Reliable Computing **7:** 129–140, 2001.

Multiplication Distributivity of Proper and Improper Intervals *

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(Received: 30 April 1999; accepted: 19 October 1999)

Abstract. The arithmetic on an extended set of proper and improper intervals presents algebraic completion of the conventional interval arithmetic allowing thus efficient solution of some interval algebraic problems. In this paper we summarize and present all distributive relations, known by now, on multiplication and addition of generalized (proper and improper) intervals.

1. Introduction

Among several extensions of the classical interval arithmetic that have been proposed, we consider that one aiming at an algebraic completion of interval arithmetic. The algebraic extension is developed by H.-J. Ortolf [9] and E. Kaucher [5], [6], further investigated by E. Gardeñes et al. [3], [4], S. Markov [7], [8], and others. The set of normal (proper) intervals is extended by improper intervals and the interval arithmetic operations and functions are extended correspondingly. The generalized interval arithmetic structure, thus obtained, possesses group properties with respect to addition and multiplication operations. Lattice operations are closed with respect to the inclusion order relation. Handling of norm and metric are very similar to norm and metric in linear spaces [6]. In order to emphasize that a generalized interval can be considered as a pair of a proper interval (in set-theoretical sense) and a "direction", sometimes the algebraic extension of the conventional interval arithmetic is called directed interval arithmetic [7], [8]. The term "modal interval analysis" [4] reflects an interpretation of generalized intervals in terms of modal logic.

The algebraic properties of the generalized interval arithmetic make it a suitable environment for solving interval algebraic problems, e. g. some interval algebraic equations, which are not linear in general, can be solved explicitly just by applying elementary algebraic transformations due to the existence of inverse elements with respect to addition and multiplication operations [13]. However, the efficient solution of some interval algebraic problems is hampered by the lack of well studied distributive relations between generalized (proper and improper) intervals.

* This work was supported by the Bulgarian National Science Fund under grant No. I-903/99.

While the existence of inverse elements with respect to addition and multiplication follows from the isomorphic embedding of the set of conventional intervals into a group [5], [7], the validity of certain distributive relations is not straightforward. The well-known subdistributivity property of normal intervals is extended in [5] for improper intervals and some special cases of distributivity for degenerated (point) intervals are discussed there. The conventional interval distributive relation

$$C \times (A_1 + \cdots + A_n) \; = \; C \times A_1 + \cdots + C \times A_n, \tag{1.1}$$

studied in [15], [16] for proper intervals, is generalized in [3] for proper and improper intervals. Gardeñes et al. [3] define four distributive domains of generalized intervals wherein the relation (1.1) holds true. Much later, in [2] a more general conditionally distributive law was formulated for intervals $A, B, C, A + B \in \mathbb{D} \setminus \mathbb{T}$

$$(A + B) \times C_{\sigma(A+B)} \; = \; A \times C_{\sigma(A)} + B \times C_{\sigma(B)}.$$

In this work we present a generalization and full characterization of the distributive relations on multiplication and addition of generalized intervals. Section 2 provides some basic concepts of the arithmetic on proper and improper intervals and introduces special functional notations which are essential for the efficient (both analytic and computer) handling of generalized intervals. Section 3 summarizes all, known by now, about the conditionally distributive relations for generalized (proper and improper) intervals and their various forms, e.g. that specifying how and in what case brackets can be disclosed in multiplying out sum of intervals, or that specifying how and in what case a common multiplier can be taken out of brackets. The complete proof of the Theorems in this section, as well as detailed comments on their application can be found in [11]. Some references to papers, containing illustrative examples for the application of the generalized interval distributive relations, are also given.

2. The Arithmetic on Proper and Improper Intervals

The set of conventional (proper) intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^- \leq a^+;\; a^-, a^+ \in \mathbb{R}\}$ is extended by the set $\{[a^-, a^+] \mid a^- \geq a^+;\; a^-, a^+ \in \mathbb{R}\}$ of improper intervals obtaining thus the set $\mathbb{D} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\} \cong \mathbb{R}^2$ of all ordered couples of real numbers called here generalized intervals. Denote the set of zero involving generalized intervals by $\mathbb{T} = \{A \in \mathbb{D} \mid a^- a^+ \leq 0\}$.

The inclusion order relation $A \subseteq B \iff (b^- \leq a^-)$ and $(a^+ \leq b^+)$ is extended for $A, B \in \mathbb{D}$.

The "dual" is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{D}$. For $A = [a^-, a^+] \in \mathbb{D}$, its dual is defined by

$$\texttt{Dual}[A] = A_- = [a^+, a^-].$$

In some papers $\mathtt{Dual}[A]$ is denoted by $\overline{A}$. Very often, consideration of one or another interval end-point or the dualization of a generalized interval depends on some binary valued variables. To avoid long branching formulae and to simplify the proofs, in our investigations we use functional notations for the interval end-points and for the dualization of intervals. Define $\Lambda = \{+, -\}$. For $\mu, \nu \in \Lambda$, define product $\lambda = \mu\nu \in \Lambda$ by $\lambda = \{+, \text{ if } \mu = \nu;\ -, \text{ if } \mu \neq \nu\}$. This product is commutative: $\mu\nu = \nu\mu$ for $\mu, \nu \in \Lambda$.

For $\lambda \in \Lambda$, define $a^{\lambda} = \begin{cases} a^{+} & \text{if } \lambda = +, \\ a^{-} & \text{if } \lambda = - \end{cases}$ and $A_{\lambda} = \begin{cases} A & \text{if } \lambda = +, \\ A_{-} & \text{if } \lambda = -. \end{cases}$

Next, we define some interval functionals, useful for describing certain classes of generalized intervals. Denote $\mathcal{L} = \{\{+\}, \{-\}, \{+, -\}\}$. For an interval $A \in \mathbb{D}$, define a functional, called "*direction set*", $\mathcal{T} : \mathbb{D} \to \mathcal{L}$ by

$$\mathcal{T}(A) = \begin{cases} \{+\} & \text{if } a^{-} < a^{+}, \\ \{-\} & \text{if } a^{-} > a^{+}, \\ \{+, -\} & \text{if } a^{-} = a^{+}, \end{cases}$$

and a functional, called "*direction*", $\tau : \mathbb{D} \to \Lambda$ by $\tau(A) \in \mathcal{T}(A)$. A generalized interval A is called *proper* if $\tau(A) = +$ and *improper* if $\tau(A) = -$. For degenerate (point) intervals $A \in \mathbb{R}$, the direction set is $\mathcal{T}(A) = \{+, -\}$. Therefore these intervals belong to both sets: the set of proper intervals and the set of improper intervals. The freedom to choose an element τ arbitrary from the direction set of the point interval $[0, 0]$ is essential for obtaining all possible distributive relations.

For an interval $A \in \mathbb{D} \setminus \mathbb{T}$, define "*sign*" $\sigma : \mathbb{D} \setminus \mathbb{T} \to \Lambda$ by

$$\sigma(A) = \begin{cases} + & \text{if } a^{-\tau(A)} > 0, \\ - & \text{if } a^{\tau(A)} < 0. \end{cases}$$

With every interval $A \in \mathbb{D}$ we can associate a proper interval $\text{pro}(A) = A_{\tau(A)} = [a^{-\tau(A)}, a^{\tau(A)}]$ wherein $a^{-\tau(A)} \leq a^{\tau(A)}$. For $A \in \mathbb{D}$, $\text{pro}(A)$ is a projection of the generalized interval A onto the conventional interval space $\mathbb{IR}$.

The definition of the well-known χ-functional, introduced by H. Ratschek in [14], is extended for generalized intervals, $\chi : \mathbb{D} \to [-1, 1]$ by

$$\chi([0, 0]) = -1 \qquad \text{and} \qquad \chi(A) = \begin{cases} a^{-} / a^{+} & \text{if } |a^{-}| \leq |a^{+}|, \\ a^{+} / a^{-} & \text{if } |a^{-}| \geq |a^{+}|. \end{cases}$$

To provide a convenient manipulation of interval formulae involving χ-functionals, we define a functional $\mathcal{N} : \mathbb{D} \to \mathcal{L}$ by

$$\mathcal{N}(A) = \begin{cases} \{+\} & \text{if } |a^{+}| > |a^{-}|, \\ \{-\} & \text{if } |a^{+}| < |a^{-}|, \\ \{+, -\} & \text{if } |a^{+}| = |a^{-}|, \end{cases}$$

and a functional $\nu : \mathbb{D} \to \Lambda$ by $\nu(A) \in \mathcal{N}(A)$. Thus, the definition of χ becomes

$$\chi(A) = a^{-\nu(A)} / a^{\nu(A)}, \qquad \text{for } A \in \mathbb{D} \setminus \{0\}.$$

Intervals A, such that $\chi(A) = -1$ are called symmetric. A symmetric interval A can be also characterized by the property $A = -A$. For a symmetric interval A, we have the freedom to choose an element ν from the set $\mathcal{N}(A) = \{+, -\}$, that is either $\nu(A) = +$ or $\nu(A) = -$, which is also essential for the distributive relations.

The arithmetic operations $+$ and $\times$ are extended from the familiar set $\mathbb{IR}$ of proper intervals to $\mathbb{D}$. In [3], [5] and [6] the definition of $\times$ is given in a table form, while using the functional "±" notations we gain a concise presentation of the interval arithmetic formulae facilitating their manipulation.

$$A + B = [a^- + b^-, a^+ + b^+], \quad \text{for } A, B \in \mathbb{D}; \tag{2.1}$$

$$A \times B = \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)},\ a^{\sigma(B)} b^{\sigma(A)}], & A, B \in \mathbb{D} \setminus \mathbb{T}; \\ [a^{\sigma(A)\tau(B)} b^{-\sigma(A)},\ a^{\sigma(A)\tau(B)} b^{\sigma(A)}], & A \in \mathbb{D} \setminus \mathbb{T},\ B \in \mathbb{T}; \\ [a^{-\sigma(B)} b^{\sigma(B)\tau(A)},\ a^{\sigma(B)} b^{\sigma(B)\tau(A)}], & A \in \mathbb{T},\ B \in \mathbb{D} \setminus \mathbb{T}; \\ [\min\{a^- b^+,\ a^+ b^-\}, & \\ \qquad \max\{a^- b^-,\ a^+ b^+\}]_{\tau(A)}, & A, B \in \mathbb{T},\ \tau(A) = \tau(B); \\ 0, & A, B \in \mathbb{T},\ \tau(A) = -\tau(B). \end{cases} \tag{2.2}$$

The occurrence of min and max functions at the end-points of the result on multiplication of two zero-involving intervals hampers the analytical derivations in interval analysis and affects the performance of corresponding computer operation. The next theorem, proven in [12], gives an explicit representation for the end-points of such products.

THEOREM 2.1. *For $A, B \in \mathbb{T}$ such that $\tau(A) = \tau(B) = \tau$*

$$A \times B = \begin{cases} [a^{-\nu(B)\tau} b^{\nu(B)},\ a^{\nu(B)\tau} b^{\nu(B)}] = A \times b^{\nu(B)}, & \text{if } \chi(A) \le \chi(B); \\ [a^{\nu(A)} b^{-\nu(A)\tau},\ a^{\nu(A)} b^{\nu(A)\tau}] = a^{\nu(A)} \times B, & \text{if } \chi(A) \ge \chi(B). \end{cases}$$

Interval subtraction and division can be expressed as composite operations: $A - B = A + (-1) \times B$ and $A / B = A \times (1 / B)$, where $1 / B = [1 / b^+, 1 / b^-]$ if $B \in \mathbb{D} \setminus \mathbb{T}$. End-pointwise:

$$A - B = [a^- - b^+,\ a^+ - b^-], \qquad A, B \in \mathbb{D};$$

$$A / B = \begin{cases} [a^{-\sigma(B)} / b^{\sigma(A)},\ a^{\sigma(B)} / b^{-\sigma(A)}], & A, B \in \mathbb{D} \setminus \mathbb{T}; \\ [a^{-\sigma(B)} / b^{-\sigma(B)\tau(A)},\ a^{\sigma(B)} / b^{-\sigma(B)\tau(A)}], & A \in \mathbb{T},\ B \in \mathbb{D} \setminus \mathbb{T}. \end{cases}$$

The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space.

The substructures $(\mathbb{D}, +, \subseteq)$ and $(\mathbb{D} \setminus \mathbb{T}, \times, \subseteq)$ are isotone groups. Hence, there exist unique inverse elements $-A_-$ and $1 / (B_-)$ with respect to the operations addition and multiplication, such that

$$A - A_- = 0 \quad \text{and} \quad B / (B_-) = 1 \quad \text{for } A \in \mathbb{D},\ B \in \mathbb{D} \setminus \mathbb{T}. \tag{2.3}$$

Addition and multiplication operations are commutative and associative, that is for $A, B, C \in \mathbb{D}$ and $\circ \in \{+, \times\}$

$$A \circ B = B \circ A, \qquad A \circ (B \circ C) = (A \circ B) \circ C.$$

The definition of norm, metric, and many topological and lattice properties of $(\mathbb{D}, +, \times, \subseteq)$ are given in [5], [6]. Some other properties and applications of the arithmetic on proper and improper intervals can be found in [3]. In this paper we are interested in the distributive relations between addition (2.1) and multiplication (2.2) operations.

3. Generalizing Interval Distributive Relations

Because for $C = c \in \mathbb{R}$, $\left(\sum_{i=1}^{n} A_i\right) \times C = \sum_{i=1}^{n}(A_i \times C)$, we exclude this case from all considerations of the generalized conditionally distributive relations below in this Section.

3.1. Conditional Distributivity for Two Additive Terms

Denote $\widehat{\mu}(I) = \begin{cases} \sigma(I) & \text{if } I \in \mathbb{D} \setminus \mathbb{T}; \\ \nu(I)\tau(I) & \text{if } I \in \mathbb{T} \setminus \{0\} \end{cases}$ and $\tilde{\mu}(I) = \begin{cases} \sigma(I) & \text{if } I \in \mathbb{D} \setminus \mathbb{T}; \\ \tau(I) & \text{if } I \in \mathbb{T} \setminus \{0\} \end{cases}$.

THEOREM 3.1. *For $A_1, A_2 \in \mathbb{D} \setminus \{0\}$ and $S = A_1 + A_2$ the following conditionally distributive relations hold true:*

1. $\underline{S \in \mathbb{D} \setminus \mathbb{T}, \quad C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})}$

$$(A_1 + A_2) \times C = A_1 \times C_{\widehat{\mu}(A_1)\widehat{\mu}(S)} + A_2 \times C_{\widehat{\mu}(A_2)\widehat{\mu}(S)}, \qquad \textit{iff}$$
$$\begin{aligned} &\textit{either} \;\; A_1, A_2 \in \mathbb{D} \setminus \mathbb{T}, \\ &\textit{or} \quad\;\; A_i \in \mathbb{T} \setminus \{0\} \;\; \textit{for some} \;\; i \in \{1, 2\} \;\; \textit{and} \\ &\qquad\qquad \chi(A_i) = 0 \;\; \textit{for all} \;\; A_i \in \mathbb{T} \setminus \{0\}; \end{aligned}$$

2. $\underline{S \in \mathbb{D} \setminus \mathbb{T}, \quad C \in \mathbb{T} \setminus \{0\}}$

$$(A_1 + A_2) \times C = A_1 \times C_{\widehat{\mu}(A_1)\widehat{\mu}(S)} + A_2 \times C_{\widehat{\mu}(A_2)\widehat{\mu}(S)}, \qquad \textit{iff}$$
$$\begin{aligned} &\textit{either} \;\; A_1, A_2 \in \mathbb{D} \setminus \mathbb{T}, \\ &\textit{or} \quad\;\; A_i \in \mathbb{T} \setminus \{0\} \;\; \textit{for some} \;\; i \in \{1, 2\} \;\; \textit{and} \\ &\qquad \textit{for all} \;\; A_i \in \mathbb{T} \setminus \{0\}, \\ &\qquad\quad \textit{either} \;\; \nu(A_i) = \tau(C)\sigma(S), \;\; \chi(C) \le \chi(A_i); \\ &\qquad\quad \textit{or} \quad\;\; \nu(A_i) \neq \tau(C)\sigma(S), \;\; \chi(A_i) = 0; \end{aligned}$$

3. $(A_1 + A_2) \times C = A_1 \times C_{\lambda} + A_2 \times C_{-\lambda}$, *iff* $S = 0$; $(\lambda \in \Lambda)$.

4. $\underline{S \in \mathbb{T}\setminus\{0\}, \quad C \in \mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R})}$

$$\begin{aligned}
(A_1 + A_2) \times C &= A_1 \times C_{\sigma(A_1)\widehat{\mu}(S)} + A_2 \times C_{\sigma(A_2)\widehat{\mu}(S)}, \\
&\qquad\qquad \textit{iff } A_1, A_2 \in \mathbb{D}\setminus\mathbb{T}, \ \ \chi(S) = 0; \\
&= A_1 \times C_{\tilde{\mu}(A_1)\tilde{\mu}(S)} + A_2 \times C_{\tilde{\mu}(A_2)\tilde{\mu}(S)}, \qquad \textit{iff } A_1, A_2 \in \mathbb{T}\setminus\{0\}; \\
&= A_{i_1} \times C_{\sigma(A_{i_1})\widehat{\mu}(S)} + A_{i_2} \times C_{\nu(A_{i_2})\tau(A_{i_2})\widehat{\mu}(S)} = A_{i_1} \times C + A_{i_2} \times C_{-}, \\
&\qquad\qquad \textit{iff } A_{i_1} \in \mathbb{D}\setminus\mathbb{T}, \ \ A_{i_2} \in \mathbb{T}\setminus\{0\}, \ \ \chi(A_{i_2}) = \chi(S) = 0;
\end{aligned}$$

5. $\underline{S, \ \ C \in \mathbb{T}\setminus\{0\} \ \ \textit{such that} \ \ \tau(C) \neq \tau(S)}$

$$\begin{aligned}
(A_1 + A_2) \times C = 0 &= A_1 \times C_{\tilde{\mu}(A_1)\tilde{\mu}(S)} + A_2 \times C_{\tilde{\mu}(A_2)\tilde{\mu}(S)}, \qquad \textit{iff } A_1, A_2 \in \mathbb{T}\setminus\{0\}; \\
&= A_1 \times C_{\widehat{\mu}(A_1)\widehat{\mu}(S)} + A_2 \times C_{\widehat{\mu}(A_2)\widehat{\mu}(S)}, \\
&\qquad\qquad \textit{iff } \chi(S) = 0 \ \ \textit{and for all } \ A_i \in \mathbb{T}\setminus\{0\} \\
&\qquad\qquad \nu(A_i) \neq \nu(S), \ \ \chi(C) \leq \chi(A_i);
\end{aligned}$$

6. $\underline{S, \ \ C \in \mathbb{T}\setminus\{0\}, \ \ \textit{such that} \ \ \tau(C) = \tau(S)}$

$$\begin{aligned}
(A_1 + A_2) \times C &= A_1 \times C_{\tilde{\mu}(A_1)\tilde{\mu}(S)} + A_2 \times C_{\tilde{\mu}(A_2)\tilde{\mu}(S)}, \\
&\qquad \textit{iff} \quad \textit{for } \ i = 1, 2, \ A_i \in \mathbb{T}\setminus\{0\}, \ \ \textit{and} \\
&\qquad\quad \textit{either } \ \chi(C) \geq \max\{\chi(A_i), \chi(S)\}; \\
&\qquad\quad \textit{or } \ \nu(A_i) = \nu(S), \ \ \chi(C) \leq \min\{\chi(A_i), \ \chi(S)\}; \\
&\qquad\quad \textit{or } \ \chi(C) = \chi(A_{i_1}) \leq \min\{\chi(A_{i_2}), \ \chi(S)\}, \\
&\qquad\qquad \tau(C) = \tau(A_{i_1}), \ \ \nu(A_{i_1}) \neq \nu(A_{i_2}) = \nu(S); \\
&= A_1 \times C_{\sigma(A_1)\widehat{\mu}(S)} + A_2 \times C_{\sigma(A_2)\widehat{\mu}(S)}, \\
&\qquad\qquad \textit{iff } A_1, A_2 \in \mathbb{D}\setminus\mathbb{T}, \ \ \chi(C) \leq \chi(S); \\
&= A_{i_1} \times C_{\nu(A_{i_1})\tau(A_{i_1})\widehat{\mu}(S)}, \\
&\qquad\qquad \textit{iff } A_1, A_2 \in \mathbb{T}\setminus\{0\}, \ \ \nu(S) = \nu(A_{i_1}) \neq \nu(A_{i_2}), \\
&\qquad\qquad \chi(A_{i_2}) = 0, \ \ \chi(C) \leq \min\{\chi(A_{i_1}), \chi(S)\}; \\
&= A_{i_1} \times C_{\sigma(A_{i_1})\widehat{\mu}(S)} + A_{i_2} \times C_{\nu(A_{i_2})\tau(A_{i_2})\widehat{\mu}(S)}, \\
&\qquad\qquad \textit{iff } A_{i_1} \in \mathbb{D}\setminus\mathbb{T}, \ \ A_{i_2} \in \mathbb{T}\setminus\{0\}, \ \ \nu(A_{i_2}) = \nu(S), \\
&\qquad\qquad \chi(C) \leq \min\{\chi(A_{i_2}), \ \chi(S)\}; \\
&= A_{i_1} \times C_{\sigma(A_{i_1})\widehat{\mu}(S)} = A_{i_1} \times C \\
&\qquad\qquad \textit{iff } A_{i_1} \in \mathbb{D}\setminus\mathbb{T}, \ \ A_{i_2} \in \mathbb{T}\setminus\{0\}, \ \ \nu(A_{i_2}) \neq \nu(S), \\
&\qquad\qquad \big(\chi(C) \leq \chi(S), \ \ \chi(A_{i_2}) = 0, \ \ \textit{or } \ \chi(C) = \chi(S) = 0\big).
\end{aligned}$$

Essential for the application of Theorem 3.1 is that in some special cases there exist two equal distributive representations due to the freedom to choose $\tau(S) \in$

$\{+,-\}$ for $S = 0$ (that is the case 3 of Theorem 3.1), and $\nu(\cdot) \in \{+,-\}$ for symmetric intervals (that is the case 6.2 of Theorem 3.1).

EXAMPLE 3.1. For $C \in \mathbb{T}\backslash\{0\}$ and $A, B \in \mathbb{D}\backslash\mathbb{T}$, such that $A+B = S \in \mathbb{T}$, $\tau(C) = \tau(S)$ and $\chi(C) = \chi(S) = -1$, from case 6.2 of Theorem 3.1 due to $\mathcal{N}(S) = \{+,-\}$ and $\sigma(A) = -\sigma(B)$, we obtain $(A+B) \times C = A \times C_{\lambda} + B \times C_{-\lambda}$, $\lambda \in \Lambda$. Indeed

$$\begin{aligned}
([3,2] + [-1,-4]) \times [7,-7] &= [14,-14], \\
[3,2] \times [7,-7]_{-} + [-1,-4] \times [7,-7] &= [-14,14] + [28,-28] = [14,-14], \\
[3,2] \times [7,-7] + [-1,-4] \times [7,-7]_{-} &= [21,-21] + [-7,7] = [14,-14].
\end{aligned}$$

Although in cases 2.2.1, 5.2, 6.3 and 6.4 of Theorem 3.1 some A_i or S may be symmetric ($\chi(\cdot) = -1$), the conditionally distributive relations are unique because their conditions fix the corresponding ν-values in an unique way.

There are some other special cases in which both relations

$$(A_1 + A_2) \times C = A_1 \times C_{\widehat{\mu}(A_1)\widehat{\mu}(S)} + A_2 \times C_{\widehat{\mu}(A_2)\widehat{\mu}(S)}, \tag{3.1}$$

$$(A_1 + A_2) \times C = A_1 \times C_{\tilde{\mu}(A_1)\tilde{\mu}(S)} + A_2 \times C_{\tilde{\mu}(A_2)\tilde{\mu}(S)} \tag{3.2}$$

hold true. Those cases are: the case 3, the case 5.2 and the case 6.2 of Theorem 3.1. In those cases there are two equal distributive representations, too. In all other cases only one of the two conditionally distributive relations holds true.

EXAMPLE 3.2. For $A, B, C \in \mathbb{T}\backslash\{0\}$ such that $A+B = S \in \mathbb{T}$, $\tau(C) \neq \tau(S)$, $\chi(S) = 0$, $\nu(A) = \nu(B) \neq \nu(S)$, and $\chi(C) \leq \min\{\chi(A), \chi(B)\}$, there are two equal distributive representations. E.g. for the intervals $A = [2,-5]$, $B = [-3,5]$ and $C = [4,-5]$, we have

$$(A+B) \times C = \begin{cases} A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)} = 0 + 0, \\ A \times C_{-\tau(A)\tau(S)} + B \times C_{-\tau(B)\tau(S)} = [25,-20] + [-25,20]. \end{cases}$$

3.2. EQUIVALENT CONDITIONALLY DISTRIBUTIVE RELATIONS

Various different forms of the conditionally distributive relations can be derived from Theorem 3.1 and the next theorem.

THEOREM 3.2. *Let $A_1, A_2, C \in \mathbb{D}\backslash\{0\}$ and $A_1 + A_2 = S \in \mathbb{D}$. Denote by (C_0) the conditions under which the relations, defined by Theorem 3.1, hold true. The relation*

$$(A_1 + A_2) \times C = A_1 \times C_{\mu(A_1)\mu(S)} + A_2 \times C_{\mu(A_2)\mu(S)},$$

that is the corresponding relation from Theorem 3.1 (μ is either $\widehat{\mu}$ or $\tilde{\mu}$ depending on the type of the relation), **is equivalent** *to the relation*

$$(A_1 + A_2) \times C_{\mu(S)} = A_1 \times C_{\mu(A_1)} + A_2 \times C_{\mu(A_2)},$$

which holds true iff the corresponding conditions (C_0)*, wherein* $\tau(C)$ *is replaced by* $\mu(S)\tau(C)$ *are satisfied.*

EXAMPLE 3.3. The relation (3.2) which holds true for $A_1, A_2 \in \mathbb{T}\setminus\{0\}$, $S \in \mathbb{T}\setminus\{0\}$ and $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$ (case 4.2 of Theorem 3.1) is equivalent to the relation

$$(A_1 + A_2) \times C_{\tau(S)} = A_1 \times C_{\tau(A_1)} + A_2 \times C_{\tau(A_2)},$$

which holds true for $A_1, A_2 \in \mathbb{T} \setminus \{0\}$, $S \in \mathbb{T} \setminus \{0\}$ and $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$.

EXAMPLE 3.4. The relation (3.1) which holds true for $A_1, A_2 \in \mathbb{D} \setminus \mathbb{T}$, $S, C \in \mathbb{T} \setminus \{0\}$, $\tau(C) = \tau(S)$, $\chi(C) \leq \chi(S)$ (case 6.2 of Theorem 3.1) is equivalent to the relation

$$(A_1 + A_2) \times C_{\nu(S)\tau(S)} = A_1 \times C_{\sigma(A_1)} + A_2 \times C_{\sigma(A_2)},$$

which holds true for $A_1, A_2 \in \mathbb{D} \setminus \mathbb{T}$, $S, C \in \mathbb{T} \setminus \{0\}$, $\tau(S) = +$, $\chi(C) \leq \chi(S)$.

The application of Theorem 3.1 and Theorem 3.2 is mainly for taking a common multiplier out of brackets in simplification of an interval expression. Due to the commutativity and associativity of + and × operations, the conditionally distributive law for two additive terms is sufficient for the simplification of any interval expression [13]. However for disclosing brackets in multiplying an interval sum we need a generalization of Theorem 3.1 for n additive terms.

3.3. Multiplication of a Finite Interval Sum

The next six theorems combine the sufficient conditions for disclosing brackets in multiplication of an interval sum without referring to the end-points of the participating intervals (conditionally distributive law for n additive terms) with the general rules for disclosing brackets in multiplication of an interval sum when there is no distributivity. The following notations will be used: $I, J, I_i \subseteq I$, $i = 1, \ldots, p$ and $J_j \subseteq J$, $j = 1, \ldots, q$ are index sets (p and q are defined in each of the theorems), such that $\bigcap_{i=1}^{p} I_i = \emptyset$, $\bigcup_{i=1}^{p} I_i = I$, $\bigcap_{j=1}^{q} J_j = \emptyset$, $\bigcup_{j=1}^{q} J_j = J$, and $A_i \in \mathbb{D}\setminus\mathbb{T}$ for all $i \in I$, $B_j \in \mathbb{T}\setminus\{0\}$ for all $j \in J$, $S = \sum_{i\in I} A_i + \sum_{j\in J} B_j$.

THEOREM 3.3. *If* $S \in \mathbb{D} \setminus \mathbb{T}$, $C \in \mathbb{D}\setminus(\mathbb{T} \cup \mathbb{R})$ *and there exist index sets* I, J_j, $j = 1, \ldots, 5$, *some of them may be empty sets, such that*

- $\chi(B_j) = 0$ *for all* $j \in J_1$*;*
- $\Sigma_2 = \sum_{j\in J_2} B_j$ *and* $\chi(\Sigma_2) = 0$*;*
- $\nu(B_j) = +$ *for all* $j \in J_3$, $\Sigma_3 = \sum_{j\in J_3} B_j$ *and* $(\Sigma_3 = 0 \vee \chi(\Sigma_3) = 0)$*;*
- $\nu(B_j) = -$ *for all* $j \in J_4$, $\Sigma_4 = \sum_{j\in J_4} B_j$ *and* $(\Sigma_4 = 0 \vee \chi(\Sigma_4) = 0)$*;*

then the following relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C \\ &= \sum_{i\in I}(A_i \times C_{\sigma(A_i)\sigma(S)}) + \sum_{j\in J_1}(B_j \times C_{\nu(B_j)\tau(B_j)\sigma(S)}) \\ &\quad + \sum_{j\in J_2}(B_j \times C_{\nu(\Sigma_2)\tau(B_j)\sigma(S)}) + \sum_{j\in J_3}(B_j \times C_{\tau(B_j)\sigma(S)}) + \sum_{j\in J_4}(B_j \times C_{-\tau(B_j)\sigma(S)}) \\ &\quad + \sum_{j\in J_5}([b_j^-, 0] \times C_{\tau(B_j)\sigma(S)} + [0, b_j^+] \times C_{\tau(B_j)\sigma(S)}).\end{aligned}$$

THEOREM 3.4. *If $S \in \mathbb{T}$, $C \in \mathbb{D}\setminus(\mathbb{T} \cup \mathbb{R})$ and there exist index sets I_i, $i = 1, \ldots, 4$ and J_j, $j = 1, 2$, some of them possibly empty sets, such that*

- $\sum_{i\in I_1} A_i = 0$,
- $\Sigma_2 = \sum_{i\in I_2} A_i$, $\chi(\Sigma_2) = 0$,
- $\Sigma_4 = \sum_{j\in J_1} B_j$, $\chi(\Sigma_4) = 0$, $\Sigma_5 = \sum_{i\in I_3} A_i + \sum_{j\in J_1} B_j$, $\chi(\Sigma_5) = 0$,

then the next relation holds true.

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= \sum_{i\in I_1}(A_i \times C_{\sigma(A_i)}) + \sum_{i\in I_2}(A_i \times C_{\sigma(A_i)\nu(\Sigma_2)\tau(S)}) \\ &\quad + \sum_{i\in I_3}(A_i \times C_{\sigma(A_i)\nu(\Sigma_5)\tau(S)}) + \sum_{j\in J_1}(B_j \times C_{\nu(B_j)\tau(B_j)\nu(\Sigma_5)\tau(S)}) \\ &\quad + \sum_{j\in J_2}(B_j \times C_{\tau(B_j)\tau(S)}) + \sum_{i\in I_4}(A_i \times c^{\sigma(C)\tau(S)}).\end{aligned}$$

THEOREM 3.5. *If $S \in \mathbb{D}\setminus\mathbb{T}$, $C \in \mathbb{T}\setminus\{0\}$ and there exist index sets I, J_j, $j = 1, \ldots, 4$, some of them may be empty sets, such that*

- $\sum_{j\in J_1} b_j^{\tau(C)\sigma(S)} = 0$ *(that is $\chi(\sum_{j\in J_1} B_j) = 0$ and $\nu(\sum_{j\in J_1} B_j) \neq \tau(C)\sigma(S)$),*
- *for all $j \in J_2$ $\chi(C) \leq \chi(B_j)$ and either $\nu(B_j) = \tau(C)\sigma(S)$, or $\nu(B_j) \neq \tau(C)\sigma(S)$, $\chi(\sum_{j\in J_2} B_j) = 0$,*
- *for all $j \in J_3$ $\tau(C) = \tau(B_j)$, $\tau(C)\sigma(S) = \nu(\sum_{j\in J_3} B_j)$, $\chi(B_j) \leq \chi(\sum_{j\in J_3} B_j) = \chi(C)$,*

then the following relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= \sum_{i\in I}(A_i \times C_{\sigma(A_i)\sigma(S)}) + \sum_{j\in J_1}(B_j \times C_{-\tau(C)\tau(B_j)}) \\ &\quad + \sum_{j\in J_2\cup J_3}(B_j \times C_{\tau(C)\tau(B_j)}) + \sum_{j\in J_4}([b_j^{\tau(C)\sigma(S)}, 0]_{-\tau(C)\sigma(S)} \times C_{\tau(B_j)\sigma(S)}).\end{aligned}$$

THEOREM 3.6. *If $S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, $\tau(C) \neq \tau(S)$ and there exist index sets I_i, $i = 1,2,3$, J_j, $j = 1,2$, some of them may be empty sets, such that*

- $\Sigma_1 = \sum_{i \in I_1} A_i$ *is such that* $(\Sigma_1 = 0 \ \vee \ \chi(\Sigma_1) = 0)$;
- $\Sigma_2 = \sum_{i \in I_2} A_i + \sum_{j \in J_2} B_j$ *is such that* $(\Sigma_2 = 0 \ \vee \ \chi(\Sigma_2) = 0)$ *and for all* $j \in J_2$, $\nu(B_j) \neq \nu(\Sigma_2)$, $\chi(C) \leq \chi(B_j)$, *then the following relation holds true*

$$\left(\sum_{i \in I} A_i + \sum_{j \in J} B_j\right) \times C = 0 = \sum_{i \in I_1} (A_i \times C_{\sigma(A_i)\nu(\Sigma_1)\tau(S)}) + \sum_{j \in J_1} (B_j \times C_{\tau(B_j)\tau(S)}) + \sum_{i \in I_2} (A_i \times C_{\sigma(A_i)\nu(\Sigma_2)\tau(S)}) + \sum_{j \in J_2} (B_j \times C_{-\tau(B_j)\tau(S)}).$$

THEOREM 3.7. *If $S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, $\tau(C) \in \mathcal{T}(S)$, $\chi(C) \leq \chi(S)$ and there exist index sets I, J_k, $k = 1, \ldots, 4$, some of them may be empty sets, such that*

- *for all* $j \in J_1$, $\chi(C) \leq \chi(B_j)$, $\left(\sum_{j \in J_1} B_j = 0 \ \vee \ \nu(B_j) = \nu(S)\right)$;
- *for all* $j \in J_2$, $\chi(B_j) \leq \chi(\Sigma_2) = \chi(C)$, *wherein* $\sum_{j \in J_2} B_j = \Sigma_2 \in \mathbb{T}$ *and either* $\Sigma_2 = 0$ *or* $\tau(\Sigma_2) = \tau(S)$, $\nu(\Sigma_2) = \nu(S)$;
- $\sum_{j \in J_3} B_j = \Sigma_3 \in \mathbb{T}$ *and for all* $j \in J_3$, $\nu(B_j) \neq \nu(S)$, $(\Sigma_3 = 0 \ \vee \ \chi(\Sigma_3) = 0)$, *then the following relation holds true*

$$\left(\sum_{i \in I} A_i + \sum_{j \in J} B_j\right) \times C = \sum_{i \in I} (A_i \times C_{\sigma(A_i)\nu(S)\tau(S)}) + \sum_{j \in J_3} (B_j \times C_{-\tau(B_j)\tau(S)}) + \sum_{j \in J_1 \cup J_2} (B_j \times C_{\nu(B_j)\tau(B_j)\nu(S)\tau(S)}) + \sum_{j \in J_4} ([0, b_j^{\nu(S)}]_{\nu(S)} \times C_{\tau(B_j)\nu(S)\tau(S)}).$$

THEOREM 3.8. *If $S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, $\tau(C) \in \mathcal{T}(S)$, $\chi(S) < \chi(C)$ and there exist index sets I_i, $i = 1,2$, J_j, $j = 1,2,3$, some of them may be empty sets, such that*

- $\sum_{i \in I_1} A_i = \Sigma_1 \in \mathbb{T}$ *is such that* $\Sigma_1 = 0$ *or* $\chi(C) = \chi(\Sigma_1)$, $\tau(C) = \tau(\Sigma_1)$;
- *for all* $j \in J_1$, $\chi(B_j) \leq \chi(C)$;
- $\sum_{j \in J_2} B_j = \Sigma_2 \in \mathbb{T}$ *is such that* $\Sigma_2 = 0$ *or for all* $j \in J_2$, $\chi(C) = \chi(\Sigma_2) \leq \chi(B_j)$, $\tau(C) = \tau(\Sigma_2)$, $\nu(B_j) = \lambda \in \Lambda$, *then the following relation holds true*

$$\left(\sum_{i \in I} A_i + \sum_{j \in J} B_j\right) \times C = \sum_{i \in I_1} (A_i \times C_{\sigma(A_i)\nu(\Sigma_1)\tau(S)}) + \sum_{j \in J_1 \cup J_2} (B_j \times C_{\tau(B_j)\tau(S)}) + \sum_{i \in I_2} (A_i \times c^{\nu(C)}) + \sum_{j \in J_3} (B_j \times c^{\nu(C)}).$$

4. Concluding Remarks

A fundamental role of the conditionally distributive law is to connect the additive and the multiplicative groups of generalized intervals. The generalized conditionally distributive law can be also of particular theoretical interest for a more complete characterization of the distributive relations in the extended interval space involving inner (nonstandard) operations [2], [8].

The application of the generalized conditionally distributive law concerns development of some numerical methods involving proper and improper intervals, explicit solution of classes of interval algebraic equations, as well as development of a methodology for true symbolic-algebraic interval computations. Even not generally valid, the distributive law for generalized intervals turned out to be an indispensable tool for the reduction of interval algebraic equations, with multi-incidence on the unknown variable, to simpler ones. The latter would be helpful for the explicit solution of the corresponding equation and/or for the reduction of the round-off errors due to the reduced number of arithmetic operations in the simplified equation. An application of interval distributive relations for finding general normal form of pseudo-linear interval expressions and equations is discussed in [10].

The existence of inverse additive and multiplicative elements in the space of generalized intervals makes it possible to find algebraic solution to certain types interval equations just by applying elementary algebraic transformations on these equations. Such equations usually come from real-life practical problems, where modelling equations involve multiple occurrences of the interval parameters. Applying a theorem for eliminating the dependency problem [3] often leads to interval equations in generalized interval arithmetic. The validity of a conditionally distributive law in this space considerably extends the class of interval algebraic equations that can be solved explicitly. Some examples, illustrating the combined application of properties (2.3) and the distributive relations for solving interval algebraic equations in one variable, can be found in [1], [13].

The algebraic and distributive-like properties of generalized interval arithmetic can be easy and effectively exploited in the environment of a computer algebra system [1]. The computer algebra implementation of the generalized interval distributive relations provides automatic simplification of symbolic-numerical interval expressions [13]. An advanced methodology for symbolic algebraic interval computations involves explicit algebraic transformations on interval formulae, automatic simplification of interval expressions and algebraic solutions to interval equations. This way, the computer algebra tools for generalized interval arithmetic and symbolic-algebraic manipulations [1], [13] greatly facilitate the application and the computations with generalized intervals, which otherwise may seem too complicated.

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