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CARDINAL INEQUALITIES FOR URYSOHN SPACES INVOLVING VARIATIONS OF THE ALMOST LINDELÖF DEGREE

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To the memory of Stoyan Nedev

ABSTRACT. Recall that for a topological space X , $t_{\theta_1}(X)$ is the smallest infinite cardinal κ such that for every $A \subset X$ and every $x \in \text{cl}(A)$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \text{cl}_{\theta}(B)$ ([6]). For every Urysohn space X we define the cardinal function $\psi_{\theta^2}(X)$, the θ^2 -pseudocharacter of X , as the smallest infinite cardinal κ such that for each $x \in X$, there is a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\text{cl}_{\theta}(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Using this new cardinal function, among other results, we show that if X is a Urysohn space and $A \subset X$ then (1) $|\text{cl}(A)| \leq |A|^{t_{\theta_1}(X)\psi_{\theta^2}(X)}$; and (2) $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)\alpha L_c(X)}$.

Since for every Urysohn space X we have $\psi_{\theta^2}(X) \leq \psi(X)L(X)$, inequality (2) sharpens, for the class of Urysohn spaces, the famous Arhangel'skiĭ–Šapironvskii inequality $|X| \leq 2^{t(X)\psi(X)L(X)}$, which is valid for every Hausdorff space X .

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Using the cardinal function bt_θ , called θ -bitightness, or recently introduced in [3] variations of the almost Lindelöf degree, other upper bounds of the cardinality of Urysohn spaces are proved which improve (2), Kočinac' inequality $|X| \leq 2^{bt_\theta(X)aL(X)}$, which is valid for every Urysohn space X , and, for special classes of Urysohn spaces, Bella-Cammaroto's inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is true for every Hausdorff space X .

1. Introduction. In 1969, Arhangel'skiĭ proved that if X is a Hausdorff space then $|X| \leq 2^{\chi(X)L(X)}$ [2]. Later he showed that for regular spaces the stronger inequality $|X| \leq 2^{t(X)\psi(X)L(X)}$ is true. In 1974, Šapirovskiĭ proved that the latter inequality was also valid for all Hausdorff spaces X [17]. Improving a result of Willard and Dissanayake from 1984 [18], in 1988, Bella and Cammaroto noticed that the inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$ is true for every Hausdorff spaces X [4]. Later Hodel showed in [11] that if X is a Uryson space then $\psi_c(X) \leq \psi(X)aL_c(X)$ and therefore Bella and Cammaroto inequality for Urysohn spaces becomes $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$.

In the same paper [4] Bella and Cammaroto showed that if X is a Urysohn space then $|X| \leq d_\theta(X)^{\chi(X)}$ and $|X| \leq 2^{\chi(X)aL(X)}$. Trying to strengthen the former inequality, in 1995, Kočinac proved that if X is a Urysohn H -closed space then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_c(X)}$ and he asked if the same result was valid for all Urysohn spaces [13]. (To the best of our knowledge this question is still open.) Kočinac' inequality could be considered as an attempt to find, for the class of Urysohn spaces, the counterpart of Bella and Cammaroto inequality that if X is a Hausdorff space then $|X| \leq d(X)^{t(X)\psi_c(X)}$ (see [4]). We note that Grizlov in [10], for the case $\psi_c(X) = \omega$, and Dow and Porter in [8], for the general case, proved the following more general result: If X is a H -closed space then $|X| \leq 2^{\psi_c(X)}$.

In this paper we introduce the cardinal function θ^2 -pseudocharacter of a Urysohn space X , denoted by $\psi_{\theta^2}(X)$, and using it we extend or sharpen some well-known cardinal inequalities for Urysohn spaces. (We recall that a space X is called Urysohn if every two distinct points in X have disjoint closed neighborhoods.)

Among other results, in Section 3, we show that if X is a Urysohn space, then $|cl_\theta(A)| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}$ (Theorem 3.4(a)) and therefore $|X| \leq d_\theta(X)^{t_\theta(X)\psi_{\theta^2}(X)}$ (Corollary 3.5). Since $\psi_c(X) = \psi_{\theta^2}(X)$ for every Urysohn H -closed space X , the latter inequality extends Kočinac' result to all Urysohn spaces. In Theorem 4.2 we use the former inequality to prove that for every Urysohn space X , we have $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$. This result and the result in Theorem 3.4 also follow from our observation that for every Urysohn

space X we have $bt_\theta(X) \leq t_\theta(X)\psi_{\theta^2}(X)$ (Theorem 3.10) and, respectively, from Cammaroto-Kočinac' inequality $|[A]_\theta| \leq |A|^{bt_\theta(X)}$ ([7]) and Kočinac' inequality $|X| \leq 2^{bt_\theta(X)aL(X)}$, which are valid for every Urysohn space X .

Recall that for a topological space X , $t_{\theta_1}(X)$ is the smallest infinite cardinal κ such that for every $A \subset X$ and every $x \in \text{cl}(A)$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \text{cl}_\theta(B)$ ([6]). In Theorem 3.8 we show that if X is a Urysohn space and $A \subset X$ then $|\text{cl}(A)| \leq |A|^{t_{\theta_1}(X)\psi_{\theta^2}(X)}$. Using a generalization of this result (Theorem 4.7), in Theorem 4.9 we show that for every Urysohn space X , $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X)}$. Since $\psi_{\theta^2}(X) \leq \psi(X)L(X)$, whenever X is a Urysohn space (Lemma 4.10), the former inequality sharpens, for the class of Urysohn spaces, Arhangel'skiĭ-Šapiroviĭ inequality $|X| \leq 2^{t(X)\psi(X)L(X)}$, which is true for every Hausdorff space X . Since for each $S(3)$ -space X (for the definition of $S(3)$ -spaces see Definition 4.12) we have $\psi_{\theta^2}(X) \leq \psi(X)aL_c(X)$ (Lemma 4.13), Theorem 4.9 improves, for the class of $S(3)$ -spaces, Bella and Cammaroto inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is valid for every Hausdorff space X and the inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$, which is valid for every Urysohn space X .

Using some variations of the almost Lindelöf degree defined recently in [3] and denoted by $\theta\text{-}aL_\theta(X)$ and $\theta\text{-}aL_c(X)$ (see Definition 5.1), we show in Theorem 5.2 that for every Urysohn space X , we have $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)\theta\text{-}aL_\theta(X)}$, and in Theorem 5.3 we establish even the stronger result $|X| \leq 2^{bt_\theta(X)\theta\text{-}aL_\theta(X)}$, whenever X is a Urysohn space. Since for every space X we have $\theta\text{-}aL_\theta(X) \leq aL(X)$, these results improve further our inequality in Theorem 4.2, while the latter result improves also Kočinac' inequality $|X| \leq 2^{bt_\theta(X)aL(X)}$.

Finally, in Theorem 5.4 we show that for every Urysohn space X , $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)\theta\text{-}aL_c(X)}$. Since for every space X we have $\theta\text{-}aL_c(X) \leq aL_c(X)$, this inequality improves the inequality $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X)}$ in Theorem 4.9, which is valid also for every Urysohn space X , and since for each $S(4)$ -space X (for the definition of $S(4)$ -spaces see Definition 4.12) $\psi_{\theta^2}(X) \leq \psi(X)\theta\text{-}aL_c(X)$ (Lemma 5.5), Theorem 5.4 improves, for the class of $S(4)$ -spaces, Bella and Cammaroto inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is valid for every Hausdorff space X and the inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$, which is valid for every Urysohn space X .

2. Preliminaries. The θ -closure of a set A in a space X , denoted by $\text{cl}_\theta(A)$, is the set of all points $x \in X$ such that for every open neighborhood U of x we have $\text{cl}(U) \cap A \neq \emptyset$. A is called θ -closed if $A = \text{cl}_\theta(A)$ and A is θ -dense if $\text{cl}_\theta(A) = X$ ([15]). The θ -density of a space X is $d_\theta(X) = \min\{|A| : A \subset$

$X, \text{cl}_\theta(A) = X\}$. We note that when $U \subset X$ is open then $\text{cl}(U) = \text{cl}_\theta(U)$ [15]. The smallest θ -closed set containing A , i.e. the intersection of all θ -closed sets containing A , is denoted by $[A]_\theta$ and is called *the θ -closed hull of A* [4].

The θ -tightness of a space X , denoted by $t_\theta(X)$, is the smallest infinite cardinal κ such that for every $A \subset X$ and every $x \in \text{cl}_\theta(A)$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \text{cl}_\theta(B)$ [7].

Recently in [5] the authors showed that even in a H -closed Urysohn space X it is possible to have $t_\theta(X) < t(X)$, $t_\theta(X) > t(X)$, or $t_\theta(X) = t(X)$, where $t(X)$ is the tightness of the space X . This fact shows that in general we do not know if we will get a better result when in a given cardinal inequality we replace $t(X)$ with $t_\theta(X)$. Therefore if one would like to get a stronger inequality, it is better to replace, if possible, $t(X)$ with $t_{\theta_1}(X)$, since $t_{\theta_1}(X) \leq t(X)$ and $t_{\theta_1}(X) \leq t_\theta(X)$.

The *closed pseudocharacter* $\psi_c(X)$ (defined only for Hausdorff spaces X) is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\text{cl}(V) : V \in \mathcal{V}_x\} = \{x\}$ [12].

For a topological space X , $k(X)$ is the smallest infinite cardinal κ such that for each point $x \in X$, there is a collection \mathcal{V}_x of closed neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and if W is a neighborhood of x then $\text{cl}(W)$ contains a member of \mathcal{V}_x [1]. As it was noted in [1], $k(X)$ is equal to the character of the semiregularization of X .

The *almost Lindelöf degree of a space X with respect to closed sets* is $aL_c(X) = \sup\{aL(F, X) : F \text{ is a closed subset of } X\}$, where $aL(F, X)$ is the minimal infinite cardinal κ such that for every open (in X) cover \mathcal{U} of F there is a subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $|\mathcal{U}_0| \leq \kappa$ and $F \subset \bigcup \{\overline{U} : U \in \mathcal{U}_0\}$. $aL(X, X)$ is called *almost Lindelöf degree of X* and is denoted by $aL(X)$.

Remark 2.1. The cardinal function $aL_c(X)$ was introduced in [18] under the name *almost Lindelöf degree* and was denoted by $aL(X)$. Here we follow the notation and terminology used in [11] and suggested in [4].

Clearly $aL(X) \leq aL_c(X) \leq L(X)$, where $L(X)$ is the Lindelöf degree of X . For examples of Urysohn spaces such that $aL(X) < aL_c(X) < L(X)$ see [18] or [11].

We finish this section with the following observation.

Lemma 2.2. *If X is a space and $F \subset X$ is θ -closed then $aL(F, X) \leq aL(X)$.*

3. Cardinal inequalities for Urysohn spaces involving the θ^2 -pseudocharacter. In 1995, Kočinac proved the following theorem and asked if the same result was valid for all Urysohn spaces.

Theorem 3.1 ([13, Theorem 2.5]). *If X is a Urysohn H -closed space then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_c(X)}$.*

We do not know the answer of his question but in order to extend the inequality in his theorem to be valid for all Urysohn spaces we introduce the following new cardinal function.

Definition 3.2. *For every Urysohn space X we define the θ^2 -pseudocharacter, denoted by $\psi_{\theta^2}(X)$, to be the smallest infinite cardinal κ such that for each $x \in X$, there is a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$.*

Remark 3.3. We note that the θ^2 -pseudocharacter is well defined for every Urysohn space X . Similarly, for every positive integer n , we can define the θ^n -pseudocharacter, denoted by $\psi_{\theta^n}(X)$, as the smallest infinite cardinal κ such that for each $x \in X$, there is a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\underbrace{\text{cl}_\theta \dots \text{cl}_\theta}_{n-1\text{-times}}(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Clearly, the θ^n -pseudocharacter will be well defined only for spaces X for which for every two distinct points $x, y \in X$ there exists an open neighborhood V of x such that $y \notin \underbrace{\text{cl}_\theta \dots \text{cl}_\theta}_{n-1\text{-times}}(\text{cl}(V))$. Such spaces, for example, are the $S(n)$ -spaces introduced by Viglino in [16] (see Definition 4.12). We only recall here that a space is $S(1)$ if and only if it is Hausdorff and a space is $S(2)$ if and only if it is Urysohn.

Since $\text{cl}_\theta(V) = \text{cl}(V)$ whenever V is an open subset of X , we have $\text{cl}_\theta(\text{cl}(V)) = \text{cl}_\theta(\text{cl}_\theta(V))$. This explains our choice of the notation $\psi_{\theta^2}(X)$ and also shows that we can use the notation $\psi_\theta(X)$ instead of $\psi_c(X)$.

We note that $\psi(x) \leq \psi_c(X) \leq \psi_{\theta^2}(X) \leq \kappa(X) \leq \chi(X)$ for every Urysohn space X . Also, for regular spaces we have $\psi(x) = \psi_c(X) = \psi_{\theta^2}(X)$.

Using the cardinal function $\psi_{\theta^2}(X)$, the θ^2 -pseudocharacter of a Urysohn space X , we can prove the following result.

Theorem 3.4. *Let X be a Urysohn space and $A \subset X$. Then*

- (a) $|\text{cl}_\theta(A)| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}$, and

$$(b) |[A]_\theta| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}.$$

Proof. (a) Let $\kappa = t_\theta(X)\psi_{\theta^2}(X)$. For each $x \in X$ we fix a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Since $t_\theta(X) \leq \kappa$, for each $x \in \text{cl}_\theta(A)$ we can fix a set $B_x \subset A$ such that $|B_x| \leq \kappa$ and $x \in \text{cl}_\theta(B_x)$. Now, let $\mathcal{F}_x = \{\text{cl}(V) \cap B_x : V \in \mathcal{V}_x\}$. Then, for every $F \in \mathcal{F}_x$ we have $x \in \text{cl}_\theta(F)$. Hence $x \in \bigcap \{\text{cl}_\theta(F) : F \in \mathcal{F}_x\} \subset \bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Thus, $x \rightarrow \mathcal{F}_x$ is an one-to-one map from $\text{cl}_\theta(A)$ into $[|A|^{\leq \kappa}]^{\leq \kappa}$. Therefore $|\text{cl}_\theta(A)| \leq |A|^\kappa$.

(b) For each $\alpha < \kappa^+$, by transfinite recursion, we define sets A_α such that $A_0 = A$ and for $\alpha > 0$, $A_\alpha = \text{cl}_\theta(\bigcup \{A_\beta : \beta < \alpha\})$. Clearly, $\bigcup \{A_\alpha : \alpha < \kappa^+\} \subseteq [A]_\theta$. To show the reverse inclusion it is sufficient to show that $\bigcup \{A_\alpha : \alpha < \kappa^+\}$ is θ -closed. For that end, let $x \in \text{cl}_\theta(\bigcup \{A_\alpha : \alpha < \kappa^+\})$. Then, there exists a subset $B_x \subset \bigcup \{A_\alpha : \alpha < \kappa^+\}$ such that $|B_x| \leq \kappa$ and $x \in \text{cl}_\theta(B_x)$. Since κ^+ is regular, there exists $\alpha < \kappa^+$ such that $B_x \subset A_\alpha$. Hence $x \in \text{cl}_\theta(A_\alpha) \subset A_{\alpha+1}$. Therefore $\bigcup \{A_\alpha : \alpha < \kappa^+\}$ is θ -closed. Thus, $\bigcup \{A_\alpha : \alpha < \kappa^+\} = [A]_\theta$.

To finish the proof we need to show that $|[A]_\theta| \leq 2^\kappa$. For that it is sufficient to prove that for every $\alpha < \kappa^+$ we have $|A_\alpha| \leq 2^\kappa$. Suppose that there exists $\alpha < \kappa^+$ such that $|A_\alpha| > 2^\kappa$ and let γ be the first ordinal number with that property. Since $A_\gamma = \text{cl}_\theta(\bigcup \{A_\beta : \beta < \gamma\})$, from (a) we have $|A_\gamma| \leq |\bigcup \{A_\beta : \beta < \gamma\}|^\kappa \leq (2^\kappa \kappa^+)^\kappa = 2^\kappa$ - contradiction. The proof is complete. \square

Corollary 3.5. *If X is a Urysohn space, then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_{\theta^2}(X)}$.*

It is well known that if X is a Urysohn H -closed space then $\text{cl}_\theta(\text{cl}(U)) = \text{cl}(U)$ for every open subset U of X ([15, Theorem 4(b)]). Hence, $\psi_{\theta^2}(X) = \psi_c(X)$ whenever X is a Urysohn H -closed space. Therefore the inequality in Corollary 3.5 coincides with the inequality in Theorem 3.1 for Urysohn H -closed spaces but is valid for all Urysohn spaces.

Since $t_\theta(X) \leq \kappa(X)$, as a corollary of Theorem 3.4, we obtain the following results of Alas and Kočinac and Bella and Cammaroto.

Corollary 3.6 ([1]). *If X is a Urysohn space, then*

- (1) $|\text{cl}_\theta(A)| \leq |A|^{\kappa(X)}$; and
- (2) $|[A]_\theta| \leq |A|^{\kappa(X)}$.

Corollary 3.7 ([4]). *If X is a Urysohn space, then*

(1) $|\text{cl}_\theta(A)| \leq |A|^{\chi(X)}$; and

(2) $|[A]_\theta| \leq |A|^{\chi(X)}$.

Notice that the inequality $|\text{cl}_\theta(A)| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}$ from Theorem 3.4(a), which is valid for every subset A of every Urysohn space X , corresponds to the inequality $|\text{cl}(A)| \leq |A|^{t(X)\psi_c(X)}$ (see [4]) which is valid for every subset A of every Hausdorff space X . Another inequality similar to the latter inequality, which is also valid only for subsets of Urysohn spaces, is given in the next theorem.

Theorem 3.8. *Let X be a Urysohn space and $A \subset X$. Then $|\text{cl}(A)| \leq |A|^{t_{\theta_1}(X)\psi_{\theta^2}(X)}$.*

Proof. Let $\kappa = t_{\theta_1}(X)\psi_{\theta^2}(X)$. For each $x \in X$ we fix a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Since $t_{\theta_1}(X) \leq \kappa$, for each $x \in \text{cl}(A)$ we can fix a set $B_x \subset A$ such that $|B_x| \leq \kappa$ and $x \in \text{cl}_\theta(B_x)$. Now, let $\mathcal{F}_x = \{\text{cl}(V) \cap B_x : V \in \mathcal{V}_x\}$. Then, for every $F \in \mathcal{F}_x$ we have $x \in \text{cl}_\theta(F)$. Hence $x \in \bigcap \{\text{cl}_\theta(F) : F \in \mathcal{F}_x\} \subset \bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Thus, $x \rightarrow \mathcal{F}_x$ is an one-to-one map from $\text{cl}(A)$ into $[|A|^{\leq \kappa}]^{\leq \kappa}$. Therefore $|\text{cl}(A)| \leq |A|^\kappa$. \square

Corollary 3.9. *If X is a Urysohn space, then $|X| \leq d(X)^{t_{\theta_1}(X)\psi_{\theta^2}(X)}$.*

We note that in the inequality in Theorem 3.8 we weaken Bella and Cammaroto inequality $|\text{cl}(A)| \leq |A|^{t(X)\psi_c(X)}$ by replacing $t(X)$ with $t_{\theta_1}(X)$ but compensate by strengthening $\psi_c(X) = \psi_\theta(X)$ to $\psi_{\theta^2}(X)$.

We recall that the θ -bitightness of a space X , denoted by $bt_\theta(X)$, is the smallest infinite cardinal κ such that for every non- θ -closed set $A \subset X$ there exists a point $x \in X \setminus A$ and a collection $\mathcal{S} \in [|[A]^{\leq \kappa}]^{\leq \kappa}$ such that $\bigcap \{\text{cl}_\theta(S) : S \in \mathcal{S}\} = \{x\}$ [7]. The θ -bitightness is not defined for every topological space X but it is well defined for every Urysohn space X . The following theorem gives the relationship between $bt_\theta(X)$ and $t_\theta(X)\psi_{\theta^2}(X)$ for every Urysohn space X .

Theorem 3.10. *If X is a Urysohn space then $bt_\theta(X) \leq t_\theta(X)\psi_{\theta^2}(X)$.*

Proof. Let $t_\theta(X)\psi_{\theta^2}(X) = \kappa$ and let A be a non- θ -closed subset of X . Then there is a point $x \in \text{cl}_\theta(A) \setminus A$. Since $t_\theta(X) \leq \kappa$, we can fix a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \text{cl}_\theta(B)$. Let \mathcal{V} be a collection of open neighborhoods of x such that $|\mathcal{V}| \leq \kappa$ and $\bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}\} = \{x\}$. Then $\text{cl}(V) \cap B \neq \emptyset$

and hence $x \in \text{cl}_\theta(B \cap \text{cl}(V))$ for every $V \in \mathcal{V}$. Thus,

$$x \in \bigcap \{ \text{cl}_\theta(B \cap \text{cl}(V)) : V \in \mathcal{V} \} \subset \bigcap \{ \text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V} \} = \{x\}.$$

This shows that $\bigcap \{ \text{cl}_\theta(B \cap \text{cl}(V)) : V \in \mathcal{V} \} = \{x\}$. The existence of the collection $\{B \cap \text{cl}(V) : V \in \mathcal{V}\}$ proves that $bt_\theta(X) \leq \kappa$. \square

Cammaroto and Kočinac noticed in [7] that for every Urysohn space X we have $bt_\theta(X) \leq \chi(X)$. Theorem 3.10 shows that the stronger inequality $bt_\theta(X) \leq t_\theta(X)\psi_{\theta^2}(X) \leq \kappa(X) \leq \chi(X)$ is true. In [7] the authors also proved that if X is a Urysohn space then for every $A \subset X$ we have $|[A]_\theta| \leq |A|^{bt_\theta(X)}$. Therefore Theorem 3.4 also follows from their result and Theorem 3.10.

4. Cardinal inequalities for Urysohn spaces involving the almost Lindelöf degree. In [9] the following result is proven.

Theorem 4.1 ([9, Corollary 6.13]). *For every Urysohn space X , $|X| \leq 2^{\kappa(X)aL(X)}$.*

The above theorem clearly generalizes Bella and Cammaroto inequality mentioned before, that for every Urysohn space X we have $|X| \leq 2^{\chi(X)aL(X)}$. Using the cardinal function $\psi_{\theta^2}(X)$ we can generalize Theorem 4.1 and therefore Bella and Cammaroto inequality as follows:

Theorem 4.2. *For every Urysohn space X , $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$.*

Proof. Let $t_\theta(X)\psi_{\theta^2}(X)aL(X) = \kappa$ and for each $x \in X$ we fix a collection $\mathcal{V}_x = \{V_\alpha(x) : \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} \text{cl}_\theta(\text{cl}(V_\alpha(x))) = \{x\}$. Also, for a non-empty subset A of X we denote by \mathcal{C}_A the set of all families $\mathcal{C} = \{\text{cl}(U_a) : a \in A, U_a \in \mathcal{V}_a\}$.

Now, let x_0 be an arbitrary point in X . Recursively we construct a family $\{F_\alpha : \alpha < \kappa^+\}$ of subsets of X with the following properties:

- (i) $F_0 = \{x_0\}$ and $\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta) \subset F_\alpha$ for every $0 < \alpha < \kappa^+$;
- (ii) $|F_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$;
- (iii) for every $\alpha < \kappa^+$, and every $F \subset \text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta)$ with $|F| \leq \kappa$ if $X \setminus \bigcup \mathcal{C} \neq \emptyset$ for some $\mathcal{C} \in \mathcal{C}_F$, then $F_\alpha \setminus \bigcup \mathcal{C} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define F_α . Since $|F_\beta| \leq 2^\kappa$ for each $\beta < \alpha$, we have $|\bigcup_{\beta < \alpha} F_\beta| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$. Then it follows from Theorem 3.4, that $|\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\alpha)| \leq 2^\kappa$. Hence, there are at most 2^κ subsets F of $\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\alpha)$ with $|F| \leq \kappa$ and for each such set F we have $|\mathcal{C}_F| \leq \kappa^\kappa = 2^\kappa$. For each $F \subset \text{cl}_\theta(\bigcup_{\beta < \alpha} F_\alpha)$ with $|F| \leq \kappa$ and each $\mathcal{C} \in \mathcal{C}_F$ for which $X \setminus \bigcup \mathcal{C} \neq \emptyset$ we choose a point in $X \setminus \bigcup \mathcal{C} \neq \emptyset$ and let E_α be the set of all such points. Clearly $|E_\alpha| \leq 2^\kappa$. Let $F_\alpha = \text{cl}_\theta(E_\alpha \cup (\bigcup_{\beta < \alpha} F_\alpha))$. Then it follows from our construction that F_α satisfies (i) and (iii) while (ii) follows from Theorem 3.4.

Now, let $G = \bigcup_{\alpha < \kappa^+} F_\alpha$. Clearly $|G| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$ and since $t_\theta(X) \leq \kappa$, G is θ -closed. To finish the proof it is sufficient to show that $G = X$. Suppose that there is $x \in X \setminus G$. Then for every $y \in G$ there is $V_y \in \mathcal{V}_y$ such that $x \notin \text{cl}(V_y)$. Since $\{V_y : y \in G\}$ is an open cover of G and G is θ -closed, it follows from Lemma 2.2 that there is $F \subset G$ with $|F| \leq \kappa$ such that $G \subset \bigcup \{\text{cl}(V_y) : y \in F\}$. Clearly $x \notin \bigcup \{\text{cl}(V_y) : y \in F\}$. Since $|F| \leq \kappa$, there is $\beta < \kappa^+$ such that $F \subset F_\beta$. Then for $\mathcal{C} = \{\text{cl}(V_y) : y \in F\}$ we have $\mathcal{C} \in \mathcal{C}_F$ and $x \in X \setminus \bigcup \mathcal{C}$. Then it follows from our construction that $F_{\beta+1} \setminus \bigcup \mathcal{C} \neq \emptyset$ which contradicts $F_{\beta+1} \subset G \subset \bigcup \mathcal{C}$. Therefore $G = X$ and the proof is completed. \square

We note that Kočinac mentioned in [13] (see also MR1205960) that for every Urysohn space X one can show that $|X| \leq 2^{bt_\theta(X)aL(X)}$. Therefore Theorem 4.2 could be deduced from that observation and Theorem 3.10.

In the following definition we introduce the concept of θ - κ -closure of a subset of a topological space. This new concept will be used later in the proof of Theorem 4.9 .

Definition 4.3. For $A \subseteq X$ and an infinite cardinal κ , let $[A]^{\leq \kappa} = \{B : B \subseteq A, |B| \leq \kappa\}$. We define the θ - κ -closure of A as $\text{cl}_{\theta\kappa}(A) = \bigcup_{B \in [A]^{\leq \kappa}} \text{cl}_\theta(B)$. A is θ - κ -closed if $\text{cl}_{\theta\kappa}(A) = A$.

The proofs of the following three observations are straightforward.

Lemma 4.4. *Let X be a Urysohn space, $A \subset X$ and $\kappa \geq t_{\theta_1}(X)$. After applying $(t_{\theta_1})^+$ -many consecutive times the operator $\text{cl}_{\theta\kappa}(\cdot)$, beginning with $\text{cl}_{\theta\kappa}(A)$, the resulting set will be closed.*

Lemma 4.5. *Let X be a Urysohn space and $G \subseteq X$ be such that $G = \bigcup_{\alpha < \kappa^+} F_\alpha$, where for each $\alpha < \beta < \kappa^+$ we have $\text{cl}_{\theta\kappa}(F_\alpha) \subseteq F_\beta$. If $\kappa \geq t_{\theta_1}(X)$ then G is a closed subset of X .*

Lemma 4.6. *If $U \subseteq X$ is open and $\kappa \geq t_{\theta_1}(X)$ then $\text{cl}_\theta(U) = \text{cl}(U) = \text{cl}_{\theta\kappa}(U)$.*

Theorem 4.7. *Let X be a Urysohn space, $A \subset X$ and κ be an infinite cardinal. Then $|\text{cl}_{\theta\kappa}(A)| \leq |A|^{\kappa \cdot \psi_{\theta^2}(X)}$.*

Proof. Let $\tau = \psi_{\theta^2}(X)$. For each $x \in X$ we fix a collection \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{V}_x| \leq \tau$ and $\bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. It follows from the definition of $\text{cl}_{\theta\kappa}(A)$ that for each $x \in \text{cl}_{\theta\kappa}(A)$ we can fix a set $B_x \subset A$ such that $|B_x| \leq \kappa$ and $x \in \text{cl}_\theta(B_x)$. Now, let $\mathcal{F}_x = \{\text{cl}(V) \cap B_x : V \in \mathcal{V}_x\}$. Then $|\mathcal{F}_x| \leq \tau$ and for every $F \in \mathcal{F}_x$ we have $x \in \text{cl}_\theta(F)$. Hence $x \in \bigcap \{\text{cl}_\theta(F) : F \in \mathcal{F}_x\} \subset \bigcap \{\text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}_x\} = \{x\}$. Thus, $x \rightarrow \mathcal{F}_x$ is an one-to-one map from $\text{cl}_{\theta\kappa}(A)$ into $[|A|^{\leq \kappa}]^{\leq \tau}$. Therefore $|\text{cl}_{\theta\kappa}(A)| \leq |A|^{\kappa \cdot \tau}$. \square

Corollary 4.8. *Let X be a Urysohn space, $A \subset X$ and $\kappa = t_{\theta_1}(X)$. Then $\text{cl}(A) \subseteq \text{cl}_{\theta\kappa}(A)$ and $|\text{cl}_{\theta\kappa}(A)| \leq |A|^{t_{\theta_1}(X)\psi_{\theta^2}(X)}$.*

We note that Theorem 3.8 follows directly from Corollary 4.8.

The following theorem gives another upper bound for the cardinality of a Urysohn space X .

Theorem 4.9. *For every Urysohn space X , $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X)}$.*

Proof. Let $t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X) = \kappa$ and for each $x \in X$ we fix a collection $\mathcal{V}_x = \{V_\alpha(x) : \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} \text{cl}_\theta(\text{cl}(V_\alpha(x))) = \{x\}$. Also, for every non-empty subset A of X we denote by \mathcal{C}_A the set of all families $\mathcal{C} = \{\text{cl}(V_a) : a \in A, V_a \in \mathcal{V}_a\}$.

Now, let x_0 be an arbitrary point in X . Recursively we construct a family $\{F_\alpha : \alpha < \kappa^+\}$ of subsets of X with the following properties:

- (i) $F_0 = \{x_0\}$ and $\text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta) \subset F_\alpha$ for every $0 < \alpha < \kappa^+$;

- (ii) $|F_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$;
- (iii) for every $\alpha < \kappa^+$, and every $F \subset \text{cl}_{\theta_\kappa}(\bigcup_{\beta < \alpha} F_\beta)$ with $|F| \leq \kappa$ if $X \setminus \bigcup \mathcal{C} \neq \emptyset$ for some $\mathcal{C} \in \mathcal{C}_F$, then $F_\alpha \setminus \bigcup \mathcal{C} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define F_α . Since $|F_\beta| \leq 2^\kappa$ for each $\beta < \alpha$, we have $|\bigcup_{\beta < \alpha} F_\beta| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$. Then it follows from Theorem 4.7 that $|\text{cl}_{\theta_\kappa}(\bigcup_{\beta < \alpha} F_\alpha)| \leq 2^\kappa$. Hence, there are at most 2^κ subsets F of $\text{cl}_{\theta_\kappa}(\bigcup_{\beta < \alpha} F_\alpha)$ with $|F| \leq \kappa$ and for each such set F we have $|\mathcal{C}_F| \leq \kappa^\kappa = 2^\kappa$. For each $F \subset \text{cl}_{\theta_\kappa}(\bigcup_{\beta < \alpha} F_\alpha)$ with $|F| \leq \kappa$ and each $\mathcal{C} \in \mathcal{C}_F$ for which $X \setminus \bigcup \mathcal{C} \neq \emptyset$ we choose a point in $X \setminus \bigcup \mathcal{C} \neq \emptyset$ and let E_α be the set of all such points. Clearly $|E_\alpha| \leq 2^\kappa$. Let $F_\alpha = \text{cl}_{\theta_\kappa}(E_\alpha \cup (\bigcup_{\beta < \alpha} F_\alpha))$. Then it follows from our construction that F_α satisfies (i) and (iii) while (ii) follows from Theorem 4.7.

Now, let $G = \bigcup_{\alpha < \kappa^+} F_\alpha$. Clearly $|G| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$ and since $t_{\theta_1}(X) \leq \kappa$, G is closed (Lemma 4.5). To finish the proof it is sufficient to show that $G = X$. Suppose that there is $x \in X \setminus G$. Then for every $y \in G$ there is $V_y \in \mathcal{V}_y$ such that $x \notin \text{cl}(V_y)$. Since $\{V_y : y \in G\}$ is an open cover of G and G is closed, there is $F \subset G$ with $|F| \leq \kappa$ such that $G \subset \bigcup \{\text{cl}(V_y) : y \in F\}$. Clearly $x \notin \bigcup \{\text{cl}(V_y) : y \in F\}$. Since $|F| \leq \kappa$, there is $\beta < \kappa^+$ such that $F \subset F_\beta$. Then for $\mathcal{C} = \{\text{cl}(V_y) : y \in F\}$ we have $\mathcal{C} \in \mathcal{C}_F$ and $x \in X \setminus \bigcup \mathcal{C}$. Then it follows from our construction that $F_{\beta+1} \setminus \bigcup \mathcal{C} \neq \emptyset$ which contradicts $F_{\beta+1} \subset G \subset \bigcup \mathcal{C}$. Therefore $G = X$ and the proof is completed. \square

Lemma 4.10. *Let X be a Urysohn space. Then $\psi_{\theta^2}(X) \leq \psi(X)L(X)$.*

Proof. Let $\psi(X)L(X) = \kappa$, $x \in X$, and $\{V_\alpha(x) : \alpha < \kappa\}$ be a collection of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$. For each $\alpha < \kappa$, let $F_\alpha(x) = X \setminus V_\alpha(x)$ and for each $y \in F_\alpha(x)$ let $U_\alpha(x, y)$ and $W_\alpha(y, x)$ be open in X neighborhoods of x and y , respectively, such that $\text{cl}(U_\alpha(x, y)) \cap \text{cl}(W_\alpha(y, x)) =$

\emptyset . Then $\{W_\alpha(y, x) : y \in F_\alpha(x)\}$ is an open cover of $F_\alpha(x)$ in X and since $L(X) \leq \kappa$, there exists a subset $A_\alpha(x)$ of $F_\alpha(x)$ such that $|A_\alpha(x)| \leq \kappa$ and $\bigcup\{W_\alpha(y, x) : y \in A_\alpha(x)\}$ covers $F_\alpha(x)$. Note that $\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) \cap W_\alpha(y, x) = \emptyset$ for each $y \in A_\alpha(x)$. Therefore $\bigcap\{\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : y \in A_\alpha(x)\} \subseteq V_\alpha(x)$. Since $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$, we have $\bigcap\{\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : \alpha < \kappa, y \in A_\alpha(x)\} = \{x\}$. Therefore $\psi_{\theta^2}(X) \leq \kappa$. \square

Corollary 4.11. *If X is a Urysohn space, then $|X| \leq 2^{t_{\theta_1}(X)\psi(X)L(X)}$.*

It follows immediately from Corollary 4.11 that for the class of Urysohn spaces Theorem 4.9 sharpens the famous Arhangel'skiĭ–Šapirovskiĭ inequality $|X| \leq 2^{t(X)\psi(X)L(X)}$, which is valid for every Hausdorff space X .

In order to compare the inequality in Theorem 4.9 with Bella and Cammaroto inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is valid for every Hausdorff space, and the inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$, which is valid for every Urysohn space, we recall the following definition:

Definition 4.12. *Let X be a topological space, $A \subset X$ and n be a positive integer. A point $x \in X$ is $S(n)$ -separated from A if there exist open sets U_i , $i = 1, 2, \dots, n$ such that $x \in U_1$, $\overline{U}_i \subset U_{i+1}$ for $i = 1, 2, \dots, n-1$ and $\overline{U}_n \cap A = \emptyset$; x is $S(0)$ -separated from A if $x \notin \overline{A}$. X is an $S(n)$ -space if every two distinct points in X are $S(n)$ -separated (see [16] and [14]).*

It follows directly from the above definition that the $S(1)$ -spaces are exactly the Hausdorff spaces and the $S(2)$ -spaces are exactly the Urysohn spaces.

Lemma 4.13. *Let X be an $S(3)$ -space. Then $\psi_{\theta^2}(X) \leq \psi(X)aL_c(X)$.*

Proof. Let $\psi(X)aL_c(X) = \kappa$, $x \in X$, and $\{V_\alpha(x) : \alpha < \kappa\}$ be a collection of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$. Let $\alpha < \kappa$ and $F_\alpha(x) = X \setminus V_\alpha(x)$. Since X is an $S(3)$ -space, for each $y \in F_\alpha(x)$ we can find open neighborhoods $U_\alpha(x, y)$, $S_\alpha(x, y)$ and $W_\alpha(x, y)$ of x such that $\text{cl}(U_\alpha(x, y)) \subset S_\alpha(x, y) \subset \text{cl}(S_\alpha(x, y)) \subset W_\alpha(x, y)$ and $y \notin \text{cl}(W_\alpha(x, y))$. Let $W'_\alpha(y, x) = X \setminus \text{cl}(W_\alpha(x, y))$. Then $\{W'_\alpha(y, x) : y \in F_\alpha(x)\}$ is an open cover of the closed set $F_\alpha(x)$ and since $aL_c(X) \leq \kappa$, there exists a subset $A_\alpha(x)$ of $F_\alpha(x)$ such that $|A_\alpha(x)| \leq \kappa$ and $\bigcup\{\text{cl}(W'_\alpha(y, x)) : y \in A_\alpha(x)\}$ covers $F_\alpha(x)$. Note that for each $y \in A_\alpha(x)$ we have $\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) \subset \text{cl}_\theta(S_\alpha(x, y)) = \text{cl}(S_\alpha(x, y)) \subset W_\alpha(x, y)$,

hence $\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) \cap \text{cl}(W'_\alpha(y, x)) = \emptyset$. Therefore

$$\bigcap \{ \text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : y \in A_\alpha(x) \} \subseteq V_\alpha(x).$$

Since $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$, we have $\bigcap \{ \text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : \alpha < \kappa, y \in A_\alpha(x) \} = \{x\}$.

Therefore $\psi_{\theta^2}(X) \leq \kappa$. \square

Corollary 4.14. *If X is an $S(3)$ -space, then $|X| \leq 2^{t_{\theta^1}(X)\psi(X)aL_c(X)}$.*

Corollary 4.14 shows that for the class of $S(3)$ -spaces, Theorem 4.9 improves Bella and Cammaroto inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is valid for every Hausdorff space X and the inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$, which is valid for every Urysohn space X .

5. Results involving variations of the almost Lindelöf degree.

Recently in [3] the authors gave the following definition:

Definition 5.1. *The θ -almost Lindelöf degree of a subset Y of a space X is $\theta\text{-}aL(Y, X) = \min\{\kappa : \text{for every cover } \mathcal{V} \text{ of } Y \text{ consisting of open subsets of } X, \text{ there exists } \mathcal{V}' \subseteq \mathcal{V} \text{ such that } |\mathcal{V}'| \leq \kappa \text{ and } \bigcup \{ \text{cl}_\theta(\text{cl}(V)) : V \in \mathcal{V}' \} = Y\}$.*

The function $\theta\text{-}aL(X, X)$ is called θ -almost Lindelöf degree of the space X and is denoted by $\theta\text{-}aL(X)$.

The θ -almost Lindelöf degree with respect to closed subsets of X is $\theta\text{-}aL_c(X) = \sup\{\theta\text{-}aL(C, X) : C \subseteq X \text{ is closed}\}$.

The θ -almost Lindelöf degree with respect to θ -closed subsets of X is $\theta\text{-}aL_\theta(X) = \sup\{\theta\text{-}aL(F, X) : F \subseteq X \text{ is } \theta\text{-closed}\}$.

It follows directly from the above definition that $\theta\text{-}aL(X) \leq aL(X)$ and Example 3.2 in [3] shows that there exist spaces X for which that inequality could be strict. Also, since every θ -closed set is closed and $aL(F, X) \leq aL(X)$ whenever $F \subset X$ is a θ -closed set (Lemma 2.2), we have $\theta\text{-}aL_\theta(X) \leq \theta\text{-}aL_c(X) \leq aL_c(X)$ and $\theta\text{-}aL(X) \leq \theta\text{-}aL_\theta(X) \leq aL(X)$.

Now, using the above variations of the almost Lindelöf degree we can strengthen the results in some of our previous theorems. For example, the following theorem improves the inequality in Theorem 4.2.

Theorem 5.2. *For every Urysohn space X , $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)\theta\text{-}aL_\theta(X)}$.*

Proof. Let $t_\theta(X)\psi_{\theta^2}(X)\theta\text{-}aL_\theta(X) = \kappa$ and for each $x \in X$ we fix a collection $\mathcal{V}_x = \{V_\alpha(x) : \alpha < \kappa\}$ of open neighborhoods of x such that

$\bigcap_{\alpha < \kappa} \text{cl}_\theta(\text{cl}(V_\alpha(x))) = \{x\}$. For a non-empty subset A of X we denote by \mathcal{K}_A the set of all families $\mathcal{K} = \{\text{cl}_\theta(\text{cl}(U_a)) : a \in A, U_a \in \mathcal{V}_a\}$.

Now, let x_0 be an arbitrary point in X . Recursively we construct a family $\{F_\alpha : \alpha < \kappa^+\}$ of subsets of X with the following properties:

(i) $F_0 = \{x_0\}$ and $\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta) \subset F_\alpha$ for every $0 < \alpha < \kappa^+$;

(ii) $|F_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$;

(iii) for every $\alpha < \kappa^+$, and every $A \subset \text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta)$ with $|A| \leq \kappa$ if $X \setminus \bigcup \mathcal{K} \neq \emptyset$

for some $\mathcal{K} \in \mathcal{K}_A$, then $F_\alpha \setminus \bigcup \mathcal{K} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define F_α . Since $|F_\beta| \leq 2^\kappa$ for each $\beta < \alpha$, we have $|\bigcup_{\beta < \alpha} F_\beta| \leq$

$2^\kappa \cdot \kappa^+ = 2^\kappa$. Then it follows from Theorem 3.4 that $|\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta)| \leq 2^\kappa$. Hence,

there are at most 2^κ subsets A of $\text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta)$ with $|A| \leq \kappa$ and for each such set

A we have $|\mathcal{K}_A| \leq \kappa^\kappa = 2^\kappa$. For each $A \subset \text{cl}_\theta(\bigcup_{\beta < \alpha} F_\beta)$ with $|A| \leq \kappa$ and each

$\mathcal{K} \in \mathcal{K}_A$ for which $X \setminus \bigcup \mathcal{K} \neq \emptyset$ we choose a point in $X \setminus \bigcup \mathcal{K} \neq \emptyset$ and let E_α be the set of all such points. Clearly $|E_\alpha| \leq 2^\kappa$. Let $F_\alpha = \text{cl}_\theta(E_\alpha \cup (\bigcup_{\beta < \alpha} F_\beta))$. Then

it follows from our construction that F_α satisfies (i) and (iii) while (ii) follows from Theorem 3.4.

Now, let $G = \bigcup_{\alpha < \kappa^+} F_\alpha$. Clearly $|G| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$ and since $t_\theta(X) \leq \kappa$,

G is θ -closed. To finish the proof it is sufficient to show that $G = X$. Suppose that there is $x \in X \setminus G$. Then for every $y \in G$ there is $V_y \in \mathcal{V}_y$ such that $x \notin \text{cl}_\theta(\text{cl}(V_y))$. Since $\{V_y : y \in G\}$ is an open cover of G and G is θ -closed, there is $A \subset G$ with $|A| \leq \kappa$ such that $G \subset \bigcup \{\text{cl}_\theta(\text{cl}(V_y)) : y \in A\}$. Clearly $x \notin \bigcup \{\text{cl}_\theta(\text{cl}(V_y)) : y \in A\}$. Since $|A| \leq \kappa$, there is $\beta < \kappa^+$ such that $A \subset F_\beta$. Then for $\mathcal{K} = \{\text{cl}_\theta(\text{cl}(V_y)) : y \in A\}$ we have $\mathcal{K} \in \mathcal{K}_A$ and $x \in X \setminus \bigcup \mathcal{K}$. Then it follows from our construction that $F_{\beta+1} \setminus \bigcup \mathcal{K} \neq \emptyset$ which contradicts $F_{\beta+1} \subset G \subset \bigcup \mathcal{K}$. The proof is completed. \square

If in the proof of Theorem 5.2 we set $\kappa = bt_\theta(X)\theta\text{-}aL_\theta(X)$ and for the estimation of the upper bounds of the cardinality of θ -closures of sets we use Cammaroto-Kočinac' inequality $|\text{cl}_\theta(A)| \leq |A|^{bt_\theta(X)}$ then we can prove the following result which is stronger than the one in Theorem 5.2.

Theorem 5.3. *For every Urysohn space X , $|X| \leq 2^{bt_\theta(X)\theta\text{-}aL_\theta(X)}$.*

The following theorem improves the result in Theorem 4.9.

Theorem 5.4. *For every Urysohn space X , $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta_2}(X)\theta\text{-}aL_c(X)}$.*

Proof. Let $t_{\theta_1}(X)\psi_{\theta_2}(X)\theta\text{-}aL_c(X) = \kappa$ and for each $x \in X$ we fix a collection $\mathcal{V}_x = \{V_\alpha(x) : \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} \text{cl}_\theta(\text{cl}(V_\alpha(x))) = \{x\}$. For every non-empty subset A of X we denote by \mathcal{C}_A the set of all families $\mathcal{C} = \{\text{cl}_\theta(\text{cl}(V_a)) : a \in A, V_a \in \mathcal{V}_a\}$.

Now, let x_0 be an arbitrary point in X . Recursively we construct a family $\{F_\alpha : \alpha < \kappa^+\}$ of subsets of X with the following properties:

(i) $F_0 = \{x_0\}$ and $\text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta) \subset F_\alpha$ for every $0 < \alpha < \kappa^+$;

(ii) $|F_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$;

(iii) for every $\alpha < \kappa^+$, and every $F \subset \text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta)$ with $|F| \leq \kappa$ if $X \setminus \bigcup \mathcal{C} \neq \emptyset$

for some $\mathcal{C} \in \mathcal{C}_F$, then $F_\alpha \setminus \bigcup \mathcal{C} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define F_α . Since $|F_\beta| \leq 2^\kappa$ for each $\beta < \alpha$, we have $|\bigcup_{\beta < \alpha} F_\beta| \leq$

$2^\kappa \cdot \kappa^+ = 2^\kappa$. Then it follows from Theorem 4.7 that $|\text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta)| \leq 2^\kappa$. Hence,

there are at most 2^κ subsets F of $\text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta)$ with $|F| \leq \kappa$ and for each such

set F we have $|\mathcal{C}_F| \leq \kappa^\kappa = 2^\kappa$. For each $F \subset \text{cl}_{\theta\kappa}(\bigcup_{\beta < \alpha} F_\beta)$ with $|F| \leq \kappa$ and each

$\mathcal{C} \in \mathcal{C}_F$ for which $X \setminus \bigcup \mathcal{C} \neq \emptyset$ we choose a point in $X \setminus \bigcup \mathcal{C} \neq \emptyset$ and let E_α be the set of all such points. Clearly $|E_\alpha| \leq 2^\kappa$. Let $F_\alpha = \text{cl}_{\theta\kappa}(E_\alpha \cup (\bigcup_{\beta < \alpha} F_\beta))$. Then it

follows from our construction that F_α satisfies (i) and (iii) while (ii) follows from Theorem 4.7.

Now, let $G = \bigcup_{\alpha < \kappa^+} F_\alpha$. Clearly $|G| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$ and since $t_{\theta_1}(X) \leq \kappa$, G is closed (Lemma 4.5). To finish the proof it is sufficient to show that $G = X$. Suppose that there is $x \in X \setminus G$. Then for every $y \in G$ there is $V_y \in \mathcal{V}_y$ such that $x \notin \text{cl}_\theta(\text{cl}(V_y))$. Since $\{V_y : y \in G\}$ is an open cover of G and G is closed, there is $F \subset G$ with $|F| \leq \kappa$ such that $G \subset \bigcup \{\text{cl}_\theta(\text{cl}(V_y)) : y \in F\}$. Clearly $x \notin \bigcup \{\text{cl}_\theta(\text{cl}(V_y)) : y \in F\}$. Since $|F| \leq \kappa$, there is $\beta < \kappa^+$ such that $F \subset F_\beta$. Then for $\mathcal{C} = \{\text{cl}_\theta(\text{cl}(V_y)) : y \in F\}$ we have $\mathcal{C} \in \mathcal{C}_F$ and $x \in X \setminus \bigcup \mathcal{C}$. Then it follows from our construction that $F_{\beta+1} \setminus \bigcup \mathcal{C} \neq \emptyset$ which contradicts $F_{\beta+1} \subset G \subset \bigcup \mathcal{C}$. Therefore $G = X$ and the proof is completed. \square

Lemma 5.5. *Let X be an $S(4)$ -space. Then $\psi_{\theta^2}(X) \leq \psi(X)\theta\text{-}aL_c(X)$.*

Proof. Let $\psi(X)\theta\text{-}aL_c(X) = \kappa$, $x \in X$, and $\{V_\alpha(x) : \alpha < \kappa\}$ be a collection of open neighborhoods of x such that $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$. Let $\alpha < \kappa$ and $F_\alpha(x) = X \setminus V_\alpha(x)$. Since X is an $S(4)$ -space, for each $y \in F_\alpha(x)$ we can find open neighborhoods $U_\alpha(x, y)$, $S_\alpha(x, y)$, $T_\alpha(x, y)$ and $W_\alpha(x, y)$ of x such that $\text{cl}(U_\alpha(x, y)) \subset S_\alpha(x, y) \subset \text{cl}(S_\alpha(x, y)) \subset T_\alpha(x, y) \subset \text{cl}(T_\alpha(x, y)) \subset W_\alpha(x, y)$ and $y \notin \text{cl}(W_\alpha(x, y))$. Let $W'_\alpha(y, x) = X \setminus \text{cl}(W_\alpha(x, y))$. Then $\{W'_\alpha(y, x) : y \in F_\alpha(x)\}$ is an open cover of the closed set $F_\alpha(x)$ and since $\theta\text{-}aL_c(X) \leq \kappa$, there exists a subset $A_\alpha(x)$ of $F_\alpha(x)$ such that $|A_\alpha(x)| \leq \kappa$ and $\bigcup \{\text{cl}_\theta(\text{cl}(W'_\alpha(y, x))) : y \in A_\alpha(x)\}$ covers $F_\alpha(x)$. Also, $\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) \subset \text{cl}_\theta(S_\alpha(x, y)) = \text{cl}(S_\alpha(x, y)) \subset T_\alpha(x, y)$ and $\text{cl}_\theta(\text{cl}(W'_\alpha(y, x))) \subset \text{cl}_\theta(X \setminus W_\alpha(x, y)) \subset \text{cl}_\theta(X \setminus \text{cl}(T_\alpha(x, y))) \subset X \setminus T_\alpha(x, y)$ whenever $y \in A_\alpha(x)$. Hence, for every $y \in A_\alpha(x)$ we have $\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) \cap \text{cl}_\theta(\text{cl}(W'_\alpha(y, x))) = \emptyset$. Thus, $\bigcap \{\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : y \in A_\alpha(x)\} \subseteq V_\alpha(x)$. Since $\bigcap_{\alpha < \kappa} V_\alpha(x) = \{x\}$, we have $\bigcap \{\text{cl}_\theta(\text{cl}(U_\alpha(x, y))) : \alpha < \kappa, y \in A_\alpha(x)\} = \{x\}$. Therefore $\psi_{\theta^2}(X) \leq \kappa$. \square

Corollary 5.6. *If X is an $S(4)$ -space, then $|X| \leq 2^{t_{\theta_1}(X)\psi(X)\theta\text{-}aL_c(X)}$.*

Corollary 5.6 shows that for the class of $S(4)$ -spaces, Theorem 5.4 improves Bella and Cammaroto inequality $|X| \leq 2^{t(X)\psi_c(X)aL_c(X)}$, which is valid for every Hausdorff space X and the inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$, which is valid for every Urysohn space X .

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