Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **44** (2018), 177–186

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON MONOTONE ORTHOCOMPACTNESS

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Communicated by V. Valov

Abstract. Junnila and Künzi defined monotone orthocompactness via transitive neighbornets, and proved that monotonically normal, monotonically orthocompact spaces must have an ortho-base. Answering one of Junnila and Künzi's questions, Shouli and Yuming claimed to have provided an example of a monotonically orthocompact space without an ortho-base. We define a version of monotone orthocompactness via interior-preserving open refinements and show that it is a strictly weaker property than monotone orthocompactness of Junnila and Künzi, and we point out an error in the paper by Shouli and Yuming, thereby indicating that the question of Junnila and Künzi appears to remain open.

1. Introduction. Monotone versions of many covering properties have been defined and studied in recent years by a number of authors. For example, a space is called monotonically compact if there is an operator r assigning to each open cover $\mathcal{U}$ a finite open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is

2010 *Mathematics Subject Classification:* 54D20, 54D70, 54F05, 54B10, 54G20, 54D65.
Key words: monotone covering properties, monotonically orthocompact, orthobase, transitive neighbornet, GO-space, regressive functions on ω_1, NSR base.

an open cover refining $\mathcal{U}$ then $r(\mathcal{V})$ refines $r(\mathcal{U})$. The monotone version of a covering property is usually much stronger than the original version, for example Gruenhage proved that monotonically compact T_2 spaces are metrizable [3, 7], and Gartside and Moody proved that monotonically paracompact spaces are protometrizable [4].

A covering property is often defined by the requirement that any open cover $\mathcal{U}$ has either a subcover or an open refinement satisfying a certain property $\mathcal{P}$. (For example, in compact spaces we find finite subcovers, and in paracompact spaces we find locally-finite open refinements.) We obtain a monotone version if we require the existence of a monotone operator r such that $r(\mathcal{U})$ is an open cover *refining* $\mathcal{U}$ (denoted $r(\mathcal{U}) \prec \mathcal{U}$) and having the required property $\mathcal{P}$, where monotonicity means that if $\mathcal{V} \prec \mathcal{U}$ then $r(\mathcal{V}) \prec r(\mathcal{U})$. For example, in this way one may define monotonically compact, monotonically Lindelöf [2, 6], or monotonically metacompact spaces [1, 11] (when the property $\mathcal{P}$ is "finite", "countable", or "point-finite", respectively).

But different characterizations of the same covering property may generate different monotone versions of that covering property. Gartside and Moody's definition of monotone paracompactness requires that $r(\mathcal{U})$ is an open star-refinement of $\mathcal{U}$ (and does not specify that $r(\mathcal{U})$ be locally-finite). If we "monotonize" the usual definition of paracompactness – by requiring that $r(\mathcal{U})$ be a locally-finite open refinement of $\mathcal{U}$ (or, as we also say, that r is a monotone locally-finite open operator) – then we obtain a larger class of spaces [12, Corollary 1.7, Example 2.1].

In the present note we compare two versions of monotone orthocompactness. We give distinguishing examples, pose a question about a certain partial order related to ω_1, and we indicate an error in [13], thus showing that a question of Junnila and Künzi, whether each monotonically orthocompact space must have an ortho-base, appears to remain open. All spaces considered are T_1 topological spaces.

2. Two versions of monotone orthocompactness. Recall that a family $\mathcal{A}$ of subsets of a topological space X is called interior-preserving if $\mathrm{Int}(\cap\mathcal{A}_1) = \cap\{\mathrm{Int}(A) : A \in \mathcal{A}_1\}$ whenever $\mathcal{A}_1 \subseteq \mathcal{A}$. For open families $\mathcal{A}$ the above condition reduces to the requirement that $\cap\mathcal{A}_1$ is open whenever $\mathcal{A}_1 \subseteq \mathcal{A}$. A space X is called orthocompact if every open cover has an open interior-preserving refinement.

Following the scheme outlined in the introduction we come up with the following:

Definition 2.1. *A space X is said to be* monotonically orthocompact via open refinements *if it has an operator $r : \mathcal{C} \to \mathcal{C}$ (called a monotone interior-preserving open operator), where $\mathcal{C}$ is the set of all open covers of X, such that $r(\mathcal{U})$ is interior-preserving and $r(\mathcal{U}) \prec \mathcal{U}$ for every $\mathcal{U} \in \mathcal{C}$, and if $\mathcal{V} \prec \mathcal{U}$ then $r(\mathcal{V}) \prec r(\mathcal{U})$.*

Junnila and Künzi defined monotone orthocompactness via transitive neighbornets. A neighbornet [8] is a relation R on a space X (i.e. $R \subseteq X \times X$) such that the set $R(x) = \{y \in X : xRy\}$ is a (not necessarily open) neighborhood of x, i.e. $x \in \operatorname{Int}(R(x))$, for each $x \in X$. The usual notion of transitivity applies, i.e. a neighbornet R is transitive if xRy and yRz implies xRz. Equivalently, $R(y) \subseteq R(x)$ whenever $y \in R(x)$. For neighbornets Q, R the condition $Q \subseteq R$ is equivalent to the condition $Q(x) \subseteq R(x)$ for all $x \in X$.

Definition 2.2 ([9, Definition 1])**.** *A topological space X is called monotonically orthocompact provided there is an operator $T : \mathcal{C} \to \mathcal{T}$, where $\mathcal{C}$ is the set of all open covers of X and $\mathcal{T}$ is the set of all transitive neighbornets of X such that $\{T(\mathcal{U})(x) : x \in X\} \prec \mathcal{U}$ and $T(\mathcal{V}) \subseteq T(\mathcal{U})$ whenever $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{V} \prec \mathcal{U}$.*

We will reserve the term "monotonically othocompact" for the property in Definition 2.2, although we may use for emphasis the term monotonically orthocompact (via transitive neighbornets), or monotonically orthocompact (via transitive ONA's) which is explained in more detail below.

If R is a transitive neighbornet then the cover $\{R(x) : x \in X\}$ is open and interior-preserving [8, Corollary 3.15], [13, Corollary 2.8]. Indeed, for any $Y \subseteq X$ and any $y \in G = \cap\{R(x) : x \in Y\}$ we have by transitivity that $R(y) \subseteq G$, thus G is open. It follows that every monotonically orthocompact space is monotonically orthocompact via open refinements, i.e. it has a monotone interior-preserving open operator as in Definition 2.1. Clearly every monotonically orthocompact via open refinements space is orthocompact.

Recall that an open neighborhood assignment (ONA for short) for a space X is a family $\mathcal{O} = \{O(x) : x \in X\}$ where each $O(x)$ is an open neighborhood of x. If R is a transitive neighbornet then $\{R(x) : x \in X\}$ is an ONA with the additional property that if $y \in R(x)$ then $R(y) \subseteq R(x)$. Conversely, let us call an ONA $\mathcal{O}$ *transitive* if $y \in O(x)$ implies $O(y) \subseteq O(x)$; then the relation R defined by xRy iff $y \in O(x)$ is a transitive neighbornet. Clearly Definition 2.2 is equivalent to the following:

Definition 2.3. *A topological space X is called monotonically orthocompact provided there is an operator $t : \mathcal{C} \to \mathcal{N}$, where $\mathcal{C}$ is the set of all open covers* *of X and $\mathcal{N}$ is the set of all transitive ONAs of X such that $\{t(\mathcal{U})(x) : x \in X\} \prec \mathcal{U}$ and $t(\mathcal{V})(x) \subseteq t(\mathcal{U})(x)$ for all x, whenever $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{V} \prec \mathcal{U}$.*

We provide a couple of examples of monotonically orthocompact via open refinements spaces that are not monotonically orthocompact. For the first of these we use the well-known observation that every point-finite open cover is interior-preserving, and therefore every monotonically metacompact space is monotonically orthocompact via open refinements. We would also use the following.

Theorem 2.4 ([9, Proposition 3]). *A space X is protometrizable if and only if it is monotonically orthocompact and monotonically normal.*

Recall that X is protometrizable if it is paracompact and has an ortho-base, where a base $\mathcal{B}$ is an *ortho-base* if for every $\mathcal{B}' \subseteq \mathcal{B}$ either $\cap\mathcal{B}'$ is open, or else $\cap\mathcal{B}'$ is a singleton $\{x\}$ and $\mathcal{B}'$ is a base at x, [10]. It is shown in [9, Corollary 1] that each space with an ortho-base is (hereditarily) monotonically orthocompact.

Example 2.5. A linearly ordered topological space (LOTS) X that is monotonically orthocompact via open refinements but not monotonically orthocompact (via transitive neighbornets).

Proof. Let $A(\omega) = \omega + 1 = [0, \omega]$ be a convergent sequence and $L(\omega_1) = [0, \omega_1]$ (with all countable ordinals isolated) be the one-point Lindelöfication of a discrete set of size ω_1. Let X denote the quotient space obtained by identifying the non-isolated point of $A(\omega)$ with the non-isolated point of $L(\omega_1)$. It is shown in [12, Example 2.1] that X is a LOTS (hence monotonically normal) with a monotone locally-finite open operator, that is not protometrizable. It follows that X is monotonically metacompact, hence monotonically orthocompact via open refinements, but X is not monotonically orthocompact, by Theorem 2.4. □

A space X is called a *lob space* if each point has a local base which is linearly ordered by inclusion.

Proposition 2.6. *Every monotonically orthocompact space is a lob space.*

Proof. This is a direct consequence of the definition of monotone orthocompactness and [9, Proposition 1] which states that a space X is a lob space if and only if there is an operator P from the set $\mathcal{C}$ of all open covers to the set of all (not necessarily transitive) neighbornets such that $\{P(\mathcal{U})(x) : x \in X\} \prec \mathcal{U}$ and $P(\mathcal{V}) \subseteq P(\mathcal{U})$ whenever $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{V} \prec \mathcal{U}$. □

Although the above proposition has a trivial proof, we stated it as it provides a more direct way to show that the space X in Example 2.5 is not monotonically orthocompact. Indeed, it is easily seen that the non-isolated point of X has no local base that is linearly ordered. We would use Proposition 2.6 in a similar way in the last section.

It follows from Theorem 2.4 that neither the Sorgenfrey line nor the ordinal space ω_1 are monotonically orthocompact, [9]. The following proposition implies that the Sorgenfrey line is monotonically orthocompact via open refinements. Recall that a GO (generalized ordered) space is a topological space X with a linear order with respect to which there is a base consisting of order-convex sets.

Proposition 2.7. *Let X be a GO-space with a σ-closed-and-discrete dense set D and such that $[x, \to)$ is open for each $x \in X$. Then X is monotonically orthocompact via open refinements.*

Proof. Write $D = \cup_{n<\omega} D_n$ where each D_n is closed-and-discrete, and $D_n \subseteq D_{n+1}$. Given any open cover $\mathcal{U}$ and any non-isolated x let $n(x,\mathcal{U}) = \min\{n : \exists U \in \mathcal{U}, \exists d \in D_n, d > x, [x, d) \subseteq U\}$. Let $e(x,\mathcal{U}) = \inf((x, \to) \cap D_{n(x,\mathcal{U})})$ (where inf is taken with respect to the linear order on X, and $e(x,\mathcal{U})$ may be a gap). Clearly $e(x,\mathcal{U}) > x$ since if $e(x,\mathcal{U}) = x$ then $x \in \overline{((x, \to) \cap D_{n(x,\mathcal{U})})}$ contradicting that each D_n is closed-and-discrete. Let Z denote the set of all isolated points of X and $r(\mathcal{U}) = \{[x, e(x,\mathcal{U})) : x \in X \setminus Z\} \cup \{\{x\} : x \in Z\}$. Then $r(\mathcal{U})$ is an open cover of X and $r(\mathcal{U}) \prec \mathcal{U}$. It is easily seen (using that $D_n \subseteq D_{n+1}$) that the operator r is monotone. We will show that $r(\mathcal{U})$ is interior-preserving. If not, then there is a non-isolated y and points x_k such that $y \in \cap_{k<\omega}[x_k, e(x_k,\mathcal{U}))$ and $y = \inf\{e(x_k,\mathcal{U}) : k < \omega\}$ with $e(x_{k+1},\mathcal{U}) < e(x_k,\mathcal{U})$ for all k. Then $x_{k+1} < x_k$ (for all k), for otherwise $x_{k+1} \in [x_k, e(x_k,\mathcal{U}))$ which would imply that $n(x_{k+1},\mathcal{U}) \le n(x_k,\mathcal{U})$ and $e(x_{k+1},\mathcal{U}) \ge e(x_k,\mathcal{U})$. There is m such that $(x_1, x_0) \cap D_m \neq \emptyset$. Since for each $k \ge 1$ there is $U_k \in \mathcal{U}$ with $(x_1, x_0) \subseteq [x_k, y) \subseteq [x_k, e(x_k,\mathcal{U})) \subseteq U_k$ if follows that $n(x_k,\mathcal{U}) \le m$ for all $k \ge 1$. Therefore $e(x_k,\mathcal{U}) = \inf((e(x_k,\mathcal{U}), \to) \cap D_m)$ for each $k \ge 1$, and $y = \inf((y, \to) \cap D_m)$, contradicting that D_m is closed-and-discrete. □

3. Is ω_1 monotonically orthocompact via open refinements?

We do not know if ω_1 is monotonically orthocompact via open refinements. It turns out that this question could be restated in purely order-theoretic terms, as follows.

Let $\mathcal{F}$ be the set of all functions f from ω_1 into ω_1 that are: (a) regressive, i.e. $f(\alpha) < \alpha$ whenever $0 < \alpha < \omega_1$, and (b) increasing, i.e. $f(\alpha) \le f(\beta)$ if $0 \le \alpha \le \beta < \omega_1$. Define a partial order $\sqsubseteq$ on $\mathcal{F}$ by $f \sqsubseteq g$ if $f(\alpha) \le g(\alpha)$ for *all* $\alpha < \omega_1$. Let $\mathcal{K}$ be the subset of $\mathcal{F}$ consisting of functions with finite range. That is, $\mathcal{K} = \{f \in \mathcal{F} : |f(\alpha) : \alpha < \omega_1| < \aleph_0\}$.

Theorem 3.1. *The following statements are either both true, or both false:*

(i) ω_1 *(with the usual order topology) is monotonically orthocompact via open refinements,*

(ii) there is a $\sqsubseteq$*-increasing map* $\psi : \mathcal{F} \to \mathcal{K}$, *i.e. if* $f \sqsubseteq g$ *then* $\psi(f) \sqsubseteq \psi(g)$, *and such that* $\psi(f) \sqsupseteq f$ *for all* $f \in \mathcal{F}$.

Proof. Assume that ω_1 were monotonically orthocompact (via open refinements) with a monotone interior-preserving open operator r. For each $f \in \mathcal{F}$ let $\mathcal{U}_f = \{\{0\}\} \cup \{(f(\alpha), \alpha] : 0 < \alpha < \omega_1\}$, an open cover of ω_1. For each $\alpha > 0$ define $\psi(f)(\alpha) = \min\{\gamma < \alpha : \text{there is } U \in r(\mathcal{U}_f),\ (\gamma, \alpha] \subseteq U\}$. Define $\psi(f)(0) = 0$. It is easily verified that $\psi(f) \in \mathcal{F}$, i.e. $\psi(f)$ is increasing and regressive. If $0 < \alpha$ then there is $U \in r(\mathcal{U}_f)$ and β such that $(\psi(f)(\alpha), \alpha] \subseteq U \subseteq (f(\beta), \beta]$. Hence $f(\beta) \le \psi(f)(\alpha)$ and $\alpha \le \beta$ which in turn implies $f(\alpha) \le f(\beta)$, to conclude $f(\alpha) \le \psi(f)(\alpha)$. If $f \sqsubseteq g$ then $\mathcal{U}_g \prec \mathcal{U}_f$ and $r(\mathcal{U}_g) \prec r(\mathcal{U}_f)$ which in turn implies $\psi(f) \sqsubseteq \psi(g)$.

Claim. $\psi(f) \in \mathcal{K}$. Proof. Suppose not, and pick α_k for $k < \omega$ with $\alpha_k < \alpha_{k+1}$ and $0 < \psi(f)(\alpha_k) < \psi(f)(\alpha_{k+1})$. Let $\beta = \sup_{k<\omega} \psi(f)(\alpha_k)$. There is k such that $\alpha_k > \beta$, for otherwise $\psi(f)(\alpha_k) \le \psi(f)(\beta) < \beta$ for all k, implying $\beta \le \psi(f)(\beta) < \beta$. Thus (removing finitely many α_k) we may assume that $\alpha_k > \beta$ for all k and hence $\beta \in \cap_{k<\omega}(\psi(f)(\alpha_k), \alpha_k]$. For each k there is an open $U_k \in r(\mathcal{U}_f)$ with $(\psi(f)(\alpha_k), \alpha_k] \subseteq U_k$ and (by minimality of $\psi(f)(\alpha_k)$) with $\psi(f)(\alpha_k) \notin U_k$. But then the open cover $r(\mathcal{U}_f)$ is not interior-preserving since $\beta \notin \operatorname{Int}(\cap\{U_k : k < \omega\})$. This contradiction proves the claim, and completes the proof that $(i) \implies (ii)$.

Now assume that condition (ii) holds, and for each open cover $\mathcal{U}$ and $\alpha > 0$ define $f_{\mathcal{U}}(\alpha) = \min\{\beta : \exists U \in \mathcal{U},\ (\beta, \alpha] \subseteq U\}$, and $f_{\mathcal{U}}(0) = 0$. Then it is easy to verify that $f_{\mathcal{U}} \in \mathcal{F}$ and if we define $r(\mathcal{U}) = \{\{0\}\} \cup \{(\psi(f_{\mathcal{U}})(\alpha), \alpha] : 0 < \alpha < \omega_1\}$ then r is the required monotone interior-preserving open operator for ω_1. □

Question 3.2. *Does there exist a map* ψ *as described in condition (ii) of the above theorem?*

It is well-known that every GO-space is orthocompact. We do not know if there is an example of a GO-space or a LOTS that is not monotonically orthocompact via open refinements.

4. On a question by Junnila and Künzi. Gruenhage [5] proved that a T_1 space X is protometrizable if and only if it is a monotonically normal space with an ortho-base. The requirement that X has an ortho-base was weakened by Junnila and Künzi to the requirement that X is monotonically orthocompact, [9, Proposition 3] (stated as Theorem 2.4 above). The question whether each monotonically orthocompact space must have an ortho-base was left open. A negative answer was claimed in [13] based on the following.

Theorem 4.1 (claimed in [13, Theorem 4.4]). *Every space with a Nötherian base of subinfinite rank is monotonically orthocompact.*

A base $\mathcal{B}$ is called a Nötherian base of subinfinite rank (NSR base for short, [10]) if (a) every strictly increasing by set inclusion sequence of members of $\mathcal{B}$ is finite, and (b) every infinite subfamily of $\mathcal{B}$ with nonempty intersection must contain two elements one of which is contained in the other.

The proof of [12, Example 2.1] shows that the space X described there (as well as in Example 2.5 here) has a NSR base. Since X is a LOTS (hence monotonically normal), Theorem 2.4 and Theorem 4.1 would imply that X is protometrizable, which it is not! Upon inspection, the proof as well as the statement of [13, Theorem 4.4] turn out to be invalid. We would present an easy example showing exactly where the proof breaks. We would also explain why the statement is invalid (based directly on the same example that the authors in [13] claimed to be a monotonically orthocompact space without an ortho-base).

Let $Z = \{a, b, c\}$ with the discrete topology (where a, b, c are distinct). While obviously Z is monotonically orthocompact, nevertheless Z could be used to show that *the proof of* [13, Theorem 4.4] is invalid. Let $\mathcal{B}$ be the base of all subsets of Z, then $\mathcal{B}$ is a NSR base. Let $\mathcal{U}_1 = \{\{a, b\}, \{a, c\}\}$ and $\mathcal{U}_2 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Clearly $\mathcal{U}_1 \prec \mathcal{U}_2$. If $\mathcal{V}_i$ is the collection of maximal (with respect to inclusion) elements of $\mathcal{U}_i$ then clearly $\mathcal{V}_i = \mathcal{U}_i$, $i = 1, 2$. The transitive neighbornet $R_{\mathcal{U}_i}$ constructed in [13] is, in essence, the same as the ONA $\{R_{\mathcal{U}_i}(x) : x \in Z\}$ where (for $i = 1, 2$) $R_{\mathcal{U}_i}(x) = \cap\{B \in \mathcal{V}_i : x \in B\}$. It is claimed without verification, in the proof of [13, Theorem 4.4], that $R_{\mathcal{U}_1}(x) \subseteq R_{\mathcal{U}_2}(x)$ for each x. But in the space Z that we consider, $R_{\mathcal{U}_1}(b) = \{a, b\}$ while $R_{\mathcal{U}_2}(b) = \{b\}$, thus $R_{\mathcal{U}_1}(b) \nsubseteq R_{\mathcal{U}_2}(b)$. In short, the proof fails because the intersection operator is not monotone: If one family (e.g. $\{\{a, b\}\}$) refines another (e.g. $\{\{a, b\}, \{b, c\}\}$) then it does not necessarily follow that the intersection of the first family is contained in the intersection of the second family.

The example presented in [13] is the Dieudonné plank D. It does indeed [10, Example 5.3] have a NSR base, and the authors (incorrectly, using [13, Theorem 4.4]) conclude that it is monotonically orthocompact. Then the authors (correctly) argue that D does not have an ortho-base since the corner point does not have a base that is linearly ordered by inclusion. But, they fail to observe that (in light of Proposition 2.6 here) the same argument shows that D is not monotonically orthocompact either. (We conclude that the statement of [13, Theorem 4.4] is invalid. Despite that at hindsight the error seems obvious, it might perhaps be subtle, as it survived not only a referee report, but also a review in Zentralblatt by one of the authors of [9].)

We searched for other examples with nice properties, but without an ortho-base, hoping to find one that is monotonically orthocompact. An ω_μ-Nagata space X (for each infinite cardinal μ) that has no ortho-base is constructed in [14, Example 4.1]. Since ω_μ-Nagata spaces are monotonically normal, such an X is just another example of a space that is not monotonically orthocompact.

Let M be the Michael line and let $\omega+1=[0,\omega]$ with the order topology (a converging sequence). Since each of M and $\omega+1$ has a NSR base, so does the product $M\times(\omega+1)$, by [10, Theorem 3.5]. It follows that $M\times(\omega+1)$ is monotonically metacompact [3, section 5], and hence monotonically orthocompact via open refinements. It is first countable, and in particular a lob space. $M\times(\omega+1)$ is not hereditarily normal since M is not perfect (the irrationals are an open set that is not F_σ), hence $M\times(\omega+1)$ is not monotonically normal, so one may ask if it is monotonically orthocompact. We modify the proof of [12, Example 2.4] to show that it is not.

Example 4.2. The space $X=M\times(\omega+1)$ is not monotonically orthocompact.

Proof. Assume X were monotonically orthocompact and let $t:\mathcal{C}\to\mathcal{N}$ be a monotone orthocompact (via transitive ONAs) operator as in Definition 2.3.

Let $\mathbb{P}=\mathbb{R}\setminus\mathbb{Q}$ denote the irrationals, and $H=X\setminus(\mathbb{Q}\times\{\omega\})$. For each $p\in\mathbb{P}$ and $n\geq 1$ let $R(p,n)=[p-\frac{1}{n},p+\frac{1}{n}]\times[n+1,\omega]$ and $C(p,n)=\big(M\setminus[p-\frac{1}{n},p+\frac{1}{n}]\big)\times[n,\omega]$. Let $U(p)=\cup_{k\geq 1}C(p,k)$, let $U(p,n)=\big(\cup_{1\leq k\leq n}C(p,k)\big)\cup R(p,n)$, let $\mathcal{U}(p)=\{U(p),H\}$ and $\mathcal{U}(p,n)=\{U(p,n),H\}$. Then $\mathcal{U}(p),\mathcal{U}(p,n)\in\mathcal{C}$ and $\mathcal{U}(p)\prec\mathcal{U}(p,n)$ for all $n\geq 1$.

For each $p\in\mathbb{P}$ there is $n_p\geq 1$ such that $\{p\}\times[n_p,\omega]\subseteq t(\mathcal{U}(p))\langle p,\omega\rangle\subseteq t(\mathcal{U}(p,n_p))\langle p,\omega\rangle$. Let $P_n=\{p\in\mathbb{P}:n_p=n\}$. Since $\mathbb{P}$ is not F_σ, we may fix $n\geq 1$ and $q\in\mathbb{Q}\cap\overline{P_n}$.

Let $V=\big([q-\frac{1}{2n},q+\frac{1}{2n}]\times[n+1,\omega]\big)\cup\Big(\big(M\setminus[q-\frac{1}{2n},q+\frac{1}{2n}]\big)\times[0,\omega]\Big)$ and $\mathcal{V}=\{V,H\}$. Fix $p\in P_n$ with $|p-q|<\frac{1}{2n}$ and $\langle p,\omega\rangle\in t(\mathcal{V})\langle q,\omega\rangle$. Then $U(p,n)\subseteq V$ and $\mathcal{U}(p,n)\prec\mathcal{V}$, hence $\langle p,n\rangle\in t(\mathcal{U}(p,n))\langle p,\omega\rangle\subseteq t(\mathcal{V})\langle p,\omega\rangle\subseteq t(\mathcal{V})\langle q,\omega\rangle$, the latter inclusion by transitivity. The only element of $\mathcal{V}$ that contains $\langle q,\omega\rangle$ is

V, hence $t(\mathcal{V})\langle q, \omega\rangle \subseteq V$. Since $\langle p, n\rangle \notin V$ we obtain a contradiction, which completes the proof. $\square$

Thus, to the best of our knowledge the following question first stated by Junnila and Künzi remains open at present.

Question 4.3. *Does every monotonically orthocompact space have an ortho-base?*

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Received May 6, 2018