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FRAGMENTABILITY OF OPEN SETS VIA SET-VALUED MAPPINGS*

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Dedicated to the memory of our dear friend and colleague
Stoyan Nedev

Abstract. We study the notion of fragmentability of nonempty open sets in a topological space X and provide several characterizations of this concept via properties of set-valued mappings taking their values in X. As a corollary we obtain that in a compact space X the nonempty open sets are fragmentable if, and only if, the set of all continuous real-valued functions in X which attain their minimum at exactly one point in X is a residual subset of the space $C(X)$ of all bounded continuous real-valued functions in X, equipped with the uniform convergence norm.

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1. Introduction. Let X be a regular topological space and suppose that d is a metric in X (which is not obliged to be related to the original topology in X). It is said that the space X is *fragmentable* by d if for any nonempty set A of X and each $\varepsilon > 0$ there is an open set V of X which intersects A and moreover, the d-diameter of $V \cap A$ is smaller than ε. An equivalent terminology is: *the metric d fragments the space X*. The notion of fragmentability was introduced by Jayne and Rogers in [12]. Different characterizations of this notion as well as various applications in topology and analysis have been investigated in a number of articles (cf. [10, 11, 14, 15, 16, 17, 20, 21]). A natural example of such a space is any metric space considered with its metric topology. Less obvious examples include some infinite dimensional Banach spaces (dual Banach spaces) which are fragmentable by the norm if we equip them with the weak (weak*) topology.

In this article we continue our investigation from [8] where we studied the topological spaces X in which there is a metric d which fragments only the nonempty open sets, i.e. in the above definition of Jayne and Rogers, A is a nonempty open set of X. We called such spaces *fos-spaces*. Evidently, this notion is weaker than the notion of fragmentability. One might trace this concept up to the paper of Thielman [22] where the author introduced the following notion: let X be a topological space, (Y, d) be a metric space and $f : X \to Y$ is a mapping between them. The mapping f was called *cliquish* in [22] if for every nonempty open set U of X and any $\varepsilon > 0$ there exists a nonempty open subset $V \subset U$ such that $d - \operatorname{diam}(f(V)) < \varepsilon$. One obtains the notion of fragmentability of nonempty open sets in X by taking $X = Y$ and the identity mapping in X. The notion of cliquishness is present also (in an implicit way) in the definition of *neighborly-prime functions* from Bledsoe [1].

In [8] we established several characterizations, both internal and external, of the fos-spaces. One of them includes involvement of a certain topological game which we will introduce in the next section and which will be used in the sequel. In this article we will give several other characterizations of the fos-spaces involving set-valued mappings which take values in such spaces.

The paper is organized as follows. In the next section we give some preliminaries, including the presentation of the topological game involved in the characterization of the fos-spaces. In particular, we formulate the characterization of the fos-spaces via the existence of a winning strategy for one of the players in this game from the paper [8]. In Section 3 we prove several characterizations of the fos-spaces using set-valued mappings taking values in such spaces. Section 4 deals with investigation of properties (like uniqueness of the solution) of optimization problems generated by bounded continuous functions in completely regular fos-spaces.

2. Preliminaries. In this section we first introduce a topological game which was considered and used in [8] for characterization of the fos-spaces. In a given regular topological space X, let us consider the following infinite topological game played between two players, denoted by Player I and Player II: Player I makes the first move by selecting a nonempty open set U_1 and Player II responds with a nonempty open set V_1 of X such that $V_1 \subset U_1$. At the nth move, $n \geq 2$, Player I selects a non empty open set U_n which is included in the previous choice of Player II and then Player II chooses a nonempty open set $V_n \subset U_n$. Proceeding in this way the players produce an infinite sequence of open sets $\{U_n, V_n\}_n$ (which satisfies $U_{n+1} \subset V_n \subset U_n$ for every n) which is called *a play* in the game. The Player II wins this play if the intersection $\cap_n U_n = \cap_n V_n$ is either empty or a singleton. Otherwise, Player I wins this play. We denote this game by $FO(X)$.

The game $FO(X)$ with different winning conditions includes one of the most known variants of the Banach-Mazur game–see e.g. Section 4 below.

Any finite sequence of nonempty open sets of the type $(U_1, V_1, \dots, U_n)$ or $(U_1, V_1, \dots, U_n, V_n)$, where U_i, $i = 1, \dots, n$, are legal choices of Player I and V_i, $i = 1, \dots, n$, are legal choices of Player II in the game $FO(X)$ is called a *partial play* in this game.

A *strategy* s for Player II in the game $FO(X)$ is inductively constructed mapping which puts into correspondence to any partial play of the type $(U_1, \dots, U_n)$, $n \geq 1$, (where, when $n \geq 2$ the V_i, $i = 1, \dots, n-1$ are obtained by s) a nonempty open set $V_n := s(U_1, \dots, U_n)$ with $V_n \subset U_n$. A strategy s for Player II is *winning* if this player wins each play $\{U_n, V_n\}_n$ in the game $FO(X)$ for which $V_n = s(U_1, \dots, U_n)$ for every $n \geq 1$. Analogously, the notion of (winning) strategy for Player I is defined.

The next result, which will be used in the sequel, is part of one of our main results from [8] and shows that the existence of a winning strategy for Player II in the game $FO(X)$ is a necessary and sufficient condition for the fragmentability of the open sets in the underlying space. Let us mention that a similar result involving a modification of the above topological game and characterizing the fragmentable spaces (that is the spaces X in which there is a metric which fragments every nonempty subset of X) is proved in [15, 16].

Theorem 2.1 ([8]). *Let X be a regular topological space. Then Player II has a winning strategy in the game $FO(X)$ if, and only if, there is a metric d in X which fragments the nonempty open sets in X (i.e. X is a fos-space).*

3. Fragmentability of open sets and properties of set-valued mappings. In this section we will investigate the relation between the fragmentability of the nonempty open sets in a regular topological space X and the properties of a class of set-valued mappings taking its values in the space X. Equivalently, we will study the relations between the existence of winning strategies for Player II in the topological game $FO(X)$ and the corresponding properties of set-valued mappings. To this end, we need to introduce some terminology for set-valued mappings.

Let $F : Z \rightrightarrows X$ be a set-valued mapping between the topological spaces Z and X. By $\mathrm{Dom}\,(F)$ we denote the *effective domain* of F which is the set $\{z \in Z : F(z) \neq \emptyset\}$ and $\mathrm{Gr}(F) := \{(z, x) \in Z \times X : x \in F(z)\}$ is the *graph* of F. When we say that F *has a closed graph*, this means that the graph $\mathrm{Gr}(F)$ is a closed subset in the product topology in $Z \times X$. When A is a subset of Z then $F(A) := \cup\{F(z) : z \in A\}$ is the image of A under the mapping F and for a subset $B \subset X$ the two possible pre-images of B are $F^{\#}(B) := \{z \in Z : F(z) \subset B\}$ and $F^{-1}(B) := \{z \in Z : F(z) \cap B \neq \emptyset\}$. The mapping F is *upper semicontinuous at some point* $z_0 \in Z$ if for every open set V containing $F(z_0)$ there is un open set U which contains z_0 and such that $F(U) \subset V$. F is *upper semicontinuous in* Z if it is upper semicontinuous at any point of Z. The mapping F is said to be *quasi-continuous* at some $z_0 \in Z$ (for single-valued mappings this notion was introduced by Kempisty [13]) if for every open set U containing z_0 and for every open set V of X such that $F(z_0) \cap V \neq \emptyset$ there is a nonempty open set $U' \subset U$ such that $F(U') \subset V$. F is *quasi-continuous in* Z if it is so at any point of Z. Another way to say that F is quasi-continuous in Z is that for every open sets U of Z and V of X such that $F(U) \cap V \neq \emptyset$ there is a nonempty open set $U' \subset U$ such that $F(U') \subset V$. For mappings that are upper semicontinuous in Z and with nonempty compact values the notion of quasi-continuity means that (see [2]) the mapping F is *minimal* among all such mappings. Here the term minimal means that the graph of F is a minimal element among all other graphs of such mappings with respect to the usual partial order on the subsets of $Z \times X$.

Recall also that a set-valued mapping $F : Z \rightrightarrows X$ is *open* if $F(U)$ is a nonempty open set in X for every nonempty open set $U \subset Z$. The mapping F is called *demi-open* if for every nonempty open set U in Z we have $\emptyset \neq \mathrm{Int}\overline{F(U)}$

is dense in $\overline{F(U)}$. Here as usual, for a subset A of X, Int$A$ is the interior of A in X and $\overline{A}$ is its closure in X. Obviosuly, every open mapping is demi-open, the converse being not true, in general.

Before presenting our first result in this section we will introduce a special complete metric space which was considered in [18] and which is related to the possible (special) plays in the game $FO(X)$. For a given regular topological space X, let $\Sigma(X)$ be the family of all sequences of nonempty open sets $(U_n)_{n\geq 1}$ such that $\overline{U}_{n+1} \subset U_n$ for each $n \geq 1$. Each element of the space $\Sigma(X)$ can be considered as a special play in the game $FO(X)$ where the odd numbers of the sequence correspond to the choices of Player I and the even numbers of the sequence correspond to the choices of Player II. In the space $\Sigma(X)$ we consider the following metric ρ: for $\sigma_1 = (U_n)_n \in \Sigma(X)$ and $\sigma_2 = (U'_n)_n \in \Sigma(X)$ we define

$$\rho(\sigma_1, \sigma_2) := \frac{1}{n}, \text{ where } n := \inf\{k \geq 1 : U_k \neq U'_k\},$$

with the convention $1/\infty = 0$. It is easily seen that this is a metric in $\Sigma(X)$. In fact the following lemma, whose proof is easy, shows that it is a complete one:

Lemma 3.1. *Let X be a regular topological space. Then, $(\Sigma(X), \rho)$ is a complete metric space.*

The space $\Sigma(X)$ determines in a natural way the following set-valued mapping $\Phi : \Sigma(X) \rightrightarrows X$:

$$\Phi((U_n)_n) := \cap_{n\geq 1} U_n, \qquad (U_n)_n \in \Sigma(X).$$

The next proposition summarizes some of the properties of the mapping Φ.

Proposition 3.2. *The mapping $\Phi : \Sigma(X) \rightrightarrows X$ has the following properties:*

(a) Φ *has a closed graph;*

(b) Φ *is an open mapping;*

(c) Φ *is quasi-continuous.*

Proof. To prove (a) let $\sigma_0 = (U_n)_n \in \Sigma(X)$ and $x_0 \in X$ be such that $x_0 \notin \Phi(\sigma_0) = \cap_{n\geq 1} U_n$. Since by definition of the elements of the space

$\Sigma(X)$ we have $\cap_n U_n = \cap_n \overline{U}_n$ then for some n_0 we have $x_0 \notin \overline{U}_{n_0}$. Therefore, there is an open neighborhood V of x_0 such that $V \cap U_{n_0} = \emptyset$. Take the open ball $B(\sigma_0, 1/n_0)$ in the space $\Sigma(X)$ and let $\sigma = (U'_n)_n$ be an arbitrary element from this ball. According to the definition of the metric ρ this will imply that $U_k = U'_k$ for every $k = 1, \ldots, n_0$. In particular, $\Phi(\sigma) \subset U_{n_0}$ which implies that $(B(\sigma_0, 1/n_0) \times V) \cap \mathrm{Gr}(\Phi) = \emptyset$. And therefore, the graph $\mathrm{Gr}(\Phi)$ is closed in the product topology in $Z \times X$.

In order to check (b), take a nonempty open subset W of $\Sigma(X)$. First we prove that $\Phi(W)$ is nonempty. Indeed, let $\sigma_0 = (U_n)_n \in W$ and let n_0 be so large that $B(\sigma_0, 1/n_0) \subset W$. Let $x_0 \in U_{n_0+1}$ and using the regularity of X, let U'_{n_0+k}, $k \geq 2$, be a sequence of open sets in X such that $\overline{U}'_{n_0+2} \subset U_{n_0+1}$, $x_0 \in U'_{n_0+k}$ and $\overline{U}'_{n_0+k+1} \subset U'_{n_0+k}$ for every $k \geq 2$. Then the sequence $\sigma := (U_1, \ldots, U_{n_0}, U_{n_0+1}, U'_{n_0+2}, \ldots)$ belongs to $B(\sigma_0, 1/n_0) \subset W$ and we have $x_0 \in \Phi(\sigma)$.

To see that $\Phi(W)$ is also open, let $x_0 \in \Phi(\sigma_0)$ for some $\sigma_0 = (U_n)_n \in W$. Let n_0 be so large that the open ball $B(\sigma_0, 1/n_0)$ be included in W. We have $x_0 \in U_{n_0+1}$. Take some arbitrary $x \in U_{n_0+1}$ and construct by induction nonempty open sets in X U'_{n_0+k}, $k \geq 2$, such that $\overline{U}'_{n_0+2} \subset U_{n_0+1}$, $x \in U'_{n_0+k}$ and $\overline{U}'_{n_0+k+1} \subset U'_{n_0+k}$ for every $k \geq 2$. This is possible because X is regular. The sequence $\sigma := (U_1, \ldots, U_{n_0}, U_{n_0+1}, U'_{n_0+2}, \ldots)$ is obviously in $\Sigma(X)$ and we have $\rho(\sigma_0, \sigma) < 1/n_0$, thus $\sigma \in W$. On the other hand, $x \in \Phi(\sigma)$ which shows that $U_{n_0+1} \subset \Phi(W)$, and thus, Φ is an open mapping.

Finally, let us prove (c). Let $\sigma_0 = (U_n)_n \in \Sigma(X)$ and for some nonempty open set V of X we have $\Phi(\sigma_0) \cap V \neq \emptyset$. Take a point $x_0 \in \Phi(\sigma_0) \cap V$. Let W be an arbitrary open set in $\Sigma(X)$ which contains σ_0. Take a ball $B(\sigma_0, \varepsilon)$ which is included in W for some $\varepsilon > 0$ and let n_0 be so large that $2/n_0 < \varepsilon$. Since $x_0 \in \Phi(\sigma_0)$ we have that $x_0 \in U_{n_0}$. Let U'_{n_0+1} be a nonempty open subset of X such that $x_0 \in U'_{n_0+1}$ and $\overline{U}'_{n_0+1} \subset U_{n_0} \cap V$. Construct further by induction a sequence of nonempty open sets U'_{n_0+k}, $k \geq 1$ such that $x_0 \in U'_{n_0+1}$ and $\overline{U}'_{n_0+k+1} \subset U'_{n_0+k}$ for every $k \geq 1$. Then the sequence $\sigma = (U_1, \ldots, U_{n_0}, U'_{n_0+1}, \ldots)$ is in $\Sigma(X)$ and one has $\rho(\sigma_0, \sigma) \leq 1/n_0$. Therefore, because of the choice of n_0 and ε, the ball $B(\sigma, 1/n_0)$ is included in W. It remains to mention that if $\sigma' = (V_n)_n \in B(\sigma, 1/n_0)$, then $\Phi(\sigma') \subset U'_{n_0+1} \subset V$ and therefore, $\Phi(B(\sigma, 1/n_0)) \subset V$. The proof is completed. $\square$

With this proposition in hand, in the next theorem we characterize the fact that X is a fos-space (which, according to Theorem 2.1 is equivalent to the existence of a winning strategy for Player II in the game $FO(X)$) by properties of set-valued mappings taking values in X. Below, in condition (d) the metric ρ is that defined above for the space $\Sigma(X)$.

Theorem 3.3. *Let X be a regular topological space. Then the following are equivalent:*

(a) *The space X is a fos-space;*

(b) *Every demi-open quasi-continuous set-valued mapping $F : (Z, \rho) \rightrightarrows X$, acting from a complete metric space (Z, ρ) into X is no more than single-valued at the points of a dense G_δ-subset of Z;*

(c) *Every open quasi-continuous set-valued mapping $F : (Z, \rho) \rightrightarrows X$, acting from a complete metric space (Z, ρ) into X is no more than single-valued at the points of a dense G_δ-subset of Z;*

(d) *The mapping $\Phi : (\Sigma(X), \rho) \rightrightarrows X$ is no more than single-valued at the points of a dense G_δ-subset of $\Sigma(X)$.*

Proof. To prove that (a) implies (b) suppose that X is a fos-space and let d be a metric in X which fragments the nonempty open sets in X and F be as in (b). For every $n \geq 1$, let

$$H_n := \bigcup \big\{W : \emptyset \neq W \text{ is open in } Z \text{ and } d - \mathrm{diam}(F(W)) < 1/n\big\}.$$

We claim that for every $n \geq 1$ the (open) set H_n is dense in Z. Indeed, fix $n \geq 1$ and take a nonempty open set W_0 in Z. The mapping F is demi-open and thus, the set $\mathrm{Int}\overline{F(W_0)}$ is nonempty and is dense in $\overline{F(W_0)}$. As d fragments the nonempty open sets in X there is a nonempty open set V of X such that $V \subset \mathrm{Int}\overline{F(W_0)}$ and $d - \mathrm{diam}(V) < 1/n$. Since $V \cap F(W_0) \neq \emptyset$ and F is quasi-continuous we can find a nonempty open set $W' \subset W_0$ such that $F(W') \subset V$. Obviously $d - \mathrm{diam}(F(W')) < 1/n$ and thus $W' \subset H_n$ showing that H_n is dense in Z. Therefore, the set $H := \cap_n H_n$ is a dense G_δ-subset of Z. Take $z_0 \in H$. Then $z_0 \in \cap_n W_n$ for some nonempty open sets W_n such that $d - \mathrm{diam}(F(W_n)) < 1/n$, $n \geq 1$. We have $F(z_0) \subset F(\cap_n W_n) \subset \cap_n F(W_n)$. And the latter intersection cannot be more than a singleton. Thus, at the points of H the mapping F is no more than a singleton. The proof of (a) implies (b) is completed.

The implications (b) implies (c) and (c) implies (d) are obvious. Therefore, let us prove that (d) implies (a).

Suppose that the mapping Φ is no more than single-valued at the points of some dense G_δ-subset G of $\Sigma(X)$. Let $G := \cap_{n\geq 1} G_n$, where G_n are open and dense subsets of $\Sigma(X)$. We construct a strategy s for Player II in the game $FO(X)$ in the following way: let U_1 be any nonempty open subset of X. It is easily seen that Φ is onto (i.e., the constant sequence $(X, X, \ldots)$ belongs to $\Sigma(X)$). Therefore, it follows by the quasi-continuity of Φ that there is a nonempty open set $W_1 \subset \Sigma(X)$ such that $\Phi(W_1) \subset U_1$. Without loss of generality we may think that $W_1 \subset G_1$ and that the ρ-diameter of W_1 is less than 1. We define the strategy s at the first step by $s(U_1) := V_1 = \Phi(W_1) \subset U_1$. The strategy s is well-defined at the first step since $\Phi(W_1)$ is a nonempty open subset of X.

Further, let (U_1, V_1, U_2) be an arbitrary partial play of length 2 in the game $FO(X)$ such that $V_1 = s(U_1)$ and let W_1 be the open set in $\Sigma(X)$ associated with the definition of s at U_1. We have $V_1 = \Phi(W_1)$ and $\emptyset \neq U_2 \subset \Phi(W_1)$. By the quasi-continuity of Φ there is a nonempty open set $W_2 \subset W_1$ such that $\Phi(W_2) \subset U_2$. We may think that $W_2 \subset G_2$, $\overline{W}_2 \subset W_1$ and that $\rho - \mathrm{diam}(W_2) < 1/2$. Put $s(U_1, V_1, U_2) := \Phi(W_2)$, the latter being a nonempty open set in X by Proposition 3.2, which is contained in U_2. Proceeding in this way we can define by induction the strategy s: suppose that the strategy has been defined in the above way for any partial play $(U_1, V_1, \ldots, V_{k-1}, U_k)$ of length k, $k = 1, \ldots, n$, $n \geq 2$. Take a partial play $(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$ of length $n+1$, where all V_k have been obtained by the strategy s and let $W_1, \ldots, W_n$ be the nonempty open sets in $\Sigma(X)$ associated with the definition of $V_k = s(U_1, \ldots, U_k)$, $k = 1, \ldots, n$. I.e. $V_k := \Phi(W_k)$, $\rho - \mathrm{diam}(W_k) < 1/k$ for every $k = 1, \ldots, n$ and $\overline{W}_{k+1} \subset W_k$ for each $k = 1, \ldots, n-1$. Since $\emptyset \neq U_{n+1} \subset V_n = \Phi(W_n)$ by the quasi-continuity of Φ we can obtain a nonempty open set $W_{n+1} \subset W_n$ such that $\Phi(W_{n+1}) \subset U_{n+1}$. We may think that $W_{n+1} \subset G_{n+1}$, $\overline{W}_{n+1} \subset W_n$ and that $\rho - \mathrm{diam}(W_{n+1}) < 1/(n+1)$. In such a way the strategy s for Player II in the game $FO(X)$ is completely defined.

Let now $\{U_n, V_n\}_{n\geq 1}$ be an s-play in the game $FO(X)$, that is a play in which each V_n is obtained via the strategy s. Let $(W_n)_n$ be the sequence of open sets in $\Sigma(X)$ associated with the play $\{U_n, V_n\}_n$ from the construction of s. We have that for every $n \geq 1$ the following is true:

(i) $V_n = \Phi(W_n)$;

(ii) $\overline{W}_{n+1} \subset W_n$ and $\rho - \mathrm{diam}(W_n) < 1/n$;

(iii) $W_n \subset G_n$.

Because of (ii) the intersection $\cap_n W_n$ is a singleton, say σ_0 and moreover, (iii) shows that $\sigma_0 \in G$. Since the graph of Φ is closed and the sets $(W_n)_n$ are a local base for σ_0 it can be seen that (see e.g. Proposition 3.1 from [6]) $\Phi(\cap_n W_n) = \cap_n \Phi(W_n)$. Thus, $\Phi(\sigma_0) = \Phi(\cap_n W_n) = \cap_n \Phi(W_n) = \cap_n V_n$, the latter equality being true because of (i). Finally, having in mind that $\sigma_0 \in G$ we see that the intersection $\cap_n V_n$ is no more than a singleton, which shows that the strategy s is winning for Player II in the game $FO(X)$. Then the fact that X is a fos-space follows by Theorem 2.1. The proof of the theorem is completed. $\square$

Remark 3.4. In fact, a close look at the proof that (a) implies (b) shows that, formally, we do not need that the mapping F is demi-open: what is enough, is that for every nonempty open set U of Z the set $\mathrm{Int}\overline{F(U)}$ be nonempty. But if the mapping F has the latter property and, in addition, F is quasi-continuous, then, it is easily seen that F is also demi-open.

Let us mention that the quasi-continuous demi-open mappings which also have closed graph were investigated in [5] from the point of view of existence of densely defined continuous selections.

Remark 3.5. Again looking carefully at the proof of (a) implies (b) one can see that in the conditions (b)–(c) the space Z is enough to be a Baire topological space.

Another remark is also in order here:

Remark 3.6. Again, a close look at the proof of the above theorem, this time at the proof (d) implies (a), shows that in this proof we do not use the specific properties of the mapping Φ. We have used only its three properties form Proposition 3.2. Therefore, the conditions from the above theorem are equivalent also to:

(d') there exist a complete metric space (Z, ρ) and an open quasi-continuous set-valued mapping $F : (Z, \rho) \rightrightarrows X$ with a closed graph, which is no more than single-valued at the points of a dense G_δ-subset of Z.

4. Fragmentability of open sets and attainability of infima of continuous functions. In this section we will give another characterization of the fragmentability of open sets in topological spaces via the attainability of infima of continuous bounded functions defined in the underlying space. This characterization again involves a certain set-valued mapping taking values in completely regular fos-spaces.

Let X be a completely regular topological space and let us denote by $C(X)$ the space of all bounded real-valued functions in X. In this space we consider the usual sup norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$, $f \in C(X)$, under which $C(X)$ is real Banach space. Consider the following set-valued mapping $M : C(X) \rightrightarrows X$ defined by $M(f) := \{x \in X : f(x) = \inf_X f\}$, $f \in C(X)$. The mapping M is called the *solution* mapping in X since it assigns to each $f \in C(X)$ the (possibly empty) set of the minima of f in X. The following proposition (the proof of which is omitted) summarizes some of the important properties of the mapping M:

Proposition 4.1. (see e.g. [4, 19]) *The mapping M is an open quasi-continuous mapping in X with closed graph and dense domain.*

In [19] a characterization was given of the generic existence of solutions for the optimization problems generated by the functions from $C(X)$ by existence of a winning strategy for one of the players in the following most known version of the Banach-Mazur game $BM(X)$ in X: two players, α and β, choose alternatively nonempty open sets (β starts first) as in the game $FO(X)$, $U_1 \supset V_1 \supset U_2 \supset V_2 \cdots$ (U_ns for β and V_ns for α). α wins the play $\{U_n, V_n\}_n$ in the game $BM(X)$ if the intersection $\cap_n U_n = \cap_n V_n$ is not empty. Otherwise, β wins. The notions of (winning) strategies for the players are defined in a complete analogy with the definition of the same notions in the game $FO(X)$.

It was proved in [19][Theorem 3.1] that the player α has a winning strategy in the Banach-Mazur game $BM(X)$ if, and only if, the set $E := \{f \in C(X) :$ the function f attains its minimum in $X\}$ contains a dense G_δ-subset of $C(X)$. In a subsequent paper [6] it was shown that the existence of special strategies in the game $BM(X)$ is equivalent to stronger conclusions on the minima for the continuous bounded functions in X (like uniqueness, or uniqueness and stability of the minima in the definition of the functions in the set E). After, in [9] the authors developed further a general approach to transform strategies between different games which allows also to prove such results. Further investigation related to variational principles for perturbations of lower semi-continuous functions can be found in [7].

Here, we have a similar (as the one above for the game $BM(X)$) characterization of the fos-spaces (equivalently for the existence of a wining strategy for Player II in the game $FO(X)$). Namely, in the next theorem we give another characterization of the fos-spaces, involving the set-valued mapping M.

Theorem 4.2. *Let X be a completely regular topological space. Then the following are equivalent:*

(a) *The space X is a fos-space;*

(b) *The set $\{f \in C(X) :$ the function f has no more than one minimum point in $X\}$ contains a dense G_δ-subset of $C(X)$ (equivalently, the mapping $M : C(X) \rightrightarrows X$ is no more than single-valued at the points of a dense G_δ-subset of $C(X)$).*

Proof. The fact that (a) implies (b) follows by Theorem 3.3 applied for the solution mapping M and having in mind its properties from Proposition 4.1. As to the inverse implication, it follows by Remark 3.6 applied for the mapping M using again the properties of the solution mapping from Proposition 4.1. □

When considering attainability of an infimum of a given function in a topological space X, a special attention is paid to the question of the uniqueness of the minimum and its stability. More precisely, given a bounded from below function $h : X \to \mathbb{R}$ it is said that the function h attains *strong minimum* in X, if there is a unique minimum point x_0 of h in X and every minimizing sequence $(x_n)_n \subset X$ for h (this means $h(x_n) \to \inf_X h$) converges to x_0. In optimization for this concept is used the term: "the problem to minimize h on X is *Tykhonov well-posed*".

In [8] we considered the case when the nonempty open sets of X are fragmentable by a complete metric d whose topology is stronger than the original topology in X and proved that this is equivalent to the fact that the space X contains a dense subset X_1 which is completely metrizable in its inherited topoology ([8]). On the other hand, in [3, 4] we proved that the completely regular topological space X contains a dense completely metrizable subspace exactly when the set $S := \{f \in C(X) : f$ attains its strong minimum in $X\}$ contains a dense G_δ-subset of $C(X)$ (see Theorem 3.5 from [4]). Therefore, we have the following immediate consequence of these two facts:

Theorem 4.3. *Let X be a completely regular topological space. Then the following are equivalent:*

(a) *The space X admits a complete metric d which fragments the nonempty open sets in X and the topology of d is stronger than the original topology of X;*

(b) *The set* $\{f \in C(X) :$ *the function* f *attains its strong minimum in* $X\}$ *contains a dense* G_δ*-subset of* $C(X)$.

It was shown in [8] that, if X is a compact fos-space, then there exists a complete metric which fragments the nonempty open subsest of X and which generates a topology which is stronger than the initial topology of X. Having this in mind we obtain the following immediate corollary of the previous theorem:

Corollary 4.4. *The compact space* X *is a fos-space if, and only if, the set* $\{f \in C(X) :$ *the function* f *attains its strong minimum in* $X\}$ *contains a dense* G_δ*-subset of* $C(X)$.

In fact, in connection with the last corollary, let us mention that in a compact topological space X a continuous bounded function (even more, a bounded below lower semi-continuous proper function) f attains its strong minimum in X exactly when f attains its minimum in X at only one point of X.

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