Then a continuous map $u : |\mathscr{N}^k(\mathscr{U})| \to Y$ is a k-skeletal selection for φ if and only if the composite mapping $u \circ |\Sigma^k_{\mathscr{U}}| : X \rightsquigarrow Y$ is a set-valued selection for $\varphi : X \rightsquigarrow Y$, see Remark 3.6.

We proceed with the following constructions of k-skeletal selections which furnish the essential part of the proof of Theorem 3.1.

Proposition 3.2. *Each lower locally constant mapping* $\varphi : X \rightsquigarrow Y$ *has a* 0*-skeletal selection.*

Proof. For each $x \in X$, take a point $y(x) \in \varphi(x)$, and set

$$U(x) = \{z \in X : y(x) \in \varphi(z)\}. \tag{3.2}$$

Then $\mathscr{U} = \{U(x) : x \in X\}$ is an open cover of X. Moreover, for each $U \in \mathscr{U}$ there is a point $x_U \in X$ with $U = U(x_U)$. Since $|\mathscr{N}^0(\mathscr{U})| = \mathscr{U}$, we may define a map $u : |\mathscr{N}^0(\mathscr{U})| \to Y$ by $u(U) = y(x_U)$, for each $U \in \mathscr{U}$. If $x \in U \in \mathscr{U}$, then $x \in U(x_U)$ and by (3.2), we get that $u(U) = y(x_U) \in \varphi(x)$. □

Proposition 3.3. *Let* X *be a paracompact space, and* $\psi : X \rightsquigarrow Y$ *be a mapping which has a* k*-skeletal selection, for some* $k \geq 0$*. Then* ψ *has a* k*-skeletal selection* $u : |\mathscr{N}^k(\mathscr{U})| \to Y$ *for some open locally finite cover* $\mathscr{U}$ *of* X.

Proof. Let $v : |\mathscr{N}^k(\mathscr{V})| \to Y$ be a k-skeletal selection for ψ, for some open cover $\mathscr{V}$ of X. Since X is paracompact, the cover $\mathscr{V}$ has an open locally finite refinement $\mathscr{U}$. Let $r : \mathscr{N}(\mathscr{U}) \to \mathscr{N}(\mathscr{V})$ be a refining map. Then by (3.1), $u = v \circ |r| \restriction |\mathscr{N}^k(\mathscr{U})| : |\mathscr{N}^k(\mathscr{U})| \to Y$ is a k-skeletal selection for ψ because $r(\Sigma^k_{\mathscr{U}}(x)) \subset \Sigma^k_{\mathscr{V}}(x)$, for every $x \in X$. □

A cover $\mathscr{V}$ of X is a *star-refinement* of a cover $\mathscr{U}$ if the cover $\mathscr{V}^* = \{V^* : V \in \mathscr{V}\}$ refines $\mathscr{U}$, where $V^* = \bigcup\{W \in \mathscr{V} : W \cap V \neq \emptyset\}$. To reflect this property, we shall say that a simplicial map $\ell : \mathscr{N}(\mathscr{V}) \to \mathscr{N}(\mathscr{U})$ is a *star-refining map* if $V^* \subset \ell(V)$, for each $V \in \mathscr{V}$. Each star-refining map $\ell : \mathscr{N}(\mathscr{V}) \to \mathscr{N}(\mathscr{U})$ has the property that

$$\bigcup \sigma \subset \bigcap \ell(\sigma), \quad \text{for each } \sigma \in \mathscr{N}(\mathscr{V}). \tag{3.3}$$

Proposition 3.4. *Let* X *be a paracompact space,* Y *be a space, and* $\psi, \varphi : X \rightsquigarrow Y$ *be such that* φ *is lower locally constant and* $\psi \overset{k}{\hookrightarrow} \varphi$ *for some* $k \geq 0$*. If* ψ *has a* k*-skeletal selection, then* φ *has a* $(k+1)$*-skeletal selection.*

Proof. By Proposition 3.3, ψ has a k-skeletal selection $u : |\mathscr{N}^k(\mathscr{U})| \to Y$ for some open locally finite cover $\mathscr{U}$ of X. For each $\sigma \in \mathscr{N}(\mathscr{U})$, let $u_\sigma =$

$u \restriction |\mathscr{N}^k(\sigma)|$ be the restriction of u over the subcomplex $|\mathscr{N}^k(\sigma)| = |\sigma| \cap |\mathscr{N}^k(\mathscr{U})|$. Then, whenever $\sigma \in \Sigma_{\mathscr{U}}^{k+1}(x)$ for some $x \in X$, the map u_σ can be extended to a continuous map $u_{(x,\sigma)} : |\sigma| \to Y$ such that

$$u_{(x,\sigma)}(|\sigma|) \subset \varphi(x). \tag{3.4}$$

Indeed, if $\sigma \in \Sigma_{\mathscr{U}}^k(x)$, then by (3.1), $u(|\sigma|) \subset \psi(x) \subset \varphi(x)$ and we can take $u_{(x,\sigma)} = u_\sigma$. If $\sigma \notin \Sigma_{\mathscr{U}}^k(x)$, then $|\mathscr{N}^k(\sigma)| = \bigcup\{|\tau| : \emptyset \neq \tau \subsetneq \sigma\}$ is homeomorphic to the k-sphere being the boundary of $|\sigma|$. Hence, u_σ has a continuous extension $u_{(x,\sigma)} : |\sigma| \to \varphi(x)$ because $u(|\mathscr{N}^k(\sigma)|) \subset u(|\Sigma_{\mathscr{U}}^k(x)|) \subset \psi(x) \overset{k}{\hookrightarrow} \varphi(x)$, see (3.1).

Now, whenever $x \in X$, set

$$K(x) = \bigcup \left\{ u_{(x,\sigma)}(|\sigma|) : \sigma \in \Sigma_{\mathscr{U}}^{k+1}(x) \right\}. \tag{3.5}$$

Then by (3.4), $K(x) \subset \varphi(x)$; moreover, $K(x)$ is compact because $\mathscr{U}$ is locally finite and, therefore, $\Sigma_{\mathscr{U}}^{k+1}(x)$ contains finitely many simplices. Since φ is lower locally constant, for each $U \in \mathscr{U}$, each point $x \in U$ is contained in the open set

$$W_{(x,U)} = \big\{ z \in U : K(x) \subset \varphi(z) \big\}. \tag{3.6}$$

Since X is paracompact, the cover $\{W_{(x,U)} : x \in U \in \mathscr{U}\}$ has an open star-refinement $\mathscr{V}$. So, there are maps $p : \mathscr{V} \to X$ and $\ell : \mathscr{V} \to \mathscr{U}$ such that

$$V^* \subset W_{(p(V),\ell(V))}, \quad \text{for every } V \in \mathscr{V}. \tag{3.7}$$

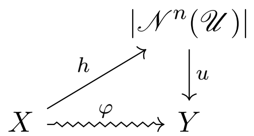
Accordingly, ℓ is a star-refining map because by (3.6), $V^* \subset W_{(p(V),\ell(V))} \subset \ell(V)$. Finally, take a map $q : \mathscr{N}(\mathscr{V}) \to \mathscr{V}$ which selects from any simplex $\sigma \in \mathscr{N}(\mathscr{V})$ a vertex $q(\sigma) \in \sigma$, and next set $\pi = p \circ q : \mathscr{N}(\mathscr{V}) \to X$. Then

$$\ell(\sigma) \in \Sigma_{\mathscr{U}}(\pi(\sigma)), \quad \sigma \in \mathscr{N}(\mathscr{V}), \tag{3.8}$$

because $\pi(\sigma) = p(q(\sigma)) \in q(\sigma) \subset \bigcup \sigma \subset \bigcap \ell(\sigma)$, see (3.3).

We complete the proof as follows. Using (3.4) and (3.8), one can define a continuous extension $v : |\mathscr{N}^{k+1}(\mathscr{V})| \to Y$ of the map $u \circ |\ell| \restriction |\mathscr{N}^k(\mathscr{V})| : |\mathscr{N}^k(\mathscr{V})| \to Y$ by $v \restriction |\sigma| = u_{(\pi(\sigma),\ell(\sigma))}$, for every $\sigma \in \mathscr{N}^{k+1}(\mathscr{V})$. This v is a $(k+1)$-skeletal selection for φ. Indeed, let $\sigma \in \Sigma_{\mathscr{V}}^{k+1}(x)$ for some $x \in X$. Then $x \in q(\sigma)$ because $q(\sigma) \in \sigma$, see (2.8). Moreover, by (3.7), $q(\sigma) \subset [q(\sigma)]^* \subset W_{(\pi(\sigma),\ell(q(\sigma)))}$. Hence, by (3.5), (3.6) and (3.8), $v(|\sigma|) = u_{(\pi(\sigma),\ell(\sigma))}(|\ell(\sigma)|) \subset K(\pi(\sigma)) \subset \varphi(x)$. □

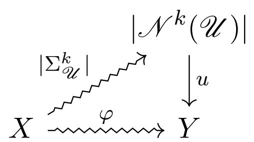
P r o o f o f T h e o r e m 3.1. According to Propositions 3.2 and 3.4, the mapping φ_n has an n-skeletal selection $u : |\mathscr{N}^n(\mathscr{U})| \to Y$, for some open cover $\mathscr{U}$ of X. Since X is paracompact and $\dim(X) \le n$, the cover $\mathscr{U}$ has an open refinement $\mathscr{V}$ with $\mathscr{N}(\mathscr{V}) = \mathscr{N}^n(\mathscr{V})$, see Remark 3.5. Let $r : \mathscr{N}(\mathscr{V}) \to \mathscr{N}^n(\mathscr{U})$ be a refining map, and $g : X \to |\mathscr{N}(\mathscr{V})|$ be a canonical map for $\mathscr{V}$ which exists because X is paracompact, see Theorems 2.1 and 2.4. Then by Corollary 2.6, the composite map $h = |r| \circ g : X \to |\mathscr{N}^n(\mathscr{U})|$ is a canonical map for $\mathscr{U}$. Finally, by (3.1) and Proposition 2.5, the composite map $f = u \circ h : X \to Y$



is a continuous selection for φ. $\square$

Remark 3.5. Let $n \ge -1$ be an integer and X be a normal space. The *order* of a cover $\mathscr{V}$ of X doesn't exceed n if $\bigcap \sigma = \emptyset$, for every $\sigma \subset \mathscr{V}$ with $\mathrm{Card}(\sigma) \ge n+2$; equivalently, if $\mathscr{N}(\mathscr{V}) = \mathscr{N}^n(\mathscr{V})$. In these terms, the covering dimension of X is at most n, written $\dim(X) \le n$, if every finite open cover of X has an open refinement $\mathscr{V}$ with $\mathscr{N}(\mathscr{V}) = \mathscr{N}^n(\mathscr{V})$. According to a result of Dowker [4, Theorem 3.5], $\dim(X) \le n$ if and only if every locally finite open cover of X has an open refinement $\mathscr{V}$ with $\mathscr{N}(\mathscr{V}) = \mathscr{N}^n(\mathscr{V})$. In particular, for a paracompact space X, we have that $\dim(X) \le n$ if and only if every open cover of X has an open refinement $\mathscr{V}$ with $\mathscr{N}(\mathscr{V}) = \mathscr{N}^n(\mathscr{V})$.

Remark 3.6. A mapping $\psi : X \rightsquigarrow Y$ is a *set-valued selection* (or *set-selection*, or *multi-selection*) for $\varphi : X \rightsquigarrow Y$ if $\psi(x) \subset \varphi(x)$, for all $x \in X$. In terms of set-valued selections, a mapping $\varphi : X \rightsquigarrow Y$ has a k-skeletal selection, $k \ge 0$, if there exists an open cover $\mathscr{U}$ of X and a continuous map $u : |\mathscr{N}^k(\mathscr{U})| \to Y$ such that the composite mapping $u \circ |\Sigma^k_{\mathscr{U}}| : X \rightsquigarrow Y$



is a set-valued selection for $\varphi : X \rightsquigarrow Y$.

4. Generating aspherical sequences of sets. For a point $y \in Y$ of a metric space (Y, d) and $\varepsilon > 0$, let

$$\mathbf{O}_\varepsilon(y) = \{z \in Y : d(z, y) < \varepsilon\}$$

be the open ε-ball centred at y; and $\mathbf{O}_\varepsilon(S) = \bigcup_{y\in S} \mathbf{O}_\varepsilon(y)$ be the ε-neighbourhood of a subset $S \subset Y$. Also, recall that a map $f : X \to Y$ is an *ε-selection* for a mapping $\varphi : X \rightsquigarrow Y$ if $f(x) \in \mathbf{O}_\varepsilon(\varphi(x))$ for every $x \in X$.

Throughout this section, $\delta : (0, +\infty) \to (0, +\infty)$ is a fixed function. To this function, we associate the sequence of iterated functions $\delta_n : (0, +\infty) \to (0, +\infty)$, $n \geq 0$, defined by

$$\delta_0(\varepsilon) = \varepsilon \quad \text{and} \quad \delta_{n+1}(\varepsilon) = \delta(\delta_n(\varepsilon)). \tag{4.1}$$

Proposition 4.1. *Let (Y, d) be a metric space and $S_0 \subset S_1 \subset \cdots \subset S_n \subset Y$ be such that $\mathbf{O}_{\delta(\varepsilon)}(y) \cap S_k \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap S_{k+1}$, for every $y \in Y$ and $k < n$. Then*

$$\mathbf{O}_{\delta_{n-k}(\varepsilon)}(y) \cap S_k \overset{k}{\hookrightarrow} \mathbf{O}_{\delta_{n-k-1}(\varepsilon)}(y) \cap S_{k+1}, \quad k < n. \tag{4.2}$$

P r o o f. Follows from the fact that $\delta_{n-k}(\varepsilon) = \delta\big(\delta_{n-k-1}(\varepsilon)\big)$, see (4.1). $\square$

We now have the following "local" version of Theorem 3.1.

Theorem 4.2. *Let (Y, d) be a metric space, X be a paracompact space with $\dim(X) \leq n$, and $\psi_k : X \rightsquigarrow Y$, $0 \leq k \leq n$, be lower locally constant mappings such that $\mathbf{O}_{\delta(\varepsilon)}(y) \cap \psi_k(x) \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap \psi_{k+1}(x)$ for every $x \in X$, $y \in Y$ and $k < n$. Then for each continuous $\delta_n(\varepsilon)$-selection $g : X \to Y$ for ψ_0, there is a continuous selection $f : X \to Y$ for ψ_n with $d(f(x), g(x)) < \varepsilon$, for all $x \in X$.*

P r o o f. Let $g : X \to Y$ be a continuous $\delta_n(\varepsilon)$-selection for ψ_0. Next, for each $k \leq n$, define a set-valued mapping φ_k by $\varphi_k(x) = \mathbf{O}_{\delta_{n-k}(\varepsilon)}(g(x)) \cap \psi_k(x)$, $x \in X$. Since g is a $\delta_n(\varepsilon)$-selection for ψ_0, the mapping φ_0 is nonempty-valued and, according to (4.2), so is each φ_k, $k \leq n$. In fact, by (4.2), the resulting sequence of mappings $\varphi_k : X \rightsquigarrow Y$, $0 \leq k \leq n$, is aspherical. Moreover, each φ_k is lower locally constant because so are ψ_k and the mapping $x \to \mathbf{O}_{\delta_{n-k}(\varepsilon)}(g(x))$, $x \in X$ (see Proposition 5.1). Hence, by Theorem 3.1, φ_n has a continuous selection $f : X \to Y$ because X is a paracompact space with $\dim(X) \leq n$. Evidently, f is a selection for ψ_n and, by (4.1), $f(x) \in \mathbf{O}_{\delta_{n-n}(\varepsilon)}(g(x)) = \mathbf{O}_{\delta_0(\varepsilon)}(g(x)) = \mathbf{O}_\varepsilon(g(x))$, $x \in X$. $\square$

We conclude this section with the following two applications of Theorem 4.2 which will provide the main interface between selections for l.s.c. mappings and Theorem 3.1, see Theorems 5.3 and 5.4.

Corollary 4.3. *Let E be a normed space, and $\emptyset \neq S \subset T \subset E$ be such that $S \overset{k}{\hookrightarrow} T$ and $\mathbf{O}_{\delta(\varepsilon)}(y) \cap S \overset{i}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap S$, for every $y \in E$ and $0 \leq i < k$. Then $\mathbf{O}_{\delta_k(\varepsilon)}(S) \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(T)$.*

P r o o f. Let $\ell : \mathbb{S}^k \to \mathbf{O}_{\delta_k(\varepsilon)}(S)$ be a continuous map from the k-sphere $\mathbb{S}^k$. Consider the constant mappings $\psi_i(x) = S$, $x \in \mathbb{S}^k$ and $i \leq k$. Then ℓ is a continuous $\delta_k(\varepsilon)$-selection for ψ_0, and $\mathbf{O}_{\delta(\varepsilon)}(y) \cap \psi_i(x) \overset{i}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap \psi_{i+1}(x)$ for every $x \in \mathbb{S}^k$, $y \in E$ and $i < k$. Hence, by Theorem 4.2, there exists a continuous $q : \mathbb{S}^k \to S$ with $\|q(x) - \ell(x)\| < \varepsilon$, for every $x \in \mathbb{S}^k$. Let h_1 be the linear homotopy between ℓ and q, i.e. $h_1(x,t) = tq(x) + (1-t)\ell(x)$, whenever $(x,t) \in \mathbb{S}^k \times [0,1]$. Then, $h_1(\mathbb{S}^k \times [0,1]) \subset \mathbf{O}_\varepsilon(S) \subset \mathbf{O}_\varepsilon(T)$. Also, let $h_2 : \mathbb{S}^k \times [0,1] \to T$ be a homotopy between q and a constant map, which exists because $S \overset{k}{\hookrightarrow} T$. Finally, take h to be the homotopy obtained by combining h_1 and h_2. Then h is a homotopy of ℓ with a constant map over a subset of $\mathbf{O}_\varepsilon(T)$. $\square$

Corollary 4.4. *Let E be a normed space, and $\emptyset \neq S \subset T \subset E$ be such that $\mathbf{O}_{\delta(\varepsilon)}(y) \cap T \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap T$ and $\mathbf{O}_{\delta(\varepsilon)}(y) \cap S \overset{i}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap S$, for every $y \in E$ and $0 \leq i < k$. Define functions*

$$\eta(\varepsilon) = \delta(\varepsilon)/2 \quad \text{and} \quad \lambda(\varepsilon, \mu) = \delta_k\big(\min\{\eta(\varepsilon), \mu\}\big), \ \varepsilon, \mu > 0. \tag{4.3}$$

Then $\mathbf{O}_{\eta(\varepsilon)}(y) \cap \mathbf{O}_{\lambda(\varepsilon,\mu)}(S) \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap \mathbf{O}_\mu(T)$, for every $y \in E$.

P r o o f. Let $\ell : \mathbb{S}^k \to \mathbf{O}_{\eta(\varepsilon)}(y) \cap \mathbf{O}_{\lambda(\varepsilon,\mu)}(S)$ be a continuous map for some $y \in E$. Then, precisely as in the previous proof, there exists a continuous map $q : \mathbb{S}^k \to S$ such that $\|q(x) - \ell(x)\| < \min\{\eta(\varepsilon), \mu\}$, for every $x \in \mathbb{S}^k$. Since $\eta(\varepsilon) = \delta(\varepsilon)/2$, see (4.3), just like before, using a linear homotopy, we get that ℓ and q are homotopic in $\mathbf{O}_{\delta(\varepsilon)}(y) \cap \mathbf{O}_\mu(S)$. Moreover q is homotopic to a constant map in $\mathbf{O}_\varepsilon(y) \cap T$ because $q : \mathbb{S}^k \to \mathbf{O}_{\delta(\varepsilon)}(y) \cap S \subset \mathbf{O}_{\delta(\varepsilon)}(y) \cap T \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(y) \cap T$. Accordingly, ℓ is homotopic to a constant map in $\mathbf{O}_\varepsilon(y) \cap \mathbf{O}_\mu(T)$. $\square$

5. Selections for equi-LC^n-valued mappings. In this section, to each $\Phi : X \rightsquigarrow Y$ we associate the mapping $\overline{\Phi} : X \to \mathscr{F}(Y)$ defined by $\overline{\Phi}(x) = \overline{\Phi(x)}$, $x \in X$. Moreover, for a pair of mappings $\Phi, \Psi : X \rightsquigarrow Y$, we will use $\Phi \wedge \Psi$ to denote their intersection, i.e. the mapping which assigns to each $x \in X$ the set $[\Phi \wedge \Psi](x) = \Phi(x) \cap \Psi(x)$. Finally, to each $\varepsilon > 0$ and a mapping $\Phi : X \rightsquigarrow Y$ in a metric space (Y, d), we will associate the mapping $\mathbf{O}[\Phi, \varepsilon] : X \rightsquigarrow Y$ defined by

$$\mathbf{O}[\Phi, \varepsilon](x) = \mathbf{O}_\varepsilon(\Phi(x)), \quad x \in X. \tag{5.1}$$

This convention will be also used in an obvious manner for usual maps $f : X \to Y$ considering f as the singleton-valued mapping $x \to \{f(x)\}$, $x \in X$. In these terms, for maps $f, g : X \to Y$ and $\varepsilon, \mu > 0$, we have that f is a μ-selection for $\Phi : X \rightsquigarrow Y$ with $d(f(x), g(x)) < \varepsilon$ for every $x \in X$, if and only if f is a selection for the mapping $\mathbf{O}[\Phi, \mu] \wedge \mathbf{O}[g, \varepsilon]$.

The following two constructions are due to Michael, see [11, Lemma 11.3] and [11, Proof that Lemma 5.1 implies Theorem 4.1, page 569]. They reduce the selection problem for l.s.c. mappings to that of lower locally constant mappings. For completeness, we sketch their proofs following the original arguments in [11].

Proposition 5.1. *Let (Y, d) be a metric space, $\Phi : X \rightsquigarrow Y$ be l.s.c. and $\varepsilon > 0$. Then the mapping $\mathbf{O}[\Phi, \varepsilon] : X \rightsquigarrow Y$ is lower locally constant.*

Proof. Take $x_0 \in X$ and a compact set $K \subset \mathbf{O}[\Phi, \varepsilon](x_0) = \mathbf{O}_\varepsilon(\Phi(x_0))$. Then $K \subset \mathbf{O}_\delta(S)$ for some finite subset $S \subset \Phi(x_0)$ and some $\delta > 0$ with $\delta < \varepsilon$. Since Φ is l.s.c., $U = \bigcap_{y \in S} \Phi^{-1}[\mathbf{O}_{\varepsilon-\delta}(y)]$ is an open set containing x_0. Moreover, $x \in U$ implies $S \subset \mathbf{O}_{\varepsilon-\delta}(\Phi(x))$ and, therefore, $K \subset \mathbf{O}_\delta(S) \subset \mathbf{O}_\varepsilon(\Phi(x)) = \mathbf{O}[\Phi, \varepsilon](x)$. $\square$

Proposition 5.2. *Let (Y, d) be a complete metric space, $\xi : (0, +\infty) \to (0, +\infty)$ be a function with $\xi(\varepsilon) \leq \varepsilon$, and $\Phi : X \rightsquigarrow Y$ be a mapping such that for each continuous $\xi(\varepsilon)$-selection $g : X \to Y$ for Φ and $\mu > 0$, then mapping $\mathbf{O}[\Phi, \mu] \wedge \mathbf{O}[g, \varepsilon]$ has a continuous selection. Then for every continuous $\xi(\varepsilon/2)$-selection $g : X \to Y$ for Φ, the mapping $\overline{\Phi} \wedge \mathbf{O}[g, \varepsilon]$ also has a continuous selection.*

Proof. Let $f_1 = g : X \to Y$ be a continuous $\xi\left(2^{-1}\varepsilon\right)$-selection for Φ. By condition with $\mu = \xi\left(2^{-2}\varepsilon\right)$, the mapping $\mathbf{O}\left[\Phi, \xi\left(2^{-2}\varepsilon\right)\right] \wedge \mathbf{O}[f_1, 2^{-1}\varepsilon]$ has a continuous selection $f_2 : X \to Y$. Thus, by induction, there exists a sequence of continuous maps $f_n : X \to E$ such that f_{n+1} is a selection for $\mathbf{O}\left[\Phi, \xi\left(2^{-(n+1)}\varepsilon\right)\right] \wedge \mathbf{O}\left[f_n, 2^{-n}\varepsilon\right]$, for every $n \in \mathbb{N}$. Then the sequence $\{f_n : n \in \mathbb{N}\}$ is uniformly Cauchy because $d(f_{n+1}(x), f_n(x)) < 2^{-n}\varepsilon$, $x \in X$. Hence, it converges uniformly to some continuous map $f : X \to Y$ because (Y, d) is complete. Since $\xi\left(2^{-n}\varepsilon\right) \leq 2^{-n}\varepsilon$, each f_n is a $2^{-n}\varepsilon$-selection for Φ being a selection for $\mathbf{O}\left[\Phi, \xi\left(2^{-n}\varepsilon\right)\right]$, see (5.1). Hence, $d(f(x), \Phi(x)) = 0$, for each $x \in X$. Finally, we also have that

$$d(f(x), g(x)) \leq \sum_{n=1}^{\infty} d(f_{n+1}(x), f_n(x)) < \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon, \quad x \in X. \qquad \square$$

Let $n \geq -1$. A family $\mathscr{S}$ of subsets of a metric space (Y,d) is called *uniformly equi-LC^n* [11] if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for every $S \in \mathscr{S}$, every continuous map of the k-sphere ($k \leq n$) in S of diameter $< \delta(\varepsilon)$ can be extended to continuous map of the $(k+1)$-ball into a subset of S of diameter $< \varepsilon$. Just as in the case of equi-LC^n families, a family $\mathscr{S}$ is uniformly equi-LC^{-1} iff it consists of nonempty sets. For such a family $\mathscr{S}$, by replacing $\delta(\varepsilon)$ with $\dfrac{\delta(\varepsilon)}{2}$, we get that $\mathscr{S}$ is uniformly equi-LC^n if there exists a function $\delta : (0,+\infty) \to (0,+\infty)$ such that

$$\mathbf{O}_{\delta(\varepsilon)}(y) \cap S \overset{k}{\hookrightarrow} \mathbf{O}_{\varepsilon}(y) \cap S, \quad \text{for every } S \in \mathscr{S},\ y \in Y \text{ and } 0 \leq k \leq n. \tag{5.2}$$

Evidently, we may further assume that $\delta(\varepsilon) \leq \varepsilon$, for every $\varepsilon > 0$. Based on this and the results of the previous section, we now have the following two applications of Theorem 3.1. The first one gives a simplified proof of [11, Theorem 4.1].

Theorem 5.3. *Let E be a Banach space and $\mathscr{S}$ be a uniformly equi-LC^n family of subsets of E. Then there exists a function $\gamma : (0,+\infty) \to (0,+\infty)$ with the following property: If X is a paracompact space with* $\dim(X) \leq n+1$, *$\Phi : X \to \mathscr{S}$ is l.s.c. and $g : X \to E$ is a continuous $\gamma(\varepsilon)$-selection for Φ, then $\overline{\Phi} \wedge \mathbf{O}[g,\varepsilon]$ has a continuous selection.*

Proof. Let $\delta(\varepsilon) \leq \varepsilon$ be as in (5.2) with respect to the family $\mathscr{S}$. Also, let $\lambda(\varepsilon,\mu)$ and $\eta(\varepsilon)$ be as in (4.3) applied to this particular function $\delta(\varepsilon)$. Next, define functions $\eta_k(\varepsilon)$ and $\lambda_k(\varepsilon,\mu)$, $0 \leq k \leq n+1$, by

$$\begin{cases} \eta_{n+1}(\varepsilon) = \varepsilon & \text{and} \quad \eta_k(\varepsilon) = \eta\big(\eta_{k+1}(\varepsilon)\big) \\ \lambda_{n+1}(\varepsilon,\mu) = \mu & \text{and} \quad \lambda_k(\varepsilon,\mu) = \lambda\big(\eta_{k+1}(\varepsilon), \lambda_{k+1}(\varepsilon,\mu)\big). \end{cases} \tag{5.3}$$

Then $\gamma(\varepsilon) = \eta_0(\varepsilon/2)$ is as required. Indeed, let X and Φ be as in the theorem. Applying Proposition 5.2 with $\xi(\varepsilon) = \eta_0(\varepsilon)$, it will be now sufficient to show that for every $\mu > 0$ and a continuous $\eta_0(\varepsilon)$-selection $g : X \to E$ for Φ, the mapping $\mathbf{O}[\Phi,\mu] \wedge \mathbf{O}[g,\varepsilon]$ has a continuous selection. To this end, for every $0 \leq k \leq n+1$, let $\varphi_k = \mathbf{O}[\Phi,\lambda_k(\varepsilon,\mu)] \wedge \mathbf{O}[g,\eta_k(\varepsilon)]$. According to Proposition 5.1, each φ_k is lower locally constant. Moreover, the resulting sequence of mappings $\varphi_k : X \rightsquigarrow E$, $0 \leq k \leq n+1$, is aspherical because by (5.2) and Corollary 4.4,

$$\begin{aligned} \varphi_k(x) &= \mathbf{O}_{\lambda_k(\varepsilon,\mu)}(\Phi(x)) \cap \mathbf{O}_{\eta_k(\varepsilon)}(g(x)) \\ &= \mathbf{O}_{\lambda(\eta_{k+1}(\varepsilon),\lambda_{k+1}(\varepsilon,\mu))}(\Phi(x)) \cap \mathbf{O}_{\eta(\eta_{k+1}(\varepsilon))}(g(x)) \\ &\overset{k}{\hookrightarrow} \mathbf{O}_{\lambda_{k+1}(\varepsilon,\mu)}(\Phi(x)) \cap \mathbf{O}_{\eta_{k+1}(\varepsilon)}(g(x)) = \varphi_{k+1}(x), \quad k \leq n. \end{aligned}$$

Hence by Theorem 3.1,

$$\varphi_{n+1} = \mathbf{O}[\Phi, \lambda_{n+1}(\varepsilon, \mu)] \wedge \mathbf{O}[g, \eta_{n+1}(\varepsilon)] = \mathbf{O}[\Phi, \mu] \wedge \mathbf{O}[g, \varepsilon]$$

has a continuous selection. The proof is complete. □

Theorem 5.4. *Let X be a paracompact space with $\dim(X) \leq n+1$, E be a Banach space, and $\Phi_k : X \rightsquigarrow E$, $0 \leq k \leq n+1$, be a sequence of l.s.c. mappings such that $\{\Phi_k(x) : x \in X\}$ is uniformly equi-LC^k and $\Phi_k \overset{k}{\hookrightarrow} \Phi_{k+1}$ for every $k \leq n$. Then Φ_{n+1} has a continuous ε-selection, for every $\varepsilon > 0$.*

Proof. According to (5.2) and Corollary 4.3, for each $0 \leq k \leq n$ there exists a function $\delta_k : (0, +\infty) \to (0, +\infty)$ such that

$$\mathbf{O}_{\delta_k(\varepsilon)}(\Phi_k(x)) \overset{k}{\hookrightarrow} \mathbf{O}_\varepsilon(\Phi_{k+1}(x)), \quad x \in X. \tag{5.4}$$

Next, define functions $\gamma_k : (0, +\infty) \to (0, +\infty)$, $0 \leq k \leq n+1$, by

$$\gamma_{n+1}(\varepsilon) = \varepsilon \quad \text{and} \quad \gamma_k(\varepsilon) = \delta_k(\gamma_{k+1}(\varepsilon)), \quad k \leq n. \tag{5.5}$$

Finally, define a sequence of mappings $\varphi_k : X \rightsquigarrow E$ by $\varphi_k = \mathbf{O}[\Phi_k, \gamma_k(\varepsilon)]$. It now follows from (5.4) and (5.5) that

$$\begin{aligned}\varphi_k(x) = \mathbf{O}_{\gamma_k(\varepsilon)}(\Phi_k(x)) &= \mathbf{O}_{\delta_k(\gamma_{k+1}(\varepsilon))}(\Phi_k(x)) \\ &\overset{k}{\hookrightarrow} \mathbf{O}_{\gamma_{k+1}(\varepsilon)}(\Phi_{k+1}(x)) = \varphi_{k+1}(x), \quad k \leq n.\end{aligned}$$

Hence, the mappings φ_k, $0 \leq k \leq n+1$, form an aspherical sequence. Moreover, by Proposition 5.1, each φ_k is lower locally constant. Since $\dim(X) \leq n+1$, by Theorem 3.1, $\varphi_{n+1} = \mathbf{O}[\Phi_{n+1}, \gamma_{n+1}(\varepsilon)] = \mathbf{O}[\Phi_{n+1}, \varepsilon]$ has a continuous selection, i.e. Φ_{n+1} has a continuous ε-selection. □

We are also ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let X, Y, $\mathscr{S} \subset \mathscr{F}(Y)$ and $\Phi : X \to \mathscr{S}$ be as in that theorem. Since $\mathscr{S}$ is equi-LC^n, by [5, Theorem 1] (see, also, [11, Proposition 2.1]), $\bigcup \mathscr{S}$ can be embedded into a Banach space E such that $\mathscr{S} \subset \mathscr{F}(E)$ is uniformly equi-LC^n. Then by Theorem 5.4, applied with $\Phi_k = \Phi$, $0 \leq k \leq n+1$, the mapping Φ has a continuous ε-selection, for every $\varepsilon > 0$. Hence, by Theorem 5.3, $\overline{\Phi} = \Phi$ has a continuous selection as well. □

Another application of Theorems 5.3 and 5.4 is the following generalisation of Theorem 1.1, see [14], [12, Theorem 7.2] and [9, Corollary 7.10].

Corollary 5.5. *Let X be a paracompact space with* $\dim(X) \leq n+1$, *Y be a completely metrizable space, and* $\Phi_k : X \to \mathscr{F}(Y)$, $0 \leq k \leq n+1$, *be a sequence of l.s.c. mappings such that* $\Phi_k \overset{k}{\hookrightarrow} \Phi_{k+1}$ *for* $k \leq n$, *while each family* $\{\Phi_k(x) : x \in X\}$, *for* $k \leq n+1$, *is equi-*LC^k. *Then* Φ_{n+1} *has a continuous selection.*

P r o o f. As before, the proof is reduced to the case when $Y = E$ is a Banach space, and each family $\{\Phi_k(x) : x \in X\} \subset \mathscr{F}(E)$, $0 \leq k \leq n+1$, is uniformly equi-LC^k. Then by Theorem 5.4, Φ_{n+1} has a continuous ε-selection, for every $\varepsilon > 0$. Finally, by Theorem 5.3, Φ_{n+1} also has a continuous selection. □

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