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Interval Analysis in the Extended Interval Space $\mathbb{I}\mathbb{R}$

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Abstract

This paper shows, how the extended Interval Space $\mathbb{I}\mathbb{R}$ can be used to write formulas, theorems, and proofs in a closed form, i.e. without using the left and right interval bounds. So a basic generalization and moreover a simplification and improvement of the theorems and proofs is achieved.

0. Introduction

The extension to generalized intervals (with negative width) retaining all important properties of interval analysis like isotoneness etc. is leading to a more closed space in algebraic as well as in lattice-theoretic sense. These advantages enable us to write formulas and proofs in a closed form without using the left and right interval bounds. So the theorems and their proofs can be shortened in many cases, the statements are more general and extended to generalized intervals. Furthermore the Interval Analysis resembles with classical analysis because the ideas of norm, metric etc. can be handled more easily. Some recently appeared papers show that this extended interval analysis facilitates or makes possible formulation and solving problems as described in [4], [8], [13], [14]. Moreover the extended interval space allows to "underestimate" interval expressions, that means a rounding "to the inner", and to transform this into the usual interval analysis. This problem occurs in safety problems, where a minimum set for the solutions instead of an inclusion is asked for.

1. The Extended Interval Space

In the following we regard only the interval analysis $\mathbb{I}\mathbb{R}$ over the field of real numbers. We can do this w.l.o.g., because the formulas in $\mathbb{I}\mathbb{R}^n$, $\mathbb{I}\mathbb{C}$ etc. are of analogous form as showed in [5] and [6].

The algebraic structure $(I\mathbb{R}, +, *, \subseteq)$ is a regular commutative semigroup with respect to addition. It can be embedded in an isotone group $(\mathbb{I}\mathbb{R}, +, \subseteq)$ as shown in [4], [5] and [6]. Moreover $(I\mathbb{R}, *, \subseteq)$ satisfies all assumptions requested in [6], so that $(I\mathbb{R}, +, *, \subseteq)$ can be embedded in the high algebraic structure $(\mathbb{I}\mathbb{R}, +, *, \subseteq)$ to be introduced now. Furthermore, $(I\mathbb{R}, \subseteq)$ can be extended to $(\mathbb{I}\mathbb{R}, \subseteq)$ so that $(\mathbb{I}\mathbb{R}, \bar{\cap}, \underline{\cup}, \subseteq)$ turns out to be a complete lattice. For special details see also [4], [5], [6], [7] and [9]. To shorten this paper some proofs and properties are neglected which are summarized in [7] and [9].

In the following $A, B, C, \ldots, Z \in \mathbb{I}\mathbb{R}$ are elements of the extended interval space and $a = [a, a] \in \mathbb{I}\mathbb{R}$ is the with a identified interval. With $A = [a, b]$ the left and right bounds of A are denoted by $\lambda(A) := a$ and $\rho(A) := b$ whereas the midpoint and radius of A are denoted by $\mu(A) := \frac{a+b}{2}$ and $\delta(A) := \frac{b-a}{2}$. The latter yields a so-called midpoint-radius designation of $A = (\mu(A), \delta(A))$. Furthermore, we define:

$$\mathscr{S}^* = \mathscr{S} \cup -\mathscr{S} \quad \mathscr{S} := \{A \in \mathbb{IR} \mid 0 \leqslant \lambda A \wedge 0 \leqslant \rho A\} \quad -\mathscr{S} := \{-A \mid A \in \mathscr{S}\}$$

$$\mathscr{T}^* = \mathscr{T} \cup \bar{\mathscr{T}} \quad \mathscr{T} := \{A \in \mathbb{IR} \mid \lambda A \leqslant 0 \leqslant \rho A\} \quad \bar{\mathscr{T}} := \{\bar{A} \mid A \in \mathscr{T}\}$$

$$\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \quad \mathscr{L}_1 := \{A \in \mathbb{IR} \mid \lambda A = 0\} \quad \mathscr{L}_2 := \{A \in \mathbb{IR} \mid \rho A = 0\}$$

(I0) $A = B :\Leftrightarrow \lambda A = \lambda B \wedge \rho A = \rho B$
$A \subseteq B :\Leftrightarrow \lambda B \leqslant \lambda A \wedge \rho A \leqslant \rho B$
$A + B := [\lambda A + \lambda B, \rho A + \rho B] = (\mu A + \mu B, \delta A + \delta B)$
$A - B := A + (-B)$ with $-B := [-\rho B, -\rho B] = (-\mu B, \delta B)$
$A \cdot B := [\lambda A \cdot \lambda B, \rho A \cdot \rho B] = (\mu A \mu B + \delta A \delta B, \mu A \delta B + \mu B \delta A)$
(the so-called hyperbolic product "$\cdot$").
$A * B :=$ Table 1

$A / B := A * 1/B$ for $B \in \mathscr{S}^* \backslash \mathscr{L}$ and $1/B := [1/\rho B, 1/\lambda B] = B \dfrac{1}{\mu(B)^2 - \delta(B)^2}$

$A \bar{\cap} B = \inf(A, B) = [\lambda A \sqcup \lambda B, \rho A \sqcap \rho B]$
$A \underline{\cup} B = \sup(A, B) = [\lambda A \sqcap \lambda B, \rho A \sqcup \rho B]$
$\bar{A} := [\rho A, \lambda A] = (\mu(A), -\delta(A))$ (conjugation)
$\mathscr{J} := [-1, 1]$, $\bar{\mathscr{J}} := [1, -1]$
$\mathscr{Q}(A) := \{T \in \mathbb{IR} \mid \bar{A} \subseteq T \subseteq A \vee A \subseteq T \subseteq \bar{A}\}$.

Table 1

$*$	$B \in \mathscr{S}$	$B \in \mathscr{T}$	$B \in -\mathscr{S}$	$B \in \bar{\mathscr{T}}$
$A \in \mathscr{S}$	$A \cdot B$	$\rho(A) \cdot B$	$\bar{A} \cdot B$	$\lambda(A) \cdot B$
$A \in \mathscr{T}$	$A \cdot \rho(B)$	$[\lambda(A)\rho(B) \sqcap \rho(A)\lambda(B), \lambda(A)\lambda(B) \sqcup \rho(A)\rho(B)]$	$\bar{A} \cdot \lambda(B)$	0
$A \in -\mathscr{S}$	$A \cdot \bar{B}$	$\lambda(A) \cdot \bar{B}$	$\bar{A} \cdot \bar{B}$	$\rho(A) \cdot \bar{B}$
$A \in \bar{\mathscr{T}}$	$A \cdot \lambda(B)$	0	$\bar{A} \cdot \rho(B)$	$[\lambda(A)\lambda(B) \sqcup \rho(A)\rho(B), \lambda(A)\rho(B) \sqcap \rho(A)\lambda(B)]$

So we get the following properties in $(\mathbb{IR}, +, *, /, \cdot, \bar{\cap}, \underline{\cup}, \subseteq)$:

Theorem 1.1.

(I1) $A + B = B + A$

(I2) $(A + B) + C = A + (B + C)$

(I3) $0 + A = A$

(I4) $A + (-\bar{A}) = 0$

(I5) $A \subseteq B \Leftrightarrow A + C \subseteq B + C$

(I6) $\left.\begin{array}{l}(A \underline{\cup} B) + C = (A + C) \underline{\cup} (B + C)\\ (A + B) \underline{\cup} C = (A \underline{\cup} (C - \bar{B})) + B\end{array}\right\}$ *(analogously for* $\bar{\cap}$*)*

(I7) $\overline{A + B} = \bar{A} + \bar{B}$, $A + \bar{A} \in \mathbb{R}$

(I8) $A * B = B * A$

(I9) $(A * B) * C = A * (B * C)$

(I10) $1 * A = A$, $0 * A = 0$

(I11) $A * 1/\bar{A} = 1 \Leftrightarrow A \in \mathscr{S}^* \backslash \mathscr{L}$

(I12) $A \subseteq B \Rightarrow A * C \subseteq B * C$

(I13) (i) $\displaystyle\bigwedge_{\substack{A, B \in \mathscr{S}^* \\ A * B \in \mathscr{S}}} \left.\begin{array}{l}(A \underline{\cup} B) * C = A * C \underline{\cup} B * C\\ (A * B) \underline{\cup} C = (A \underline{\cup} (C / \bar{B})) * B\end{array}\right\}$ *(analogously for* $\bar{\cap}$*)*

in general: $(A \underline{\cup} B) * C \subseteq A * C \underline{\cup} B * C$, $(A \bar{\cap} B) * C \supseteq A * C \bar{\cap} A * C$

(ii) $\displaystyle\bigwedge_{A, B \in \mathbb{IR}} A \subseteq B \Rightarrow \bigwedge_{u, v \in \mathbb{R}} \bigvee_{x \in \mathbb{R}} A * u \bar{\cap} B * v \subseteq A * x \subseteq B * v$

(I14) $\overline{A * B} = \bar{A} * \bar{B}$, $A * \bar{A} \in \mathbb{R}$

(I15) $A * B \in \mathscr{T} \Leftrightarrow A \in \mathscr{T} \vee B \in \mathscr{T}$
$A * B \in \bar{\mathscr{T}} \Leftrightarrow A \in \bar{\mathscr{T}} \vee B \in \bar{\mathscr{T}}$

(I16) $\displaystyle\bigwedge_{\substack{A, B \in \mathbb{IR} \\ a \in \mathbb{R}}} a * (A + B) = a * A + a * B$

(I17) $\displaystyle\bigwedge_{\substack{A \in \mathbb{IR} \\ a, b \in \mathbb{R} \\ a \cdot b \geqslant 0}} A * (a + b) = A * a + A * b$

(I18) (i) $\displaystyle\bigwedge_{A, B, C \in \mathbb{IR}} A * (B + C) \subseteq A * B + A * C$

(ii) $\displaystyle\bigwedge_{A, B, C \in \mathbb{IR}} A * (B + C) \supseteq A * B + A * C$

(iii) $\displaystyle\bigwedge_{A, C \in \mathbb{IR}} \bigwedge_{B + C \in \mathbb{IR}} A * (B + \bar{C}) \supseteq A * B + \bar{A} * \bar{C}$

(I19) $\displaystyle\bigwedge_{A, B \in \mathscr{S}^* \backslash \mathscr{L}} 1/(A * B) = 1/A * 1/B$

(I20) $\displaystyle\bigwedge_{A, B \in \mathscr{S}^* \backslash \mathscr{L}} 1/(A / B) = B / A$

(I21) $\displaystyle\bigwedge_{A \in \mathscr{S}^* \backslash \mathscr{L}} 1/A = A/(A * \bar{A})$

(I22) $\displaystyle\bigwedge_{\substack{A, B \in \mathscr{S}^* \backslash \mathscr{L} \\ A * B \in \mathscr{S}}} \begin{array}{l}1/(A \underline{\cup} B) = 1/A \underline{\cup} 1/B\\ 1/(A \bar{\cap} B) = 1/A \bar{\cap} 1/B\end{array}$

(I23) $A \cdot B = B \cdot A$

(I24) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

(I25) $1 \cdot A = A$

(I26) $\bigwedge_{A \notin \mathscr{L}} A \cdot 1/\bar{A} = 1$

(I27) $\overline{A \cdot B} = \bar{A} \cdot \bar{B},\ A \cdot \bar{A} \in \mathbb{R}$

(I28) $(A + B) \cdot C = A \cdot C + B \cdot C$

(I29) $A \cdot B = 0 \Leftrightarrow (A = a \cdot \bar{B} \in \mathscr{L} \wedge a \in \mathbb{R})$

(I30) $A \subseteq B \Leftrightarrow \bar{A} \supseteq \bar{B}$

(I31) $\overline{A \bar{\cap} B} = \bar{A} \veebar \bar{B}$

(I32) $(\mathbb{IR}, \bar{\cap}, \veebar, \subseteq)$ *conditionally complete lattice*

(I33) $(\mathbb{IR}, +, \subseteq)$ *isotone group*

(I34) $(\mathbb{IR}, *, \subseteq)$ *isotone semigroup, but* $(\mathbb{IR} \backslash \mathscr{T}^*, *, \subseteq)$ *isotone group*

(I35) $\mathbb{IR} = \mathrm{IR} \cup \overline{\mathrm{IR}}, \quad \mathbb{R} = \mathrm{IR} \cap \overline{\mathrm{IR}}$, *with* $\overline{\mathrm{IR}} = \{\bar{A} \mid A \in \mathrm{IR}\}$

(I36) $A \subseteq B \subseteq C \Rightarrow \mathscr{Q}(B) \subseteq \mathscr{Q}(A) \cup \mathscr{Q}(C)$
especially: $B \in \mathscr{Q}(A) \Rightarrow \mathscr{Q}(B) \subseteq \mathscr{Q}(A)$.

With arithmetical expressions we can construct the so-called interval functions. A further method to get interval functions is the direct extension of continuous real functions from $\mathbb{R}$ to $\mathbb{IR}$. For this purpose we give the following definitions:

Definition 1.2. Let $\tilde{f} \in C^0_{\mathscr{D}}$ with $\mathscr{D} = Y \times Y \times \cdots \times Y \times A_1 \times A_2 \times \cdots \times A_m$ be a continuous function, then $\tilde{f}$ can be extended on $\mathscr{D}^* \in \mathbb{IR}$ with $\mathscr{D}^* = \mathscr{Q}(Y) \times \cdots \times \mathscr{Q}(Y) \times \mathscr{Q}(A_1) \times \cdots \times \mathscr{Q}(A_m)$ and $(X, \mathscr{B}) \in \mathscr{D}^*$:

$$f(X, \mathscr{B}) = \tilde{f}(X, \ldots, X, B_1, \ldots, B_m) = \mathop{\mathscr{W}}_{\substack{x \in \mathscr{Q}(X) \times \cdots \times \mathscr{Q}(X) \\ a \in \mathscr{Q}(B_1) \times \cdots \times \mathscr{Q}(B_m)}} \{\tilde{f}(x, a)\} \tag{1.1}$$

For arbitrary $Z \in \mathbb{IR}$ the operator $\mathscr{W}$ is defined as:

$$\mathop{\mathscr{W}}_{z \in \mathscr{Q}(Z)} :\equiv \begin{cases} \displaystyle\bigcup_{z \in Z} & \text{for} \quad Z \in \mathrm{IR} \\ \displaystyle\bar{\bigcap_{z \in Z}} & \text{for} \quad Z \in \overline{\mathrm{IR}} \end{cases} \tag{1.2}$$

This operator is dependent on the order of evaluation, i.e.

$$\mathop{\mathscr{W}}_{(u,v) \in \mathscr{Q}(U) \times \mathscr{Q}(V)} :\equiv \mathop{\mathscr{W}}_{u \in \mathscr{Q}(U)} \cdot \mathop{\mathscr{W}}_{v \in \mathscr{Q}(V)}. \tag{1.3}$$

2. Topological Properties of $\mathbb{IR}$, Norm, Metric, Sequence, Width

$(\mathbb{IR}, +, *, /)$ is a normed space in the following sense:

Definition 2.1.

$$\bigwedge_{A \in \mathbb{IR}} |A| := \inf\{t \in \mathbb{R}^+ \mid t * \bar{\mathscr{J}} \subseteq A \subseteq t * \mathscr{J}\} =$$

$$= \inf\{t \in \mathbb{R}^+ \mid A \in t * \mathscr{Q}(\mathscr{J})\}. \tag{2.1}$$

This is a norm with the following properties:

Lemma 2.2.

(N1) $|A| \geqslant 0,\ |A| = 0 \Leftrightarrow A = 0$

(N2) $|A + B| \leqslant |A| + |B|$

(N3) $|a * A| = |a|\,|A|$

(N4) $|A * B| \leqslant |A|\,|B|$

(N5) $|1/A| = \dfrac{|A|}{A * \bar{A}}$ *for* $A \in \mathscr{S}^* \backslash \mathscr{L}$

(N6) $|A + B| = |A| + |B|$ *if* $A = -A \wedge B = -B \wedge A, B \in \mathscr{T}$ *or* $A, B \in \bar{\mathscr{T}}$
or if $A * B \in \mathscr{S}^* \backslash \mathscr{L}$, *resp.*

(N7) $|A * B| = |A|\,|B|$ *for* $A, B \in \mathrm{IR}$ *or* $A, B \in \overline{\mathrm{IR}}$

(N8) $A * B = |A| * B$ *for* $B = -B \wedge A, B \in \mathrm{IR}$ *or* $A, B \in \overline{\mathrm{IR}}$

(N9) $|A \cdot B| \leqslant |A|\,|B|$

(N10) $A \subseteq B \subseteq C \Rightarrow |B| \leqslant |A| \sqcup |C|$, *especially*: $a \subseteq A \Rightarrow |a| \leqslant |A|$

(N11) $|A \veebar B| \leqslant |A| \sqcup |B|, |A \bar{\cap} B| \leqslant |A| \sqcup |B|$, *but*
$|A \bar{\cap} B| \leqslant |A| \sqcap |B| \wedge |A \veebar B| = |A| \sqcup |B|$, *for* $A, B \in \mathrm{IR}$ *and*
$|A \veebar B| \leqslant |A| \sqcap |B| \wedge |A \bar{\cap} B| = |A| \sqcup |B|$, *for* $A, B \in \overline{\mathrm{IR}}$

(N12) $|\pm A| = |A|, |\bar{A}| = |A|$

(N13) $|A| \subseteq A * \operatorname{sign}(A)$ *for* $A \in \mathrm{IR}$; $|A| \supseteq A * \operatorname{sign}(A)$ *for* $A \in \overline{\mathrm{IR}}$

$$\textit{with} \quad \operatorname{sign}(A) = \begin{cases} +1 & A \in \mathscr{S} \\ \mathscr{J} & A \in \mathscr{T} \\ \bar{\mathscr{J}} & A \in \bar{\mathscr{T}} \\ -1 & A \in -\mathscr{S} \end{cases}$$

For the proof we use the following properties:

Lemma 2.3.

$$|W|\bar{\mathscr{J}} \subseteq W \subseteq |W|\mathscr{J} \tag{2.2}$$

$$t\bar{\mathscr{J}} \subseteq W \subseteq t\mathscr{J} \Leftrightarrow \bigwedge_{\substack{w \in \mathscr{Q}(W) \\ w \in \mathbb{R}}} t\bar{\mathscr{J}} \subseteq w \subseteq t\mathscr{J} \tag{2.3}$$

$$|W| = \sup_{w \in \mathscr{Q}(W)} |w| = |\lambda W| \sqcup |\rho W| \tag{2.4}$$

Proof. See [9].

Proof (to Lemma 2.2).

(N1) $|A| \geqslant 0$ by definition
$A = 0 \Leftrightarrow 0 \subseteq A \subseteq 0 \Leftrightarrow t = 0 \wedge t\bar{\mathscr{J}} \subseteq A \subseteq t\mathscr{J} \Leftrightarrow |A| = t = 0$

(N2) $|A|\bar{\mathscr{J}} \subseteq A \subseteq |A|\mathscr{J},\ |B|\bar{\mathscr{J}} \subseteq B \subseteq |B|\mathscr{J} \overset{(I5)(I17)}{\Rightarrow} (|A|+|B|)\bar{\mathscr{J}} \subseteq A+B \subseteq (|A|+|B|)\mathscr{J} \Rightarrow |A+B| \leqslant |A|+|B|$

(N3) $|A|\bar{\mathscr{J}} \subseteq A \subseteq |A|\mathscr{J} \overset{(I12)}{\Leftrightarrow}$

$a*|A|*\bar{\mathscr{J}} \subseteq a*A \subseteq a*|A|*\mathscr{J} \Leftrightarrow$
$|a||A|*\bar{\mathscr{J}} \subseteq a*A \subseteq |a|*|A|*\mathscr{J} \Leftrightarrow |a*A| \leqslant |a|\,|A|$

on the other hand we have

$|a*A|\bar{\mathscr{J}} \subseteq a*A \subseteq |a*A|\mathscr{J} \overset{(I12)}{\Leftrightarrow}$

$\frac{|a*A|}{a}\bar{\mathscr{J}} \subseteq A \subseteq \frac{|a*A|}{a}\mathscr{J} \Leftrightarrow$ for $\pm\mathscr{J} = \mathscr{J}$

$\frac{|a*A|}{|a|}\bar{\mathscr{J}} \subseteq A \subseteq \frac{|a*A|}{|a|}\mathscr{J} \Leftrightarrow |A| \leqslant \frac{|a*A|}{|a|} \Leftrightarrow |a|\,|A| \leqslant |a*A|$

(N4) $|A|\bar{\mathscr{J}} \subseteq A \subseteq |A|\mathscr{J} \wedge |B|\bar{\mathscr{J}} \subseteq B \subseteq |B|\mathscr{J} \overset{(I12)}{\Rightarrow}$

$|A|\,|B|*\bar{\mathscr{J}} \subseteq A*B \subseteq |A|\,|B|\mathscr{J} \Rightarrow |A*B| \leqslant |A|\,|B|$

Equality does not hold in generality, as shown by the following example for $I\!R \times \overline{I\!R}$:

$0 = |[0,1]*[1,0]| \neq |[0,1]|\,|[1,0]| = 1$ cf. (N7).

(N5) From (I21) we get at once

$$|1/A| = \left|\frac{A}{A*\bar{A}}\right| = \frac{|A|}{A*\bar{A}} \quad \text{for} \quad A*\bar{A} \in R^+$$

(N6) See [9].

(N7) See [9].

(N8) For $A, B \in I\!R$ we have with $B = B*\mathscr{J} = |B|\mathscr{J}$

$A \subseteq |A|\mathscr{J} \Rightarrow A*B \subseteq |A|\,|B|\mathscr{J}*\mathscr{J} = |A|\,|B|*\mathscr{J} = |A|B$ and with (N13)
$|A| \subseteq A*\mathrm{sign}(A) \subseteq A*\mathscr{J} \Rightarrow |A|B \subseteq A*B*\mathscr{J} = A*B$, i.e.

$$A*B = |A|B \text{ (N12)}$$

For $A, B \in \overline{I\!R}$ we have $\bar{A}*\bar{B} = |\bar{A}|*\bar{B} = |A|*\bar{B}$, hence

$A*B = \overline{|A|*\bar{B}} = |A|*B$

(N9) $|A \cdot B| = |\lambda A \lambda B| \sqcup |\rho A \rho B| = |\lambda A|\,|\lambda B| \sqcup |\rho A|\,|\rho B| \leqslant$
$\leqslant (|\lambda A| \sqcup |\rho A|)(|\lambda B| \sqcup |\rho B|) = |A|\,|B|$

(N10) With $|A|\bar{\mathscr{J}} \subseteq A \subseteq |A|\mathscr{J}$ and $|C|\bar{\mathscr{J}} \subseteq C \subseteq |C|\mathscr{J}$ follows $|A|\bar{\mathscr{J}} \subseteq A \subseteq B \subseteq C \subseteq |C|\mathscr{J}$ and therefore $|B| \leqslant |A| \sqcup |C|$.
If $a \subseteq A$ then $\bar{A} \subseteq a \subseteq A$ and with (N12) $|a| \leqslant |A| \sqcup |\bar{A}| = |A|$

(N11) With $|A|\bar{\mathscr{J}} \subseteq A \subseteq |A|\mathscr{J}$ and $|B|\bar{\mathscr{J}} \subseteq B \subseteq |B|\mathscr{J}$
we get $|A|\bar{\mathscr{J}} \subseteq |B|\bar{\mathscr{J}} \subseteq A \subseteq B \subseteq |A|\mathscr{J} \subseteq |B|\mathscr{J} \Leftrightarrow$

$|A|*\mathscr{J} \cup |B|*\mathscr{J} \overset{(I13)}{=} (|A| \cup |B|)*\mathscr{J}$ and in the same way

$|A|*\bar{\mathscr{J}} \cup |B|*\bar{\mathscr{J}} = (|A| \cup |B|)*\bar{\mathscr{J}}$, so that with (N10) holds:

$|A \cup B| \underset{(N4)}{\leqslant} |(|A| \cup |B|)*\mathscr{J}| \sqcup |(|A| \cup |B|)*\bar{\mathscr{J}}|$
$\leqslant ||A| \cup |B|| = |A| \sqcup |B|$

The other properties are proved in [9] analogously.

(N12) $\pm T = T$ and $t\bar{\mathscr{J}} \subseteq A \subseteq t\mathscr{J} \overset{(I30)}{\Leftrightarrow} t\bar{\mathscr{J}} \subseteq \bar{A} \subseteq t\mathscr{J}$ proves the assertion

(N13) see [9].

The norm defined in Definition 2.1 is inducing a metric in $I\!R$ in the following well-known way:

Definition 2.4.

$$\bigwedge_{A,B \in I\!R} q(A,B) := |A - \bar{B}| \qquad (2.5)$$

This is a metric with the following properties:

Lemma 2.5.

(M1) $q(A,B) \geqslant 0,\ q(A,B) = 0 \Leftrightarrow A = B,\ q(A,B) = q(B,A)$

(M2) $q(A,B) \leqslant q(A,C) + q(C,B)$

(M3) $q(A+B, A+C) = q(B,C)$

(M4) $q(A+B, C+D) \leqslant q(A,C) + q(B,D)$

(M5) $q(a*B, a*C) = |a| q(B,C),\ a \in R$
$q(A*b, A*c) = |A|\, q(b,c)$ *for* $b \cdot c \geqslant 0$

(M6) $q(A*B, A*C) \leqslant |A|\, q(B,C)$

(M7) $q(A/B, A/C) \leqslant \frac{|A|}{|B*C|} q(B,C)$

(M8) $A \subseteq B \subseteq C \Rightarrow q(B,C) \leqslant q(A,C) \wedge q(A,B) \leqslant q(A,C) \Leftrightarrow$
$q(B,C) \sqcup q(A,B) \leqslant q(A,C)$

(M9) $A_1 \subseteq X \subseteq A_2 \wedge B_1 \subseteq Y \subseteq B_2 \Rightarrow q(X,Y) \leqslant q(A_1,B_2) \sqcup q(A_2,B_1)$
For $X \in \mathscr{Q}(A)$ *and* $Y \in \mathscr{Q}(B)$ *we have*: $q(X,Y) \leqslant q(A,B)$

(M10) $q(A,B) = q(\bar{A}, \bar{B})$

(M11) $|A| - |B| \leqslant q(A,B)$

(M12) $q(A,B) \leqslant x \Leftrightarrow \bigvee_{C \in I\!R} \begin{cases} \bigwedge_{a \subseteq A+C} \bigvee_{b \subseteq B+C} q(a,b) \leqslant x \ \wedge \\ \bigwedge_{b \subseteq B+C} \bigvee_{a \subseteq A+C} q(a,b) \leqslant x \end{cases}$

(M13) *The set* $\mathscr{R}(A, B) := \{Z \mid q(A, B) = q(A, Z) + q(Z, B)\}$ *is non-empty for arbitrary* $A, B \in I\mathbb{R}$ *and is representing a rectangle in the* μ-δ-*plane with* A *and* B *as diagonal corners.*

(M14) $Z \in \mathscr{R}(A, 0) \Rightarrow q(A, Z) = |A| - |Z|$

Proof.

(M1) follows from (N1) and (N12)

(M2) $|A - \bar{B}| \overset{(I4)}{=} |A - \bar{C} + C - \bar{B}| \overset{(N2)}{\leqslant} |A - C| + |C - \bar{B}|$

(M3) $|A + B - \overline{A + C}| = |A - \bar{A} + B - \bar{C}| \overset{(I4)}{=} |B - \bar{C}|$

(M4) $|A + B - \overline{C + D}| = |A - \bar{C} + B - \bar{D}| \overset{(N2)}{\leqslant} |A - \bar{C}| + |B - \bar{D}|$

(M5) $|a * B - \overline{a * C}| = |a * B - a * \bar{C}| \overset{(I16)}{=} |a(B - \bar{C})| \overset{(N3)}{=} |a|\,|B - C|$

For the second formula we can assume $c \leqslant b$ w.l.o.g., so that with $b \cdot c \geqslant 0$ follows $0 \leqslant \dfrac{c}{b} \leqslant 1$. Hence with the just proved assertion we have

$$|A * b - \overline{A * c}| = |b| \left| A * 1 - \overline{A * \frac{c}{b}} \right| = |b| \left| A * \left(\frac{c}{b} + 1 - \frac{c}{b}\right) - \overline{A * \frac{c}{b}} \right| =$$

$$\overset{(I17)}{=} |b| \left| A * \frac{c}{b} + A * \left(1 - \frac{c}{b}\right) - \overline{A * \frac{c}{b}} \right| =$$

$$\overset{(I4)}{=} |b| \left| A * \left(1 - \frac{c}{b}\right) \right| \overset{(N3)}{=} |A|\,|b| \left| 1 - \frac{c}{b} \right| = |A|\,|b - c|$$

(M6) with $f(x, a) = a \cdot x$ and the interval function $f(X, A) = A * X$ we derive from Theorem 2.6

$$|ax - ay| \leqslant \sup_{a \in \mathscr{Q}(A)} |a|\,|x - y| = |A|\,|x - y|$$

and so

$$q(A * X, A * Y) \leqslant |A|\, q(X, Y).$$

(M7) With (M6) we have $q(A/B, A/C) \leqslant |A|\, q(1/B, 1/C)$. Furthermore

$$|1/B - 1/\bar{C}| = |\bar{C}/(B * C) - \overline{B/(B * C)}|$$

$$\overset{(M6)}{\leqslant} \frac{1}{|B * C|} |\bar{C} - B| = \frac{1}{|B * C|} q(B, C)$$

(M8) From $A \subseteq B \subseteq C \Leftrightarrow A - \bar{C} \subseteq B - \bar{C} \subseteq 0 \wedge 0 \subseteq B - \bar{A} \subseteq C - \bar{A}$ we have with (N10)

$$|B - C| \leqslant |A - \bar{C}| \wedge |A - \bar{B}| \leqslant |A - \bar{C}|.$$

(M9) $A_1 - \bar{B}_2 \subseteq X - \bar{Y} \subseteq A_2 - \bar{B}_1 \overset{(N10)}{\Rightarrow} |X - \bar{Y}| \leqslant |A_1 - \bar{B}_2| \sqcup |A_2 - \bar{B}_1|$

(M10) follows from (N12)

(M11) see [9]

(M12) "$\Rightarrow$" proves by $q(a, b) \overset{(M9)}{\leqslant} q(A + C, B + C) \overset{(M3)}{=} q(A, B) \leqslant x$

"$\Leftarrow$": $A + C \in I\mathbb{R}$ and $B + C \in I\mathbb{R}$ always holds and therefore with $|a - b| \leqslant x \Leftrightarrow a - b \subseteq \mathscr{J}x \wedge b - a \subseteq \mathscr{J}x$ we have

$$\bigwedge_{a \subseteq A + C} \bigvee_{b \subseteq B + C} a \subseteq \mathscr{J}x + b \wedge \bigwedge_{b \subseteq B + C} \bigvee_{a \subseteq A + C} b \subseteq \mathscr{J}x + a \Rightarrow$$

$$\bigwedge_{a \subseteq A + C} a \subseteq \mathscr{J}x + B + C \wedge \bigwedge_{b \subseteq B + C} b \subseteq \mathscr{J}x + A + C.$$

Therefore we have $A + C \subseteq \mathscr{J}x + B + C$ and $B + C \subseteq \mathscr{J}x + A + C$ such that

$A - \bar{B} \subseteq \mathscr{J}x$ and $B - \bar{A} \subseteq \mathscr{J}x$. Hence

$A - \bar{B} \subseteq \mathscr{J}x$ and $A - \bar{B} \supseteq \bar{\mathscr{J}}x$; therefore $\bar{\mathscr{J}}x \subseteq A - \bar{B} \subseteq x\mathscr{J}$

and with (N10) finally $|A - \bar{B}| \leqslant x$.

(M14) We have $|A| = q(A, 0) = q(A, Z) + q(Z, 0) \Rightarrow q(A, Z) = |A| - |Z|$

The following two theorems state important estimations for the interval analysis theoretically as well as in practical applications. They show, roughly spoken, that the topological properties of real functions hold in their interval extension, too.

Theorem 2.6. *Let* $f(x, a)\colon D_1 \times \cdots \times D_n \times A_1 \times \cdots \times A_m \to \mathbb{R}$ *with* $D_i \in I\mathbb{R}$, $A_j \in I\mathbb{R}$ *for* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ *be a continuous function in the sense of Lipschitz, which is (see Definition 1.2) extended on the interval domain* $\mathscr{D} = \mathscr{Q}(D_1) \times \cdots \times \mathscr{Q}(D_n)$ *with* $\mathscr{X} = (X_1, \ldots, X_n)$ *and* $\mathscr{A} = (A_1, \ldots, A_m)$ *(parameter domain) as in* (1.1):

$$f(\mathscr{X}, \mathscr{A}) = \mathop{\mathscr{W}}_{\substack{x \in \mathscr{Q}(X_1) \times \cdots \times \mathscr{Q}(X_n) \\ a \in \mathscr{Q}(A_1) \times \cdots \times \mathscr{Q}(A_m) = \mathscr{Q}(\mathscr{A})}} \{f(x, a)\}$$

The continuousity in the sense of Lipschitz shall hold for each variable x_i, $1 \leqslant i \leqslant n$ *in the form*

$$|f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n, \mathscr{A}) - f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n, \mathscr{A})| \leqslant l_i |s - t| \tag{*}$$

for $x_j \in D_j$, $1 \leqslant j \leqslant n$, $j \neq i$ *and* $a \in \mathscr{Q}(\mathscr{A})$. *Then we have for all* $\mathscr{X}, \mathscr{Y} \in \mathscr{D}$:

$$q(f(\mathscr{X}, \mathscr{A}), f(\mathscr{Y}, \mathscr{A})) \leqslant \sum_{i=1}^{n} l_i q(X_i, Y_i) \tag{2.6}$$

i.e. the interval extension is continuous in the sense of Lipschitz, too.

Proof. With $(*)$ we have immediately from the triangle inequation

$$q(f(x,a),f(y,a)) = |f(x,a)-f(y,a)| \overset{(*)}{\leqslant} \sum_{i=n}^{1} l_i|x_i - y_i| \overset{(M9)}{\leqslant} \sum_{i=1}^{n} l_i q(X_i, Y_i) \qquad (**)$$

Let now $C \in \mathbb{I}\mathbb{R}$ such that

(i) for every $u = f(x,a) \subseteq f(\mathcal{X},\mathcal{A}) + C$ with $x \in \mathcal{Q}(\mathcal{X}) \wedge a \in \mathcal{Q}(\mathcal{A})$ there always exists a $v = f(y,a) \subseteq f(\mathcal{Y},\mathcal{A}) + C$ with $y \in \mathcal{Q}(\mathcal{Y})$. Then

$$q(u,v) \overset{(**)}{\leqslant} \sum_{i=1}^{n} l_i q(X_i, Y_i),$$

(ii) and for every $v = f(y,b) \subseteq f(\mathcal{Y},\mathcal{A}) + C$ with $y \in \mathcal{Q}(\mathcal{Y}) \wedge b \in \mathcal{Q}(\mathcal{A})$ there always exists a $u = f(x,b) \subseteq f(\mathcal{X},\mathcal{A}) + C$ with $x \in \mathcal{Q}(\mathcal{X})$. Then

$$q(u,v) \overset{(**)}{\leqslant} \sum_{i=1}^{n} l_i q(X_i, Y_i).$$

So we get with (M12)

$$q(f(\mathcal{X},\mathcal{A}),f(\mathcal{Y},\mathcal{A})) \overset{(M3)}{=} q(f(\mathcal{X},\mathcal{A})+C, f(\mathcal{Y},\mathcal{A})+C) \overset{(M12)}{\leqslant}$$

$$\overset{(i)(ii)}{\leqslant} \sum_{i=1}^{n} l_i q(X_i, Y_i).$$

Theorem 2.7. *If in Theorem 2.6 $D_1 = \cdots = D_n = D$ and holds*

$$f(X,\mathcal{A}) = \mathop{\mathcal{W}}_{\substack{x \in \mathcal{Q} \\ a \in \mathcal{Q}(\mathcal{A})}} \{\bar{f}(x,a)\} = \bar{f}(X,\ldots,X,\mathcal{A})$$

then for every $X, Y \in \mathcal{D}$

$$q(f(X,\mathcal{A}),f(Y,\mathcal{A})) \leqslant \sum_{i=1}^{n} l_i q(X,Y) = lq(X,Y).$$

After introducing a metric $\mathbb{I}\mathbb{R}$ becomes a metric and therefore a topological space. Therefore we can use Cauchy sequences, convergence and continuousity as usually.

Lemma 2.8.

(2.7) *Every sequence in $\mathbb{I}\mathbb{R}$ has at least one limes point in $\mathbb{I}\mathbb{R}$.*

(2.8) *$\{A_k\}_{k\geqslant 0}$ is a Cauchy sequence if $\{\lambda A_k\}_{k\geqslant 0}$ and $\{\rho A_k\}_{k\geqslant 0}$ are both Cauchy sequences, i.e.* $\lim_{k\to\infty} A_k = A \Leftrightarrow \lim_{k\to\infty} \lambda A_k = \lambda A$ $\lim_{k\to\infty} \rho A_k = \rho A$.

(2.9) *The operations $+, -, *, /, \bar{\cap}, \underline{\cup}$ and conjugation are continuous operators. Moreover $f(\mathcal{X},A)$ is a continuous function if $f(x,a)$ like in (1.1) is one.*

(2.10) *Every sequence $\{A_k\}_{k\geqslant 0}$ with $A_0 \supseteq A_1 \supseteq \cdots \supseteq B$ is a Cauchy sequence and convergent against the interval $A = \bigcap A_k \supseteq B$.*

Proof.

(2.7) Follows from the homeomorphism of $(\mathbb{I}\mathbb{R}, |\cdot|)$ and $(\mathbb{R}^2, \|\cdot\|)$ with respect to the maximum norm and (2.4).

(2.8) $|A_k - \bar{A}| \overset{(2.4)}{=} |\lambda A_k - \lambda A| \sqcup |\rho A_k - \rho A|$ proves the assertion.

(2.9) see [9].

(2.10) The sequence $\{A_i - \bar{B}\}_{i\geqslant 0}$ is lower bounded by 0 and therefore convergent and so a Cauchy sequence with limes A. For $\bigcap$ is isotone we have for every n

$\bigcap_{i\geqslant 0}^{n} A_i \supseteq A_n \supseteq A \supseteq B$ and hence the assertion for $n \to \infty$.

For the quality of numerical algorithms in interval analysis the width is an important criterion.

Definition 2.9. The functional

(2.11) $\bigwedge_{A \in \mathbb{I}\mathbb{R}} d(A) := |A - A| = q(A, \bar{A})$

is called the width (diameter) of the interval A.

For d the following properties hold:

Lemma 2.10.

(2.12) $d(A) = |\rho A - \lambda A|, \quad d(\bar{A}) = d(A)$

(2.13) $d(A) = 0 \Leftrightarrow A = a \in \mathbb{R}$

(2.14) $A \subseteq B \subseteq C \Rightarrow d(B) \leqslant d(A) \sqcup d(C)$

(2.15) $d(A+B) = d(A) + d(B)$

(2.16) $d(A*B) \leqslant d(A)|B| + |A|\,d(B)$

(2.17) (i) $d(A*B) \geqslant |A|\,d(B) \sqcup d(A)\,|B|$ *for $A, B \in \mathbb{I}\mathbb{R}$ or $A, B \in \overline{\mathbb{I}\mathbb{R}}$*

(ii) $d(A*B) \geqslant d(A)\,d(B)$ *for $A, B \in \mathcal{T}^*$*

(2.18) (i) $d(a*B) = |a|\,d(B)$ *for $a \in \mathbb{R}$*

(ii) $d(A*B) = |B|\,d(A)$ *for $A, B \in \mathbb{I}\mathbb{R}$, $A \in \mathcal{T}$, $B \in \mathcal{S}^*$*
or $A, B \in \overline{\mathbb{I}\mathbb{R}}$, $A \in \mathcal{T}$, $B \in \mathcal{S}^*$

(2.19) $d(A^n) \leqslant n|A|^{n-1} d(A)$ *if* $A^n := \underbrace{A * A * \cdots * A}_{n \text{ factors}}$

(2.20) $d((A-B)^n) \leqslant 2\,d(A)^n$ *for $B \in \mathcal{Q}(A)$, $n \geqslant 1$*
$\leqslant 2^{1-n} d(A)^n$ *for $\mu(A) \subseteq B$, $n \geqslant 1$*

(2.21) $|A| \leqslant d(A) \leqslant 2|A|$ *for $A \in \mathcal{T}^*$*

(2.22) (i) $d(1/A) = \dfrac{d(A)}{A * \bar{A}}$ *for $A \notin \mathcal{T}^*$*

$$\text{(ii)}\quad A \subseteq X/Y \subseteq B \Rightarrow d(X/Y) \leqslant \left(\frac{d(X)}{|X|} + \frac{d(Y)}{|Y|}\right)(|A| \sqcup |B|)$$

for $Y \notin \mathscr{T}^*$ *and* $|X * Y| = |X|\,|Y|$ (*cf.* N7, N8).

(2.23) $\bar{B} \subseteq \bar{A} \subseteq A \subseteq B \Rightarrow q(A, B) \leqslant q(\bar{A}, B) \leqslant q(B, \bar{B}) = d(B)$

$$\bar{B} \subseteq \bar{A} \subseteq A \subseteq B \Rightarrow \frac{d(B) - d(A)}{2} \leqslant q(A, B) \leqslant d(B) - d(A)$$

(2.24) $d(A \bar{\sqcap} B) \leqslant d(A) \sqcap d(B)$ *for* $A, B \in \mathrm{I}\mathbb{R}$

Proof.

(2.12) $A - A = [\lambda A - \rho A, \rho A - \lambda A] \overset{(2.4)}{\Rightarrow} |A - A| = |\rho A - \lambda A|$

$$d(\bar{A}) = |\bar{A} - \bar{A}| \overset{(N12)}{=} |A - A| = d(A)$$

(2.13) $d(A) = 0 \overset{(2.12)}{\Rightarrow} \rho A = \lambda A \Rightarrow A \in \mathbb{R}$

(2.14) $A \subseteq B \subseteq C \overset{(I5)}{\Rightarrow} A - A \subseteq B - A \subseteq B - B \subseteq C - B \subseteq C - C \overset{(N10)}{\Rightarrow}$

$$d(B) = |B - B| \leqslant |C - C| \sqcup |A - A| = d(C) \sqcup d(A)$$

(2.15) $|(A + B) - (A + B)| = |A - A + B - B| \overset{(N2)}{\leqslant}$

$$|A - A| + |B - B| = d(A) + d(B)$$

(2.16) $d(A * B) = q(A * B, \overline{A * B}) \overset{(M2)}{\leqslant} q(A * B, \bar{A} * B) + q(\bar{A} * B, \overline{A * B}) \leqslant$

$$\overset{(M6)}{\leqslant} |B| q(A, \bar{A}) + |A| q(B, \bar{B}) = |B|\, d(A) + |A|\, d(B)$$

(2.17) (i) For $A, B \in \mathrm{I}\mathbb{R}$ we have

$$\left.\begin{array}{ll} & 0 \subseteq A * (B - B) \overset{(I18,i)}{\subseteq} A * B - A * B \\ \text{as well as} & \\ & 0 \subseteq B * (A - A) \overset{(I18,i)}{\subseteq} A * B - A * B \end{array}\right\} \overset{(N4)(N10)}{\Rightarrow}$$

$$|A|\, d(B) = |A|\,|B - B| \overset{(N7)}{=} |A * (B - B)| \overset{(N10)}{\leqslant} |A * B - A * B| = d(A * B)$$

as well as

$$|B|\, d(A) = |B|\,|A - A| \overset{(N7)}{=} |B * (A - A)| \overset{(N10)}{\leqslant} |B * A - B * A| = d(A * B),$$

and therefore $|A|\, d(B) \sqcup |B|\, d(A) \leqslant d(A * B)$.

For $A, B \in \overline{\mathrm{I}\mathbb{R}}$ the assertion follows with $d(A * B) = d(\overline{A * B}) = d(\bar{A} * \bar{B})$

(ii) proves with (i) and (2.21)

(2.18) (i) $d(a * B) = |a * B - a * B| \overset{(I16)}{=} |a * (B - B)| =$

$$\overset{(N3)}{=} |a|\,|B - B| = |a|\, d(B)$$

$$\text{(ii)}\ d(A * B) \overset{(N8)}{=} d(A * |B|) \overset{(i)}{=} |B|\, d(A)$$

(2.19) True for $n = 0$, so take induction over n:

$$d(A^{n+1}) = d(A^n * A) \overset{(2.16)}{\leqslant} |A|\, d(A^n) + |A|^n d(A) \overset{(N4)}{\leqslant}$$

$$\leqslant |A| n |A|^{n-1} d(A) + |A|^n d(A) = (n + 1)|A|^n d(A)$$

(2.20) see [9]

(2.21) see [9]

(2.22) (i) $d(1/A) \overset{(I11)}{=} d(1/A * A/\bar{A}) = d(A * 1/(A * \bar{A})) \overset{(2.18)}{=} \dfrac{d(A)}{A * \bar{A}}$

(ii) $A \subseteq X/Y \subseteq B \Rightarrow Y * \bar{Y} * A \subseteq Y * X \subseteq Y * \bar{Y} * B \Rightarrow$

$$\Rightarrow |Y * X| \overset{(N7)}{=} |Y|\,|X| \overset{(N10)}{\leqslant} Y * \bar{Y} * (|A| \sqcup |B|) \text{ and so}$$

$$|X| \leqslant \frac{Y * \bar{Y} * (|A| \sqcup |B|)}{|Y|} \text{ as well as } |Y| \leqslant \frac{Y * \bar{Y}(|A| \sqcup |B|)}{|X|} \qquad (*)$$

On the other hand

$$d\left(\frac{X}{Y}\right) = \frac{d(X * Y)}{Y * \bar{Y}} \overset{(2.16)}{\leqslant} \frac{d(X)|Y|}{Y * \bar{Y}} + \frac{d(Y)|X|}{Y * \bar{Y}} \overset{(*)}{\leqslant} \left(\frac{d(X)}{|X|} + \frac{d(Y)}{|Y|}\right)(|A| \sqcup |B|).$$

(2.23) $\bar{B} \subseteq \bar{A} \subseteq A \subseteq B \Rightarrow q(A, B) \leqslant q(\bar{A}, B) \leqslant q(\bar{B}, B) = d(B)$

on the other hand

$$d(B) = q(B, \bar{B}) \leqslant q(B, A) + q(A, \bar{A}) + q(A, \bar{B}) \overset{(M13)(M10)}{=} 2q(A, B) + d(A),$$

i.e. $\dfrac{d(B) - d(A)}{2} \leqslant q(A, B)$.

(2.24) see [9].

To conclude this chapter we give some important estimations for interval functions, at first an analogous estimation to (2.6) for the width.

Theorem 2.11. *Under the assumption of Theorem* 2.6 *we have for all* $\mathscr{X} \in \mathscr{D}$

$$d(f(\mathscr{X})) \leqslant \sum_{i=1}^{n} l_i\, d(X_i) \qquad (2.25)$$

where $\mathscr{A} = a \in \mathbb{R}^m$ *are no intervals.*

Proof.

$$d(f(\mathscr{X})) = q(f(\mathscr{X}), \overline{f(\mathscr{X})}) = q(f(\mathscr{X}), f(\bar{\mathscr{X}})) \overset{(2.6)}{\leqslant} \sum_{i=1}^{n} l_i q(X_i, \bar{X}_i) \overset{(2.11)}{=} \sum_{i=1}^{n} l_i d(X_i).$$

Furthermore we give estimations for the distance of interval functions with properties depending on the width of the arguments. Let be in the following $\mathscr{D} \subset \mathrm{I}\mathbb{R}$.

Theorem 2.12. *Let $g(X, \mathscr{A})$ be the interval extension of $g(x, a)$ like in Theorem 2.6 and let $f(X, \mathscr{A})$ be an arbitrary interval function satisfying*

$$\bigwedge_{X \in \mathscr{D}} \bigvee_{T \in \mathscr{Q}(X)} g(T, \mathscr{A}) \subseteq f(X, \mathscr{A}) \subseteq g(X, \mathscr{A}) \tag{2.26}$$

Then for all $X \in \mathscr{D}$

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant \sum_{i=1}^{n} l_i d(X) = l\, d(X) \tag{2.27}$$

If in (2.26) $\mu(X) \subseteq T$ or $f(X, \mathscr{A}) = \mathscr{W}_{\substack{x \in \mathscr{Q}(X) \\ a \in \mathscr{Q}(A)}} \{g(x, a)\}$

then we have

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant l/2\, d(X) \tag{2.28}$$

Proof.

(2.27): From (2.26) we get

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant q(g(T, \mathscr{A}), g(X, \mathscr{A})) \leqslant$$

$$\overset{(2.6)}{\leqslant} \sum_{i=1}^{n} l_i q(T, X) \leqslant \tag{$*$}$$

(respecting $\bar{X} \subseteq T \subseteq X$) $\quad \overset{(M8)}{\leqslant} \sum_{i=1}^{n} l_i q(\bar{X}, X) = l\, d(X)$

(2.28): From $\mu(X) \subseteq T \subseteq X$ we derive

$$q(T, X) \leqslant q(\mu(X), X) = q(X/2 + \bar{X}/2, X) \overset{(M3)}{=} q(\bar{X}/2, X/2) = d(X)/2,$$

so $(*)$ is estimated by $l/2\, d(X)$.

If $f(X, \mathscr{A}) = \mathscr{W}_{\substack{x \in \mathscr{Q}(X) \\ a \in \mathscr{Q}(\mathscr{A})}} \{g(x, a)\}$ then trivially

$g(\mu(X), \mathscr{A}) \subseteq f(X, \mathscr{A}) \subseteq g(X, \mathscr{A})$ holds.

Theorem 2.13. *Let $h(X, \mathscr{A})$ with $X \in \mathscr{D}$ be the interval extension of $h(x, a)$ for $a \in \mathscr{Q}(\mathscr{A})$ and let h be continuous in the sense of Lipschitz with constant l. Let for $G \in \mathrm{I}\mathbb{R}$*

$$g(X, \mathscr{A}) = G + h(X, \mathscr{A}) * (X - Z)$$

be an interval function in (generalized) centralized form [3] *with arbitrary $Z \in \mathscr{Q}(X)$*

and let $f(X, \mathscr{A})$ be an arbitrary interval function with the following property:

$$\bigwedge_{X \in \mathscr{D}} \bigwedge_{T \in \mathscr{Q}(X)} G + h(T, \mathscr{A}) * (X - \mathscr{A}) \subseteq f(X, \mathscr{A}) \subseteq g(X, \mathscr{A}) \tag{2.29}$$

then

$$\text{for all } X \in \mathscr{D}\colon q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant l \cdot d(X)^2 \tag{2.30}$$

holds. Moreover for $\mu(X) \subseteq T$ or $\mu(X) = Z$ in (2.29) we have

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant (l/2) \cdot d(X)^2. \tag{2.31}$$

If $\mu(X) \subseteq T$ and $\mu(X) = Z$ then

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant (l/4) \cdot d(X)^2. \tag{2.32}$$

Proof.

(2.30): From (2.29) we get

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \overset{(M8)}{\leqslant} q(G + h(T, \mathscr{A}) * (X - Z), G + h(X, \mathscr{A}) * (X - Z))$$

$$\overset{(M3)}{=} q(h(T, \mathscr{A}) * (X - Z), h(X, \mathscr{A}) * (X - Z)) \leqslant$$

$$\overset{(M6)}{\leqslant} |X - Z|\, q(h(T, \mathscr{A}), h(X, \mathscr{A})) \leqslant |X - Z| \cdot l \cdot q(T, X) \leqslant$$

$$\overset{(N10,M8)}{\leqslant} |X - X| \cdot l \cdot q(\bar{X}, X) \overset{(2.11)}{\leqslant} d(X) \cdot l \cdot d(X) = l \cdot d(X)^2 \tag{$*$}$$

with $\bar{X} - \bar{X} \subseteq X - Z \subseteq X - X$ and $\bar{X} \subseteq T \subseteq X$.

(2.31): From $\mu(X) = (X + \bar{X})/2 \subseteq T \subseteq X$ we get with (M8)

$$q(T, X) \leqslant q(X/2 + \bar{X}/2, X) \overset{(M3)}{=} q(\bar{X}/2) = d(X)/2$$

or in the case $\mu(X) = Z$ and

$$|X - Z| = |X - X/2 - \bar{X}/2| = |X/2 - X/2| = d(X)/2$$

with line $(*)$ the assertion.

(2.32): With both estimations in (2.31) we achieve the factor $l/4$.

Lemma 2.14. *If $\mathscr{A} = a \in \mathbb{R}^m$, $Z \in \mathscr{Q}(X)$ and $f(X) = \mathscr{W}_{x \in X}\{g(Z) + h(x) * (x - Z)\}$, then with $G = g(Z)$ we have $f(X) \subseteq g(X)$ and from*

$$\mathscr{W}_{x \in X}\{g(Z) + h(x) * (x - Z)\} = g(Z) + h(w) * (X - Z)$$

with $|h(w)| = \min_{x \in X} |h(x)|$, $T = w \in \mathbb{R}$ and $0 \subseteq X - Z$ on the other hand

$$G + h(w) * (X - Z) \subseteq f(X)$$

so that (2.29) holds. Therefore we have

$$q(\mathscr{W}_{x \in X}\{g(x)\}, g(Z) + h(X) * (X - Z)) \leqslant l\, d(X)^2. \tag{2.33}$$

Remarks. Both the Theorems 2.12 and 2.13 can be generalized (to several variables) as well as in the properties (2.26) and (2.29). For instance in (2.26) suffices the weaker assumption

$$\bigwedge_{X \in \mathscr{D}} \bigvee_{T \subseteq X \subseteq U} g(T, \mathscr{A}) \subseteq f(X, \mathscr{A}) \subseteq g(U, \mathscr{A}). \tag{2.26'}$$

Finally we can state, that if an interval function f can be included in this way by the interval function g, then f and g have the same topological properties.

If g is continuous, then so is f, if g is continuous in the sense of Lipschitz, then so is f and this of the same degree. This theorem is a very important instrument in the interval analysis.

Theorem 2.15. *Let $g \in C^1_{\mathscr{D}}$, g' continuous in the sense of Lipschitz with constant l, $g(X) = g(Z) + g'(X) * (X - Z)$ the interval extension of g with $Z \in \mathscr{Q}(X)$ and $g'(X)$ the interval extension of g' for $X \in \mathscr{D}$. If then for an arbitrary function $f(X)$*

$$\bigwedge_{X \in \mathscr{D}} \bigvee_{T \in \mathscr{Q}(X)} g(Z) + g'(T)(X - Z) \subseteq f(X) \subseteq g(X) \tag{2.34}$$

holds, then

$$q(f(X), g(X)) \leqslant l\, d(X)^2 \tag{2.35}$$

For $z = \mu(X)$ or $\mu(X) \subseteq T$ we have

$$q(f(X), g(X)) \leqslant l/2\, d(X)^2 \tag{2.36}$$

If $z = \mu(X)$ and $\mu(X) \subseteq T$ then

$$q(f(X), g(X)) \leqslant l/4\, d(X)^2. \tag{2.37}$$

The following theorem is a generalization of Theorem 2.15 and introduces the application of Taylor-expansions.

Theorem 2.16. *Let $L(X, \mathscr{A})$ be as in Theorem 2.13 and for arbitrary $Z \in \mathscr{Q}(X)$ and $G_i \in \mathbb{IR}$, $0 \leqslant i \leqslant n - 1$*

$$g(X, \mathscr{A}) = \sum_{i=0}^{n-1} G_i * (X - Z)^i + h(X, \mathscr{A}) * (X - Z)^n$$

be an interval function. Moreover for the interval function $f(X, \mathscr{A})$

$$\bigwedge_{X \in \mathscr{D}} \bigvee_{T \in \mathscr{Q}(X)} \sum_{i=0}^{n-1} G_i * (X - Z)^i + h(T, \mathscr{A}) * (X - Z)^n \subseteq f(X, \mathscr{A}) \subseteq g(X, \mathscr{A}) \tag{2.38}$$

let be satisfied. Then for all $X \in \mathscr{D}$

$$q(f(X, \mathscr{A}), g(X, \mathscr{A})) \leqslant l \cdot d(X)^{n+1}$$

holds. If $Z = \mu(X)$ or $\mu(X) \subseteq T$, then the constant can be halved as in Theorem 2.15.

Proof. Like for Theorem 2.13, where $|(X - Z)^n| \leqslant |X - Z|^n \leqslant d(X)^n$ and $q(T, X) = \frac{1}{2} d(X)$ have now to be estimated, resp.

In many cases we have with $z \in \mathbb{R}$, $z \in X$:

$$G_i = f^{(i)}(z)/i! \text{ and } h(X, \mathscr{A}) = \frac{f^{(n)}(X, \mathscr{A})}{n!}$$

where $z = \mu(X)$ is to be preferred in general.

3. Conclusion

The methods used here improve the estimations of the Theorems 3, 4, and 6 in [1] with a factor 2 and, moreover the assertions were widely generalized and reduced to few, clear lines. So the Interval Analysis now becomes a calculus which is comparable to those of classical Analysis: The handling of norm and metric are very similar to norm and metric in linear spaces. As a by-product of the extension to generalized intervals the Interval Analysis turns out to be more independent of set theory because many results are derivable without using set theory.

Some further assertions are summarized in [9]. Nevertheless a lot of properties have to be investigated in the future to complete the theory.

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