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# Algebraic Solutions to a Class of Interval Equations

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**Abstract:** The arithmetic on the extended set of proper and improper intervals is an algebraic completion of the conventional interval arithmetic and thus facilitates the explicit solution of certain interval algebraic problems. Due to the existence of inverse elements with respect to addition and multiplication operations certain interval algebraic equations can be solved by elementary algebraic transformations. The conditionally distributive relations between extended intervals allow that complicated interval algebraic equations, multi-incident on the unknown variable, be reduced to simpler ones. In this paper we give the general type of "pseudo-linear" interval equations in the extended interval arithmetic. The algebraic solutions to a pseudo-linear interval equation in one variable are studied. All numeric and parametric algebraic solutions, as well as the conditions for nonexistence of the algebraic solution to some basic types pseudo-linear interval equations in one variable are found. Some examples leading to algebraic solution of the equations under consideration and the extra functionalities for performing true symbolic-algebraic manipulations on interval formulae in a *Mathematica* package are discussed.

**Key Words:** Extended interval arithmetic, interval equation, algebraic solutions, algebraic transformations.
**Category:** G.1.5, I.1.1

## 1 Introduction

The extended set $\mathcal{D}$ of proper and improper intervals together with the corresponding extension of the inclusion order relation and the arithmetic operations presents an algebraic completion of the conventional interval arithmetic [Alefeld and Herzberger 1974, Moore 1966]. This algebraic completion, studied in most details by E. Kaucher [Kaucher 1973, Kaucher 1977, Kaucher 1980] and E. Gardeñes [Gardeñes and Trepat 1980], is more closed in algebraic and set-theoretic sense resembling to the classical analysis, and retain all properties of interval analysis. It is of particular theoretical and practical interest to exploit the abundant algebraic properties of the extended interval arithmetic in finding explicit solutions to certain interval problems.

Contrary to conventional interval arithmetic, the equations $A + X = 0$ and $A \times Y = 0$ possess unique algebraic solutions which define the inverse additive, resp. multiplicative elements in $\mathcal{D}$. Due to the existence of inverse elements we can solve certain interval equations by elementary algebraic transformations, or to transform some equations into simple "formally" linear interval equations. For example, the interval equation

$$\frac{[7,-11] + [1,5] \times X}{X} = [3,2], \qquad 0 \notin X$$

is algebraically equivalent to the "linear" interval equation

$$[1,5] \times X + [-3,-2] \times X = [-7,11], \qquad 0 \notin X. \tag{1}$$

However in $\mathcal{D}$, like in conventional interval arithmetic, there are only conditionally valid distributive relations and therefore the equation

$$\sum_{i=1}^{n} A_i \times X \; = \; B, \qquad\quad A_i, B \in \mathcal{D} \tag{2}$$

is not linear. We call such equations, which look like linear, *"pseudo-linear"*. In Section 3 of this paper, the general normal form of pseudo-linear interval equations in the algebraic extension $\mathcal{D}$ of conventional interval arithmetic is presented. In Section 4, the algebraic solutions to the general pseudo-linear interval equation in one variable are studied.

*Interval algebraic solution* is an interval (interval vector) that substituting it into the equation(s) and performing all interval operations results in valid equality(ies). The algebraic solutions have close relations to the solutions of tolerance and control problems, as well as to the united solution set of a linear interval problem. Therefore a straightforward way for finding the algebraic solutions would facilitate the solution of the corresponding tolerance or control problem. However, the relations between different solution set of a linear interval problem will not be discussed here.

In [Kaucher 1977], the algebraic solutions to the equation (2) are considered. The algebraic solutions not involving zero are defined there by the solutions of certain linear systems of equations in $\mathbb{R}^2$. It is proposed, that algebraic solutions involving zero be found by solving a big number of linear inequalities. However, no explicit solutions are given in [Kaucher 1977] and the equation (2) is only a special case of pseudo-linear interval equation in $\mathcal{D}$.

The aim of this paper is to present all numeric and parametric algebraic solutions, as well as the conditions for nonexistence of the algebraic solution to some basic types pseudo-linear interval equations in one variable.

## 2 The Algebraic Completion of $I\mathbb{R}$

The set of conventional (proper) intervals $I\mathbb{R} = \{[a^-, a^+] \mid a^- \leq a^+, a^-, a^+ \in \mathbb{R}\}$ is extended by the set $\{[a^-, a^+] \mid a^- > a^+, a^-, a^+ \in \mathbb{R}\}$ of improper intervals obtaining thus the set $\mathcal{D} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\} \cong \mathbb{R}^2$ of all ordered couples of real numbers called extended (or directed) intervals. Directed intervals are denoted by capital letters and $a^\lambda \in \mathbb{R}$, with $\lambda \in \Lambda = \{+, -\}$, is the first or second end-point of $A \in \mathcal{D}$ depending on the value of $\lambda$. The binary variable $\lambda$ is sometimes expressed as a "product" of two or more binary variables, $\lambda = \mu\nu$, $\mu, \nu \in \Lambda$, defined by $++ = -- = +$ and $+- = -+ = -$. Degenerate (point) intervals are those for which $a^- = a^+$.

The inclusion order relation between normal intervals is extended for $A, B \in \mathcal{D}$ by

$$A \subseteq B \Longleftrightarrow (b^- \leq a^-) \;\text{ and }\; (a^+ \leq b^+).$$

Dual is an important operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathcal{D}$. For $A = [a^-, a^+] \in \mathcal{D}$ "dual" is defined by

$$\mathtt{Dual}[A] = \overline{A} = [a^+, a^-].$$

Very often the dualization of an extended interval depends on the value of some binary-valued interval functionals. In order to avoid long branching formulae and to simplify the proofs, we use also the notation $A_\lambda$ with $\lambda \in \Lambda$ and $A_\lambda = \{A, \text{ if } \lambda = +;\ \overline{A}, \text{ if } \lambda = -\}$.

The following interval functionals are useful for describing certain classes of extended intervals. For an interval $A \in \mathcal{D}$ "direction" $\tau : \mathcal{D} \to \Lambda$ is defined by

$$\tau(A) = \begin{cases} +, & \text{if } a^- \le a^+, \\ -, & \text{otherwise} . \end{cases}$$

An extended interval $A$ is called *proper*, if $\tau(A) = +$ and *improper* otherwise. With every interval $A \in \mathcal{D}$ we can associate a proper interval $\text{pro}(A) = A_{\tau(A)} = [a^{-\tau(A)}, a^{\tau(A)}]$ where $a^{-\tau(A)} \le a^{\tau(A)}$. For $A \in \mathcal{D}$, $\text{pro}(A) = A_{\tau(A)}$ is a projection of the extended interval $A$ onto the conventional interval space $I\mathbb{R}$.

Denote $\mathcal{T} = \{A \in \mathcal{D} \mid A = [0,0] \text{ or } a^-a^+ < 0\}$. For an interval $A \in \mathcal{D} \setminus \mathcal{T}$ "sign" $\sigma : \mathcal{D} \setminus \mathcal{T} \to \Lambda$ is defined by

$$\sigma(A) = \begin{cases} +, & \text{if } a^{-\tau(A)} \ge 0, \\ -, & \text{otherwise.} \end{cases}$$

In particular, $\sigma$ is well defined over $\mathbb{R} \setminus 0$.

The definition [Ratschek 1970] of the well-known $\chi$-functional is extended in [Popova 1997] for directed intervals, $\chi : \mathcal{D} \to [-1, 1]$

$$\chi_A = \begin{cases} -1, & \text{if } A = [0,0] \\ a^{-\nu(A)}/a^{\nu(A)}, & \text{otherwise} , \end{cases}$$

where $\nu(A) = \{+, \text{ if } |a^+| = |a^-|;\ \sigma(|a^+| - |a^-|), \text{ otherwise}\}$. It is obvious that $a^{\nu(A)} = \{a^+, \text{ if } \mid a^+ \mid \ge \mid a^- \mid;\ a^-, \text{otherwise}\}$. Functional $\chi$ admits the geometric interpretation [Ratschek 1970] that $A$ is more symmetric than $B$ iff $\chi_A \le \chi_B$.

The arithmetic operations $+$ and $\times$ are extended from the familiar set $I\mathbb{R}$ of normal intervals to $\mathcal{D}$. In [Kaucher 1973], [Gardeñes and Trepat 1980] and [Kaucher 1980] the definition of $\times$ is given in a table form, while using the "$\pm$" notations we gain a concise presentation of the interval arithmetic formulae facilitating their manipulation.

$$A + B = [a^- + b^-, a^+ + b^+], \text{ for } A, B \in \mathcal{D};$$

$$A \times B = \begin{cases} [a^{-\sigma(B)}b^{-\sigma(A)}, a^{\sigma(B)}b^{\sigma(A)}], & A, B \in \mathcal{D} \setminus \mathcal{T}; \\ [a^{\sigma(A)\tau(B)}b^{-\sigma(A)}, a^{\sigma(A)\tau(B)}b^{\sigma(A)}], & A \in \mathcal{D} \setminus \mathcal{T},\ B \in \mathcal{T}; \\ [a^{-\sigma(B)}b^{\sigma(B)\tau(A)}, a^{\sigma(B)}b^{\sigma(B)\tau(A)}], & A \in \mathcal{T},\ B \in \mathcal{D} \setminus \mathcal{T}; \\ [\min\{a^-b^+, a^+b^-\}, \max\{a^-b^-, a^+b^+\}]_{\tau(A)}, & A, B \in \mathcal{T}, \tau(A) = \tau(B); \\ 0, & A, B \in \mathcal{T}, \tau(A) = -\tau(B). \end{cases}$$

Interval subtraction and division can be expressed as composite operations $A - B = A + (-1) \times B$ and $A/B = A \times (1/B)$, where $1/B = [1/b^+, 1/b^-]$ if $B \in \mathcal{D} \setminus \mathcal{T}$. End-pointwise:

$$A - B = [a^- - b^+, a^+ - b^-], \qquad A, B \in \mathcal{D};$$
$$A/B = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & A, B \in \mathcal{D} \setminus \mathcal{T}; \\ [a^{-\sigma(B)}/b^{-\sigma(B)\tau(A)}, a^{\sigma(B)}/b^{-\sigma(B)\tau(A)}], & A \in \mathcal{T},\ B \in \mathcal{D} \setminus \mathcal{T}. \end{cases}$$

The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space.

Some basic properties of the extended interval arithmetic [Kaucher 1973] are:

1. The operations $+$ and $\times$ are commutative and associative in $\mathcal{D}$.
2. $X = [0, 0] = 0$ and $Y = [1, 1] = 1$ are the unique neutral elements with respect to $+$ and $\times$ operations.
3. The substructures $(\mathcal{D}, +, \subseteq)$ and $(\mathcal{D} \setminus \mathcal{T}, \times, \subseteq)$ are isotone groups. Hence, there exist unique inverse elements $-\overline{A}$ and $1/\overline{B}$ with respect to the operations $+$ and $\times$ such that
$$A - \overline{A} = 0 \quad \text{and} \quad B/\overline{B} = 1. \tag{3}$$
4. $A \subseteq B \Longleftrightarrow \overline{A} \supseteq \overline{B}$; $\qquad \overline{A \circ B} = \overline{A} \circ \overline{B}$ for $\circ \in \{+, -, \times, /\}$.

Definition of norm and metric, as well as many topological and lattice properties of $(\mathcal{D}, +, \times, \subseteq)$ are given in [Kaucher 1973], [Kaucher 1980]. Some other properties and applications of the extended interval arithmetic can be found in [Gardeñes and Trepat 1980].

The conditionally distributive law for multiplication and addition of extended intervals is proven in its general form in [Popova 1997]. Next two equivalent theorems specify how to multiply a sum of extended intervals and how and when a common multiplier can be taken out of brackets.

For $A \in \mathcal{D}$ define $\mu(A) = \{\sigma(A),\ \text{if}\ A \in \mathcal{D} \setminus \mathcal{T};\quad \tau(A),\ \text{if}\ A \in \mathcal{T}\}$.

***Theorem 1.*** *Let* $A_i,\ i = 1, \ldots, n$ *and* $C$ *be extended intervals. Denote* $\sum_{i=1}^{n} A_i = S$. *The equality*

$$\left(\sum_{i=1}^{n} A_i\right) \times C = \sum_{i=1}^{n} \left(A_i \times C_{\mu(A_i)\mu(S)}\right)$$

*holds true iff exactly one of the assumptions i) to v) holds true.*

***Theorem 2.*** *Let* $A_i,\ i = 1, \ldots, n$ *and* $C$ *be extended intervals. Denote* $\sum_{i=1}^{n} A_i = S$. *The equality*

$$\sum_{i=1}^{n} \left(A_i \times C_{\mu(A_i)}\right) = \left(\sum_{i=1}^{n} A_i\right) \times C_{\mu(S)}$$

*holds true iff exactly one of the assumptions i) to v) holds true.*

i) $A_i, S \in \mathcal{D} \setminus \mathcal{T},\ i = 1, \ldots, n$ and $C \in \mathcal{D}$;

ii) $A_i \in \mathcal{D} \setminus \mathcal{T}, i = 1, \ldots, n,\ S \in \mathcal{T}$, and

either $C = c \in \mathbb{R}$,

or $C \in \mathcal{T},\ \begin{cases} S = 0 & \textit{or} \\ \tau(C) = \begin{cases} +, & \text{for Th2;} \\ \tau(S), & \text{for Th1,} \end{cases} & \chi_C \le \chi_S,\ \nu(S) = +; \end{cases}$

iii) $A_i, S \in \mathcal{T},\ i = 1, \ldots, n$ and

either $C \in \mathcal{D} \setminus \mathcal{T}$,
or $C \in \mathcal{T},\ \{\tau(C) = -, \text{ for Th2; } \tau(C) \neq \tau(S), \text{ for Th1}\}$,
or $C \in \mathcal{T},\ \tau(C) = \{+, \text{ for Th2; } \tau(S), \text{ for Th1}\}$ and
either for all $i, j = 1, ..., n\ \ \tau(A_i) = \tau(A_j),\ \begin{cases} \chi_C \ge \chi_{A_i}, \text{ or} \\ \chi_C \le \chi_{A_i}, \nu(A_i) = \nu(A_j), \end{cases}$
or there exist indexes $p, q$ such that $\tau(A_p) \neq \tau(A_q)$ and
$\chi_C \le \min\{\chi_{A_i}, \chi_S\},\ \nu(A_i) = \nu(S)$ for all $i = 1, ..., n$;

iv) $A_i \in \mathcal{T},\ i = 1, \ldots, n,\ S \in \mathcal{D} \setminus \mathcal{T}$ and

either $C = c \in \mathbb{R}$,
or $C \in \mathcal{T},\ \begin{cases} \tau(C) = -, & \text{for Th2;} \\ \tau(C) \neq \sigma(S), & \text{for Th1,} \end{cases}\ s^- = 0,$
or $C \in \mathcal{T},\ \tau(C) = \begin{cases} +, & \text{for Th2;} \\ \sigma(S), & \text{for Th1,} \end{cases}\ \nu(A_i) = +,\ \chi_C \le \chi_{A_i}, i = 1, ..., n;$

v) there exist index sets $P, Q \neq \emptyset,\ P \cup Q = \{1, \ldots, n\},\ P \cap Q = \emptyset$ such that $A_p \in \mathcal{D} \setminus \mathcal{T}$ for $p \in P$, $A_q \in \mathcal{T}$ for $q \in Q$, and

either $C = c \in \mathbb{R}$,
or $C \in \mathcal{T},\ \tau(C) = \begin{cases} +, & \text{for Th2;} \\ \mu(S), & \text{for Th1,} \end{cases}\ \chi_C \le \min_{q \in Q}\{\chi_{A_q}\},\ \nu(A_q) = +,$
or $C \in \mathcal{T},\ \begin{cases} \tau(C) = -, & \text{for Th2;} \\ \tau(C) \neq \mu(S), & \text{for Th1,} \end{cases}$ and $\begin{cases} \sum_{q \in Q} a_q^- = 0, & \text{if } S \in \mathcal{D} \setminus \mathcal{T}, \\ \sum_{p \in P} a_p^- = 0, & \text{if } S \in \mathcal{T}. \end{cases}$

The above distributive relations are essential for simplification of interval arithmetic expressions, especially in performing elementary algebraic transformations.

In [Kaucher 1973] the so-called *hyperbolic product* is introduced by

$$A \times_h B = [a^- b^-, a^+ b^+], \qquad A, B \in \mathcal{D}.$$

The inverse elements $-\overline{A}$ and $1/\overline{A}$ generate operations

$$A -_h B = A - \overline{B} = [a^- - b^-, a^+ - b^+], \quad A, B \in \mathcal{D},$$
$$A \,/_h\, B = A/\overline{B} = [a^-/b^-, a^+/b^+], \quad A \in \mathcal{D},\ B \in \mathcal{D} \setminus \mathcal{T},$$

called *hyperbolic subtraction*, resp. *hyperbolic division*. The interval arithmetic addition together with the hyperbolic product form a field $\{\mathcal{D}, +, \times_h\}$ [Kaucher 1977], where a distributive law

$$A \times_h C + B \times_h C = (A + B) \times_h C$$

holds true for arbitrary $A, B, C \in \mathcal{D}$.

In [Kaucher 1973, Kaucher 1977, Kaucher 1980] the result of interval multiplication is expressed by the hyperbolic product of the arguments, except for the case of arguments $A, B \in \mathcal{T},\ \tau(A) = \tau(B) = \tau$, when $A \times B = [\min\{a^-b^+, a^+b^-\}, \max\{a^-b^-, a^+b^+\}]_\tau$. In fact, min / max form of interval multiplication hampers most of the interval arithmetic investigations and the efficient implementation of this operation. In [Popova 1997] the result of interval multiplication is presented explicitly by the end-points of the arguments. Thus, we have the following transition formula between interval multiplication and the hyperbolic product

$$A \times B = \begin{cases} A_{\sigma(B)} \times_h B_{\sigma(A)}, & A, B \in \mathcal{D} \setminus \mathcal{T}; \\ a^{\sigma(A)\tau(B)} \times_h B_{\sigma(A)}, & A \in \mathcal{D} \setminus \mathcal{T},\ B \in \mathcal{T}; \\ A_{\sigma(B)} \times_h b^{\tau(A)\sigma(B)}, & A \in \mathcal{T},\ B \in \mathcal{D} \setminus \mathcal{T}; \\ a^{\nu(A)} \times_h B_{\nu(A)\tau}, & A, B \in \mathcal{T},\ \tau = \tau(A) = \tau(B),\ \chi_B \le \chi_A; \\ A_{\nu(B)\tau} \times_h b^{\nu(B)}, & A, B \in \mathcal{T},\ \tau = \tau(A) = \tau(B),\ \chi_B \ge \chi_A. \end{cases} \quad (4)$$

Formula (4) will be used for finding algebraic solutions to a pseudo-linear interval equation in $\mathcal{D}$. Latter two cases of this formula are essential for the explicit presentation of zero algebraic solutions.

## 3 Expressions Having Normal Form in $\mathcal{D}$

Consider the interval expression in one variable

$$\sum_{i=1}^{r} (A_i \times X) + \sum_{j=r+1}^{n} (A_j \times \overline{X}) \quad (5)$$

where $X, A_i \in \mathcal{D},\ i = 1, ..., n$. We shall find the general form of an expression in $\mathcal{D}$ to which the above expression (5) can be simplified.

**Definition 3.** An interval expression in one variable is in normal form if it cannot be simplified.

Divide $\mathcal{D}$ into four disjoint nonempty subsets $\mathcal{S}_i,\ i = 1, 2, 3, 4$

$$\mathcal{S}_1 = \{A \in \mathcal{D} \setminus \mathcal{T} \mid \sigma(A) = +\}, \qquad \mathcal{S}_2 = \{A \in \mathcal{D} \setminus \mathcal{T} \mid \sigma(A) = -\},$$

$$\mathcal{S}_3 = \{A \in \mathcal{T} \mid \tau(A) = +\}, \qquad \mathcal{S}_4 = \{A \in \mathcal{T} \mid \tau(A) = -\}.$$

The expression (5) can be rewritten in the form

$$\sum_{k=1}^{4} \left( \sum_{A_i \in \mathcal{S}_k} (A_i \times X) + \sum_{A_j \in \mathcal{S}_k} (A_j \times \overline{X}) \right).$$

Let $X \in \mathcal{D} \setminus \mathcal{T}$. Substituting $\overline{X} = Y$ last expression is equivalently transformed to

$$\begin{aligned}
&\sum_{A_i \in \mathcal{S}_1} (A_i \times X) + \sum_{A_i \in \mathcal{S}_2} (A_i \times \overline{Y}) + \sum_{A_i \in \mathcal{S}_3} (A_i \times X) + \sum_{A_i \in \mathcal{S}_4} (A_i \times \overline{Y}) \\
&+ \sum_{A_j \in \mathcal{S}_1} (A_j \times Y) + \sum_{A_j \in \mathcal{S}_2} (A_j \times \overline{X}) + \sum_{A_j \in \mathcal{S}_3} (A_j \times Y) + \sum_{A_j \in \mathcal{S}_4} (A_j \times \overline{X}) \\
\overset{Th2}{=} & \left(\sum_{A_i \in \mathcal{S}_1} A_i\right) \times X + \left(\sum_{A_i \in \mathcal{S}_2} A_i\right) \times \overline{Y} + \left(\sum_{A_i \in \mathcal{S}_3} A_i\right) \times X + \left(\sum_{A_i \in \mathcal{S}_4} A_i\right) \times \overline{Y} \\
+ & \left(\sum_{A_j \in \mathcal{S}_1} A_j\right) \times Y + \left(\sum_{A_j \in \mathcal{S}_2} A_j\right) \times \overline{X} + \left(\sum_{A_j \in \mathcal{S}_3} A_j\right) \times Y + \left(\sum_{A_j \in \mathcal{S}_4} A_j\right) \times \overline{X} \\
= & \sum_{k=1}^{4} (B_k \times X + C_k \times \overline{X}), \qquad (6)
\end{aligned}$$

where $B_k = \sum_{A_i \in \mathcal{S}_k} A_i$ and $C_k = \sum_{A_j \in \mathcal{S}_k} A_j$ for $k = 1, 2, 3, 4$.

If $B_1 + C_2,\ B_2 + C_1 \in \mathcal{T}$ and $B_3 + C_4,\ B_4 + C_3 \in \mathcal{D} \setminus \mathcal{T}$, then (6) cannot be further simplified unless $X = x \in \mathbb{R}$ when (6) is equivalent to $\sum_{k=1}^{4} (B_k + C_k) \times x$.

If $B_1 + C_2,\ B_2 + C_1 \in \mathcal{D} \setminus \mathcal{T}$ and $B_3 + C_4,\ B_4 + C_3 \in \mathcal{T}$, then (6) is equivalent to

$$\begin{aligned}
&(B_1 + C_2) \times X_{\sigma(B_1 + C_2)} + (B_2 + C_1) \times X_{-\sigma(B_2 + C_1)} \\
&+ (B_3 + C_4) \times X_{\tau(B_3 + C_4)} + (B_4 + C_3) \times X_{-\tau(B_4 + C_3)},
\end{aligned}$$

which cannot be further simplified for $X \notin \mathbb{R}$.

For $X \in \mathcal{T}$ the expression (5) is equivalent to

$$\begin{aligned}
&\sum_{k=1}^{2} (B_k \times X + C_k \times \overline{X}) \\
&+ \sum_{A_i \in \mathcal{S}_3} (A_i \times X) + \sum_{A_i \in \mathcal{S}_4} (A_i \times X) + \sum_{A_j \in \mathcal{S}_3} (A_j \times \overline{X}) + \sum_{A_j \in \mathcal{S}_4} (A_j \times \overline{X}),
\end{aligned}$$

where $B_k, C_k,\ k = 1, 2$ are as above. If $\tau(X) = +$ we have

$$\sum_{k=1}^{2} (B_k \times X + C_k \times \overline{X}) + \sum_{A_i \in \mathcal{S}_3} (A_i \times X) + \sum_{A_j \in \mathcal{S}_4} (A_j \times \overline{X}),$$

and if $\tau(X) = -$ we have

$$\sum_{k=1}^{2} (B_k \times X + C_k \times \overline{X}) + \sum_{A_i \in \mathcal{S}_4} (A_i \times X) + \sum_{A_j \in \mathcal{S}_3} (A_j \times \overline{X}).$$

It is obvious from the conditions *iii)–v)* of Theorem 2 that a further simplification of the latter two expressions will depend on the values of the $\chi$-functional for $X$ and the corresponding coefficients $A_i, A_j \in \mathcal{T}$.

The above results can be summarized in the following

***Theorem 4.*** *The normal form of a pseudo-linear interval expression in one variable $X \in \mathcal{D}$ is*

$$\sum_{k=1}^{2}(B_k \times X + C_k \times \overline{X}) + \sum_{i=1}^{r} A_i \times X + \sum_{i=r+1}^{n} A_i \times \overline{X},$$

*where $B_k, C_k \in \mathcal{S}_k$ and $A_i \in \mathcal{T}$.*
*The normal form of a pseudo-linear interval expression in one variable $X \in \mathcal{D}\backslash\mathcal{T}$ is*

$$\sum_{k=1}^{4}(B_k \times X + C_k \times \overline{X}),$$

*where $B_k, C_k \in \mathcal{S}_k$.*

*The normal form of a pseudo-linear interval expression in one variable $X \in \mathcal{T}$ is*

$$\sum_{k=1}^{2}(B_k \times X + C_k \times \overline{X}) + \sum_{\tau(A_i)=\tau(X)} A_i \times X + \sum_{\tau(A_i)\neq\tau(X)} A_i \times \overline{X},$$

*where $B_k, C_k \in \mathcal{S}_k$ and $A_i \in \mathcal{T}$.*

In other words Theorem 4 says that the normal form of a pseudo-linear interval expression in one variable $X \in \mathcal{D} \setminus \mathcal{T}$ cannot contain more than eight additive terms, while the number of additive terms in a pseudo-linear interval expression in one variable $X \in \mathcal{T}$ depends on the number of coefficients from $\mathcal{T}$. If for some $k = 1, 2, 3, 4$ there are no coefficients from $\mathcal{S}_k$ in the expression under consideration, then its normal form contains less additive terms.

*Example 1.* The normal form of the expression

$$\sum_{t=1}^{4}([t,a] \times X) + [-3,-20] \times X + [-1,2] \times X + [4,-3] \times X$$
$$+\sum_{t=1}^{3}([t,t^2] \times \overline{X}) + \sum_{t=1}^{3}([-t^2,0] \times \overline{X}) + \sum_{t=1}^{3}([-1,t] \times \overline{X}) + [5,-2] \times \overline{X},$$

where $a \geq 0$, is

$$[10,4a] \times X + [-3,-20] \times X + [6,14] \times \overline{X} + [-14,0] \times \overline{X} +$$
$$[-1,2] \times X + [4,-3] \times X + \sum_{t=1}^{3}([-1,t] \times \overline{X}) + [5,-2] \times \overline{X},$$

or

$$\begin{cases} [10,4a]\times X+[-3,-20]\times X+[6,14]\times \overline{X}+[-14,0]\times \overline{X} \\ [-1,2]\times X+[4,-3]\times X+[-3,6]\times \overline{X}+[5,-2]\times \overline{X}, \qquad X\in \mathcal{D}\setminus \mathcal{T} \\ \\ [10,4a]\times X+[-3,-20]\times X+[6,14]\times \overline{X}+[-14,0]\times \overline{X} \\ +[-1,2]\times X+[5,-2]\times \overline{X}, \qquad X\in \mathcal{T},\ \tau(X)=+ \\ \\ [10,4a]\times X+[-3,-20]\times X+[6,14]\times \overline{X}+[-14,0]\times \overline{X} \\ +[4,-3]\times X+\sum_{t=1}^{3}([-1,t]\times \overline{X}) \qquad X\in \mathcal{T},\ \tau(X)=- \end{cases}$$

*Example 2.* The normal form of the expression

$$\sum_{t=1}^{5}([t,1]\times X)+[-3,-20]\times X+\sum_{t=1}^{3}([t,t^2]\times \overline{X})+\sum_{t=1}^{3}([-t^2,0]\times \overline{X}),$$

where there are no coefficients from $\mathcal{T}$, is

$$[1,5]\times X+[3,12]\times \overline{X}.$$

Theorem 4 determines the general type of a pseudo-linear interval equation in one variable

$$\sum_{k=1}^{2}(B_k\times X+C_k\times \overline{X})+\sum_{i=1}^{r}A_i\times X+\sum_{i=r+1}^{q}A_i\times \overline{X}=V,$$

where $X,V\in\mathcal{D}$, $B_k,C_k\in\mathcal{S}_k$, $k=1,2$ and $A_i\in\mathcal{T}$, $i=1,...,q$.

Hence, the general form of one pseudo-linear interval equation in $n$ variables is

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{2}(B_j^{(k)}\times X_j+C_j^{(k)}\times \overline{X_j})+\sum_{k=1}^{r}A_j^{(k)}\times X_j+\sum_{k=r+1}^{q}A_j^{(k)}\times \overline{X_j}\right)=V,$$

where $X_j,V\in\mathcal{D}$; $B_j^{(k)},C_j^{(k)}\in\mathcal{S}_k, k=1,2$ and $A_j^{(k)}\in\mathcal{T}$ for $j=1,...,n$.
In vector form latter equation can be written as

$$\sum_{k=1}^{2}(\mathcal{B}^{(k)}\times \mathcal{X}+\mathcal{C}^{(k)}\times \overline{\mathcal{X}})+\sum_{k=1}^{r}\mathcal{A}^{(k)}\times \mathcal{X}+\sum_{k=r+1}^{q}\mathcal{A}^{(k)}\times \overline{\mathcal{X}}=V,$$

with interval vectors $\mathcal{B}^{(k)}=\left(B_1^{(k)},\ldots,B_n^{(k)}\right),\mathcal{C}^{(k)}=\left(C_1^{(k)},\ldots,C_n^{(k)}\right)\in\mathcal{S}_k^n$, $\mathcal{A}^{(k)}=\left(A_1^{(k)},\ldots,A_n^{(k)}\right)\in\mathcal{T}^n$, $\mathcal{X}=(X_1,\ldots,X_n)^\top\in\mathcal{D}^n$ and $V\in\mathcal{D}$.

A general system of $m$ pseudo-linear interval equations in $n$ variables is

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{2}(B_{ij}^{(k)}\times X_j + C_{ij}^{(k)}\times \overline{X_j}) + \sum_{k=1}^{r} A_{ij}^{(k)}\times X_j + \sum_{k=r+1}^{q} A_{ij}^{(k)}\times \overline{X_j}\right) = V_i, \quad i=1,...,m,$$

where $X_j, V \in \mathcal{D}$, $B_{ij}^{(k)}, C_{ij}^{(k)} \in \mathcal{S}_k$, and $A_{ij}^{(k)} \in \mathcal{T}$ for $i=1,...,m$, $j=1,...,n$. The corresponding pseudo-linear interval equation system in $\mathcal{D}^{m\times n}$ is:

$$\sum_{k=1}^{2}(\mathcal{B}^{(k)}\times \mathcal{X} + \mathcal{C}^{(k)}\times \overline{\mathcal{X}}) + \sum_{k=1}^{r} \mathcal{A}^{(k)}\times \mathcal{X} + \sum_{k=r+1}^{q} \mathcal{A}^{(k)}\times \overline{\mathcal{X}} = \mathcal{V},$$

with $\mathcal{B}^{(k)} = \left(B_{ij}^{(k)}\right), \mathcal{C}^{(k)} = \left(C_{ij}^{(k)}\right) \in \mathcal{S}_k^{m\times n}$, $\mathcal{A}^{(k)} = \left(A_{ij}^{(k)}\right) \in \mathcal{T}^{m\times n}$ and $\mathcal{V} = (V_1,\ldots,V_m)^\top \in \mathcal{D}^m$, $\mathcal{X} = (X_1,\ldots,X_n)^\top, \overline{\mathcal{X}} = (\overline{X_1},\ldots,\overline{X_n})^\top \in \mathcal{D}^n$.

## 4 Algebraic Solutions to a Pseudo-Linear Interval Equation

Consider the interval equation in $\mathcal{D}$

$$\frac{\sum_{k=1}^{2}(B_k'\times X + C_k'\times \overline{X}) + \sum_{i=1}^{r_1} A_i'\times X + \sum_{i=r_1+1}^{q_1} A_i'\times \overline{X} + V'}{\sum_{k=1}^{2}(B_k''\times X + C_k''\times \overline{X}) + \sum_{i=1}^{r_2} A_i''\times X + \sum_{i=r_2+1}^{q_2} A_i''\times \overline{X} + V''} = V,$$

where the nominator and denominator in the left-hand side of the equation are pseudo-linear interval expressions in normal form and the denominator is from $\mathcal{D}\setminus\mathcal{T}$. We shall transform this equation into a pseudo-linear interval equation in normal form by applying to the equation successive algebraic transformations, based on the equalities (3). We call that a pseudo-linear interval equation in one variable is in normal form if its left-hand side is a pseudo-linear interval expression in normal form. Multiplying both sides of the equation by dual of the denominator, we obtain the following equivalent equation:

$$\sum_{k=1}^{2}(B_k'\times X + C_k'\times \overline{X}) + \sum_{i=1}^{r_1} A_i'\times X + \sum_{i=r_1+1}^{q_1} A_i'\times \overline{X} + V' \;=\; V\times \overline{\left(\sum_{k=1}^{2}(B_k''\times X + C_k''\times \overline{X}) + \sum_{i=1}^{r_2} A_i''\times X + \sum_{i=r_2+1}^{q_2} A_i''\times \overline{X} + V''\right)}$$

Subtracting dual of the right-hand side of latter equation from its both sides we obtain next equivalent equation:

$$\sum_{k=1}^{2}(B_k'\times X + C_k'\times \overline{X}) + \sum_{i=1}^{r_1} A_i'\times X + \sum_{i=r_1+1}^{q_1} A_i'\times \overline{X} + V' - \overline{V}\times \left(\sum_{k=1}^{2}(B_k''\times X + C_k''\times \overline{X}) + \sum_{i=1}^{r_2} A_i''\times X + \sum_{i=r_2+1}^{q_2} A_i''\times \overline{X} + V''\right) \;=\; 0.$$

To transform latter equation into normal form we need to disclose brackets in the left-hand side of the equation by applying Theorem 1. Since we do not know to which class $\mathcal{S}_k,\ k = 1, 2, 3, 4$ belongs $X$ in the general case, we have to consider eight cases varying $X \in \mathcal{S}_k$ and $\Sigma \in \mathcal{S}_l$ for $k = 1, 2, 3, 4,\ l = 1, 2$, where $\Sigma$ is the expression in brackets. For every one of the eight cases, according to Theorem 4 latter equation is equivalent to a pseudo-linear interval equation in normal form which algebraic solutions will be found below.

For example, if $X \in \mathcal{S}_1$ and $\Sigma \in \mathcal{S}_2$ latter equation is equivalent to the equation

$$\begin{aligned}
&\sum_{k=1}^{2}(B'_k \times X + C'_k \times \overline{X}) + \sum_{i=1}^{r_1} A'_i \times X + \sum_{i=r_1+1}^{q_1} A'_i \times \overline{X} - \\
&\sum_{k=1}^{2}(B''_k \times V_{\sigma(B''_k)} \times X + C''_k \times V_{\sigma(C''_k)} \times \overline{X}) - \\
&\sum_{i=1}^{r_2} A''_i \times V_{\tau(A''_i)} \times X - \sum_{i=r_2+1}^{q_2} A''_i \times V_{\tau(A''_i)} \times \overline{X} = \overline{V'' \times V_{\mu(V'')} - V'},
\end{aligned} \tag{7}$$

which according to Theorem 4 is equivalent to a pseudo-linear interval equation in normal form

$$\sum_{k=1}^{4}(P_k \times X + Q_k \times \overline{X}) = \overline{V'' \times V_{\mu(V'')} - V'},$$

where the particular values of $P_k, Q_k \in \mathcal{S}_k$ depend on the characteristic $\mu$ of $V \in \mathcal{D}$.

Consider the general pseudo-linear interval equation

$$\sum_{k=1}^{2}(B_k \times X + C_k \times \overline{X}) + \sum_{i=1}^{r} A_i \times X + \sum_{i=r+1}^{q} A_i \times \overline{X} = V, \tag{8}$$

where $V \in \mathcal{D}$, $B_k, C_k \in \mathcal{S}_k, k = 1, 2$ and $A_i \in \mathcal{T}, i = 1, ..., q$.

The general scheme we shall follow in finding the algebraic solutions to (8) is:

1. Transform the initial equation into a finite number of linear systems of equations in $\mathbb{R}^2$ by using the transition formula (4);
2. Find all solutions to the corresponding linear systems of equations using the well-known methods for solving linear systems of equations over $\mathbb{R}$;
3. Restrict (project) the solutions we have found in step 2 to the extended interval subspace corresponding to the class interval algebraic solutions we are looking for.

It is obvious from Theorem 4 that the three classes interval algebraic solutions to the equation (8) ($X \in \mathcal{D} \setminus \mathcal{T}$; $X \in \mathcal{T}$ and $\tau(X) = +$; $X \in \mathcal{T}$ and $\tau(X) = -$) generate three different special cases of this equation. That is why we split the initial general problem into two subproblems which we shall solve following the above general scheme. First, find all nonzero algebraic solutions to the equation (8) and second, find all zero algebraic solutions to the same equation.

**Algebraic solutions $X \in \mathcal{D} \setminus \mathcal{T}$ to equation (8).**
According to Theorem 4, for $X \in \mathcal{D}\setminus\mathcal{T}$ this equation is equivalent to the equation

$$\sum_{k=1}^{4}(B_k \times X + C_k \times \overline{X}) = V, \tag{9}$$

where $B_k, C_k \in \mathcal{S}_k, k = 1,2,3,4;$ $B_k = \sum^{r}_{\substack{i=1 \\ A_i \in \mathcal{S}_k}} A_i,$ $C_k = \sum^{q}_{\substack{i=r+1 \\ A_i \in \mathcal{S}_k}} A_i$ for $k = 3,4$. Applying transition formula (4) we obtain the following equivalent equation

$$\begin{aligned}
&(B_1)_{\sigma(X)} \times_h X + (C_1)_{\sigma(X)} \times_h \overline{X} + \\
&(B_2)_{\sigma(X)} \times_h \overline{X} + (C_2)_{\sigma(X)} \times_h X + \\
&(B_3)_{\sigma(X)} \times_h x^{\sigma(X)} + (C_3)_{\sigma(X)} \times_h x^{-\sigma(X)} + \\
&(B_4)_{\sigma(X)} \times_h x^{-\sigma(X)} + (C_4)_{\sigma(X)} \times_h x^{\sigma(X)} \;=\; V,
\end{aligned}$$

which, due to the distributivity of the hyperbolic product, is equivalent to

$$\begin{aligned}
&(B_1 + C_2)_{\sigma(X)} \times_h X + (B_2 + C_1)_{\sigma(X)} \times_h \overline{X} + \\
&(B_3 + C_4)_{\sigma(X)} \times_h x^{\sigma(X)} + (B_4 + C_3)_{\sigma(X)} \times_h x^{-\sigma(X)} \;=\; V.
\end{aligned}$$

Last equation is equivalent to a couple systems of linear equations in $\mathbb{R}^2$

$$\mathcal{U}X = \mathcal{V}, \qquad \text{with } \mathcal{X} = (x^-, x^+)^\top,\ \mathcal{V} = (v^-, v^+)^\top. \tag{10}$$

Matrices of both systems differ

$$\mathcal{U} = \mathcal{U}(\sigma(X)) = \begin{cases} \begin{pmatrix} u_{11}, u_{21} \\ u_{12}, u_{22} \end{pmatrix}, & \text{if } \sigma(X) = +, \\ \begin{pmatrix} u_{22}, u_{12} \\ u_{21}, u_{11} \end{pmatrix}, & \text{if } \sigma(X) = -, \end{cases}$$

where

$$\begin{aligned}
u_{11} &= b_1^- + c_2^- + c_3^- + b_4^-, \qquad & u_{21} &= c_1^- + b_2^- + b_3^- + c_4^-, \\
u_{12} &= c_1^+ + b_2^+ + c_3^+ + b_4^+, \qquad & u_{22} &= b_1^+ + c_2^+ + b_3^+ + c_4^+.
\end{aligned}$$

It is obvious that both matrices $\mathcal{U}(\sigma(X))$ have one and the same determinant. Denote

$$\begin{aligned}
&d = u_{11}u_{22} - u_{12}u_{21} \\
&d_1 = \det\begin{pmatrix} u_{11}, v^- \\ u_{12}, v^+ \end{pmatrix}, \qquad d_2 = \det\begin{pmatrix} u_{21}, v^- \\ u_{22}, v^+ \end{pmatrix}, \\
&\tilde{d}_1 = \det\begin{pmatrix} u_{22}, v^- \\ u_{21}, v^+ \end{pmatrix}, \qquad \tilde{d}_2 = \det\begin{pmatrix} u_{12}, v^- \\ u_{11}, v^+ \end{pmatrix}.
\end{aligned} \tag{11}$$

Apply the standard rules of algebra for finding the solutions to the systems (10) in $\mathbb{R}^2$:

1. If $d \neq 0$ both systems (10) have unique solutions $\mathcal{X}_1 = (-d_2/d, d_1/d)^\top$ and $\mathcal{X}_2 = (-\tilde{d}_2/d, \tilde{d}_1/d)^\top$. The solutions $\mathcal{X}_1$ and $\mathcal{X}_2$ generate numeric algebraic solutions to the equation (9), resp. (8)

$$X_1 = [-d_2/d, d_1/d], \qquad \text{if } X_1 \in \mathcal{S}_1,$$
$$X_2 = [-\tilde{d}_2/d, \tilde{d}_1/d], \qquad \text{if } X_2 \in \mathcal{S}_2.$$

$X_1 \in \mathcal{S}_1$ if $-d_2/d \geq 0$ and $d_1/d \geq 0$ and ($d_1 \neq 0$ or $d_2 \neq 0$). The conditions $-d_2/d \geq 0$ and $d_1/d \geq 0$ are equivalent to $dd_2 \leq 0 \leq dd_1$.
Analogously, $X_2 \in \mathcal{S}_2$, if $d\tilde{d}_1 \leq 0 \leq d\tilde{d}_2$ and ($\tilde{d}_1 \neq 0$ or $\tilde{d}_2 \neq 0$).

2. If $d = 0$ and $d_1 = d_2 = 0$, then the system $\mathcal{U}(\sigma(X) = +)\mathcal{X} = \mathcal{V}$ has parametric solutions depending on one real parameter. Since the parameter may occur either in the first or in the second component of the solution vector, we have two types parametric solutions: $\mathcal{X}_1' = ((v^- - u_{21}\alpha)/u_{11},\ \alpha)^\top$ and $\mathcal{X}_1'' = (\beta,\ (v^- - u_{11}\beta)/u_{21})^\top$, where $\alpha, \beta \in \mathbb{R}$. The solution vectors $\mathcal{X}_1'$ and $\mathcal{X}_1''$ generate positive parametric algebraic solutions $X_1' = [(v^- - u_{21}\alpha)/u_{11},\ \alpha] \in \mathcal{S}_1$, $X_1'' = [\beta,\ (v^- - u_{11}\beta)/u_{21}] \in \mathcal{S}_1$ to the equation (9), resp. (8) if both their components are non negative. Last requirement imposes the following restrictions on the parameters:
   - if $(u_{21}, u_{11} \geq 0,\ v^- \leq 0)$ or $(u_{21}, u_{11} \leq 0,\ v^- \geq 0)$, then there is no positive parametric solution;
   - if $(u_{21}, v^- \geq 0,\ u_{11} \leq 0)$ or $(u_{21}, v^- \leq 0,\ u_{11} \geq 0)$, then $0 \leq v^-/u_{21} \leq \alpha,\ \ 0 \leq \beta$;
   - if $(u_{21} \geq 0,\ u_{11}, v^- \leq 0)$ or $(u_{21} \leq 0,\ u_{11}, v^- \geq 0)$, then $0 \leq \alpha,\ \ 0 \leq v^-/u_{11} \leq \beta$;
   - if $(u_{21}, u_{11}, v^- \geq 0)$ or $(u_{21}, u_{11}, v^- \leq 0)$, then $0 \leq \alpha \leq v^-/u_{21},\ \ 0 \leq \beta \leq v^-/u_{11}$

   If $d = 0$ and $\tilde{d}_1 = \tilde{d}_2 = 0$, then the system $\mathcal{U}(\sigma(X) = -)\mathcal{X} = \mathcal{V}$ has two types parametric solutions $\mathcal{X}_2' = ((v^+ - u_{11}\delta)/u_{21}, \delta)^\top$ and $\mathcal{X}_2'' = (\gamma, (v^+ - u_{21}\gamma)/u_{11})^\top$, where $\gamma, \delta \in \mathbb{R}$. The solution vectors $\mathcal{X}_2'$ and $\mathcal{X}_2''$ generate negative parametric algebraic solutions $X_2' = [(v^+ - u_{11}\delta)/u_{21}, \delta] \in \mathcal{S}_2$, $X_2'' = [\gamma,\ (v^+ - u_{21}\gamma)/u_{11}] \in \mathcal{S}_2$ to the equation (9), resp. (8) if both their components are non positive. Last requirement imposes the following restrictions on the parameters $\gamma$ and $\delta$:
   - if $(u_{21}, u_{11} \geq 0,\ v^+ \leq 0)$ or $(u_{21}, u_{11} \leq 0,\ v^+ \geq 0)$, then $v^+/u_{11} \leq \delta \leq 0,\ \ v^+/u_{21} \leq \gamma \leq 0$;
   - if $(u_{21}, v^+ \geq 0,\ u_{11} \leq 0)$ or $(u_{21}, v^+ \leq 0,\ u_{11} \geq 0)$, then $\delta \leq v^+/u_{11} \leq 0,\ \ \gamma \leq 0$;
   - if $(u_{21} \geq 0,\ u_{11}, v^+ \leq 0)$ or $(u_{21} \leq 0,\ u_{11}, v^+ \geq 0)$, then $\delta \leq 0,\ \ \gamma \leq v^+/u_{21} \leq 0$;
   - if $(u_{21}, u_{11}, v^+ \geq 0)$ or $(u_{21}, u_{11}, v^+ \leq 0)$, then there is no negative parametric solution.
3. If $d = 0$ and ($d_1 \neq 0$ or $d_2 \neq 0$), then the system $\mathcal{U}(\sigma(X) = +)\mathcal{X} = \mathcal{V}$ has no solution and in this case the equation (9), resp. (8), possesses no positive solutions.

If $d = 0$ and ($\tilde{d}_1 \neq 0$ or $\tilde{d}_2 \neq 0$), then the system $\mathcal{U}(\sigma(X) = -)\mathcal{X} = \mathcal{V}$ has no solution and in this case the equation (9), resp. (8), possesses no negative solutions.

In general, equation (8) has no nonzero algebraic solution if

either $d \neq 0$ and ($dd_1 < 0$ or $0 < dd_2$ or $d_1 = d_2 = 0$) and ($d\tilde{d}_2 < 0$ or $0 < d\tilde{d}_1$ or $\tilde{d}_1 = \tilde{d}_2 = 0$);

or $d = 0$ and ($d_1 \neq 0$ or $d_2 \neq 0$) and ($\tilde{d}_1 \neq 0$ or $\tilde{d}_2 \neq 0$).

**Algebraic solutions $X \in \mathcal{T}$ to equation (8).**
According to Theorem 4, for $X \in \mathcal{T}$ this equation is equivalent to the equation

$$\sum_{k=1}^{2}(B_k \times X + C_k \times \overline{X}) + \sum_{\substack{i=1 \\ \tau(A_i)=\tau(X)}}^{r} A_i \times X + \sum_{\substack{i=r+1 \\ \tau(A_i)\neq\tau(X)}}^{q} A_i \times \overline{X} = V, \quad (12)$$

where $B_k, C_k \in \mathcal{S}_k$ and $A_i \in \mathcal{T}$.

An obvious criterion for nonexistence of zero algebraic solutions to the last equation is $V \in \mathcal{D} \setminus \mathcal{T}$. Thus for $V \in \mathcal{T}$ we transform the equation (12) into a number systems of linear equations (10) by using transition formula (4). To facilitate this process we define several classes of zero algebraic solutions. The direction $\tau(X) \in \{+, -\}$ of the algebraic solution generates two such classes. Let us reorder the coefficients $A_i \in \mathcal{T}$, $i = 1, ..., q$ according to their $\chi$-values and find the corresponding sequence

$$A_{i_1}, ..., A_{i_m} \quad \text{such that} \quad \chi(A_{i_1}) < \chi(A_{i_2}) < ... < \chi(A_{i_m})$$

where $m \leq q$. To the above sequence of $m$ coefficients, having distinct $\chi$-values, we correspond a sequence of $m+1$ disjoint intervals

$$(-\infty, \chi(A_{i_1}))\,,\ [\chi(A_{i_1}), \chi(A_{i_2}))\,,\ \ldots\ [\chi(A_{i_m}), \infty)\,.$$

Last sequence of intervals determines $m+1$ classes of zero algebraic solutions with a fixed direction $\tau(X) = \tau$. To every one of these $m+1$ classes of zero algebraic solutions with fixed direction $\tau(X) = \tau$ and fixed $\chi$-characteristic $\chi(X) \in \left[\chi(A_{i_k}), \chi(A_{i_{k+1}})\right)$, $k = 1, ..., m-1$ correspond two subclasses of algebraic solutions with different $\nu$-values $\nu(X) = \nu \in \{+, -\}$. This way we defined $2(2m+1)$ disjoint classes of zero algebraic solutions to the equation (12)

$$\begin{aligned} X(\tau, \chi, \nu) = \{X \in \mathcal{T} \mid\ & \tau(X) = \tau \in \{+, -\}; \\ & \chi(X) \in \left[\chi(A_{i_{k-1}}), \chi(A_{i_k})\right), k = 1, ..., m; \\ & \nu(X) = \nu \in \{+, -\}\}. \end{aligned}$$

For every $X \in X(\tau, \chi, \nu)$ there exists a unique system of linear equations (10) which is determined by the transition formula (4). Due to the representation (4) we can find the corresponding matrix $\mathcal{U}$ in an explicit form:

$$\mathcal{U} = \begin{cases} \begin{pmatrix} \alpha, \beta \\ \beta, \alpha \end{pmatrix}, & \text{if } X \in X(\tau, \chi \in (-\infty, \chi(A_{i_1})), \nu), \\ \begin{pmatrix} \alpha + \begin{cases} \delta^-, \text{if } \nu = -; \\ 0, \quad \text{if } \nu = + \end{cases}, & \beta + \begin{cases} 0, \quad \text{if } \nu = -; \\ \delta^-, \text{if } \nu = + \end{cases} \\ \beta + \begin{cases} \delta^+, \text{if } \nu = -; \\ 0, \quad \text{if } \nu = + \end{cases}, & \alpha + \begin{cases} 0, \quad \text{if } \nu = -; \\ \delta^+, \text{if } \nu = + \end{cases} \end{pmatrix}, & \text{otherwise,} \end{cases} \quad (13)$$

where

$$\alpha = b_1^\tau + c_2^\tau + \sum_{\substack{\chi(A_i) \geq \chi \\ \nu(A_i) = \tau(A_i) = \tau}} a_i^\tau + \sum_{\substack{\chi(A_i) \geq \chi \\ \nu(A_i) = \tau \neq \tau(A_i)}} a_i^\tau$$

$$\beta = b_2^{-\tau} + c_1^{-\tau} + \sum_{\substack{\chi(A_i) \geq \chi \\ \tau(A_i) = \tau \neq \nu(A_i)}} a_i^{-\tau} + \sum_{\substack{\chi(A_i) \geq \chi \\ \tau(A_i) \neq \tau = \nu(A_i)}} a_i^{-\tau}$$

$$[\delta] = \sum_{\substack{\tau(A_i) = \tau \\ \chi(A_i) < \chi}} (A_i)_{\tau\nu} + \sum_{\substack{\tau(A_i) \neq \tau \\ \chi(A_i) < \chi}} (A_i)_{\tau\nu}.$$

All solutions $\mathcal{X} = (x^-, x^+)^\top$ to the above $2(2m+1)$ systems, such that $[x^-, x^+] \in X(\tau, \chi, \nu)$, define algebraic solutions $X \in \mathcal{T}$ to the equation (12), resp. (8).

The above results can be summarized in the following

***Theorem 5.*** *All algebraic solutions $X \in \mathcal{D} \setminus \mathcal{T}$ to a pseudo-linear interval equation (8) are*

$$X_1 = [-d_2/d, d_1/d] \in \mathcal{S}_1, \qquad \text{if } \; d \neq 0, \;\; dd_2 \leq 0 \leq dd_1, \;\; (d_1 \neq 0 \text{ or } d_2 \neq 0);$$

$$X_2 = [-\tilde{d}_2/d, \tilde{d}_1/d] \in \mathcal{S}_2, \qquad \text{if } \; d \neq 0, \;\; d\tilde{d}_1 \leq 0 \leq d\tilde{d}_2, \;\; (\tilde{d}_1 \neq 0 \text{ or } \tilde{d}_2 \neq 0);$$

*where $d, d_i, \tilde{d}_i$, $i = 1, 2$ are defined by (11);*

$$X_1' = [(v^- - u_{21}\alpha)/u_{11}, \; \alpha], \;\; X_1'' = [\beta, \; (v^- - u_{11}\beta)/u_{21}], \qquad X_1', X_1'' \in \mathcal{S}_1, \quad \text{if } \; d = 0, \;\; d_1 = d_2 = 0;$$

$$X_2' = [(v^+ - u_{11}\delta)/u_{21}, \delta], \;\; X_2'' = [\gamma, \; (v^+ - u_{21}\gamma)/u_{11}], \qquad X_2', X_2'' \in \mathcal{S}_2, \quad \text{if } \; d = 0, \;\; \tilde{d}_1 = \tilde{d}_2 = 0;$$

*where the parameters $\alpha, \beta, \gamma, \delta$ are defined as follows:*

- *if $(u_{21}, u_{11} \geq 0;\ v^-, v^+ \leq 0)$ or $(u_{21}, u_{11} \leq 0;\ v^-, v^+ \geq 0)$, then*
  *there is no positive parametric solution;*
  $v^+/u_{11} \leq \delta \leq 0, \;\; v^+/u_{21} \leq \gamma \leq 0;$
- *if $(u_{21}, v^-, v^+ \geq 0;\ u_{11} \leq 0)$ or $(u_{21}, v^-, v^+ \leq 0;\ u_{11} \geq 0)$, then*
  $0 \leq v^-/u_{21} \leq \alpha, \;\; 0 \leq \beta, \;\; \delta \leq v^+/u_{11} \leq 0, \;\; \gamma \leq 0;$
- *if $(u_{21} \geq 0;\ u_{11}, v^-, v^+ \leq 0)$ or $(u_{21} \leq 0;\ u_{11}, v^-, v^+ \geq 0)$, then*
  $0 \leq \alpha, \;\; 0 \leq v^-/u_{11} \leq \beta, \;\; \delta \leq 0, \;\; \gamma \leq v^+/u_{21} \leq 0;$
- *if $u_{21}, u_{11}, v^-, v^+ \geq 0$ or $u_{21}, u_{11}, v^-, v^+ \leq 0$, then*
  $0 \leq \alpha \leq v^-/u_{21}, \;\; 0 \leq \beta \leq v^-/u_{11},$
  *there is no negative parametric solution.*

*All solutions $\mathcal{X} = (x^-, x^+)^\top$ to $2(2m+1)$ linear systems (10), determined by (13), such that $[x^-, x^+] \in X(\tau, \chi, \nu)$, define algebraic solutions $X \in \mathcal{T}$ to a general pseudo-linear interval equation (8) in $\mathcal{D}$.*

Next we give explicitly all algebraic solutions to two special types pseudo-linear interval equations in one variable, which solutions are most frequently sought.

### 4.1 Pseudo-linear equation involving nonzero coefficients

Consider the general pseudo-linear interval equation (8), where there are no coefficients from $\mathcal{T}$. The equation

$$\sum_{k=1}^{2}(B_k \times X + C_k \times \overline{X}) = V, \tag{14}$$

where $B_k, C_k \in \mathcal{S}_k$, $V \in \mathcal{D}$ and $B_1 + C_2, B_2 + C_1 \in \mathcal{T}$ is the general type of a pseudo-linear equation involving nonzero coefficients. Equations (14), where $B_1 + C_2 \in \mathcal{T}$ and/or $B_2 + C_1 \in \mathcal{T}$, are reducible to have less than four additive terms and therefore are special cases of equation (14). Denote $P = [p^-, p^+] = B_1 + C_2$, $Q = [q^-, q^+] = B_2 + C_1$. As a corollary from Theorem 5 we obtain all algebraic solutions to the equation (14).

The numeric algebraic solutions to this equation are:

$$X_{1,2} = [\frac{p^{\sigma(X)}v^- - q^{-\sigma(X)}v^+}{p^-p^+ - q^-q^+},\ \frac{p^{-\sigma(X)}v^+ - q^{\sigma(X)}v^-}{p^-p^+ - q^-q^+}],$$

$$\text{if } \ p^-p^+ \neq q^-q^+,\ \ x^-, x^+ \geq^{\sigma(X)} 0,\ \ (x^- \neq 0 \text{ or } x^+ \neq 0);$$

$$X_{3,4} = [\frac{p^{\tau(X)}v^- - q^{-\tau(X)}v^+}{(p^{\tau(X)})^2 - (q^{-\tau(X)})^2},\ \frac{p^{\tau(X)}v^+ - q^{-\tau(X)}v^-}{(p^{\tau(X)})^2 - (q^{-\tau(X)})^2}],$$

$$\text{if } \ V \in \mathcal{T},\ |p^{\tau(X)}| \neq |q^{-\tau(X)}|,\ (x^{-\tau(X)} < 0 < x^{\tau(X)} \text{ or } x^- = x^+ = 0).$$

$X_1$ and $X_2$ above are nonzero algebraic solutions, positive and negative respectively, while $X_3, X_4 \in \mathcal{T}$, $\tau(X_3) \neq \tau(X_4)$.

For $\lambda \in \Lambda$, $\leq^{\lambda} = \{\leq, \text{ if } \lambda = +;\ \geq, \text{ if } \lambda = -\}$.

All parametric algebraic solutions to the equation (14) are:

$$X'_{1,2} = [\frac{v^- - q^{-\sigma(X)}\alpha}{p^{-\sigma(X)}},\ \alpha], \qquad X''_{1,2} = [\beta,\ \frac{v^- - p^{-\sigma(X)}\beta}{q^{-\sigma(X)}}],$$

$$\text{if } \ p^-p^+ = q^-q^+ \text{ and } q^{-\sigma(X)}v^+ = p^{\sigma(X)}v^- = p^{-\sigma(X)}v^+ = q^{\sigma(X)}v^-$$

wherein the parameters $\alpha$ and $\beta$ are subjected to the following constrains:

– if $p^{-\sigma(X)}q^{-\sigma(X)} \geq 0$, then

$$0 \leq^{\sigma(X)} \alpha \leq^{\sigma(X)} \frac{v^-}{q^{-\sigma(X)}}, \qquad 0 \leq^{\sigma(X)} \beta \leq^{\sigma(X)} \frac{v^-}{p^{-\sigma(X)}};$$

– if $p^{-\sigma(X)}q^{-\sigma(X)} \le 0$, then

$$(0 \le^{\sigma(X)} \alpha \text{ and } \frac{v^-}{q^{-\sigma(X)}} \le^{\sigma(X)} \alpha), \qquad (0 \le^{\sigma(X)} \beta \text{ and } \frac{v^-}{p^{-\sigma(X)}} \le^{\sigma(X)} \beta);$$

$$X'_{3,4} = [\frac{v^- - q^{-\tau(X)}\delta}{p^{\tau(X)}},\ \delta], \quad X''_{3,4} = [\gamma,\ \frac{v^- - p^{\tau(X)}\gamma}{q^{-\tau(X)}}],$$

if $V \in \mathcal{T}$, $|p^{\tau(X)}| = |q^{-\tau(X)}| \neq 0$, $q^{-\tau(X)}v^+ = p^{\tau(X)}v^- = p^{\tau(X)}v^+ = q^{-\tau(X)}v^-$

wherein the parameters $\delta$ and $\gamma$ are subjected to the following constrains:

– if $p^{\tau(X)}q^{-\tau(X)} > 0$, then

$$(0 <^{\tau(X)} \delta \text{ and } \frac{v^-}{q^{-\tau(X)}} <^{\tau(X)} \delta), \qquad (\gamma <^{\tau(X)} 0 \text{ and } \gamma <^{\tau(X)} \frac{v^-}{p^{\tau(X)}});$$

– if $p^{\tau(X)}q^{-\tau(X)} < 0$, then

$$0 <^{\tau(X)} \delta <^{\tau(X)} \frac{v^-}{q^{-\tau(X)}}, \qquad \frac{v^-}{p^{\tau(X)}} <^{\tau(X)} \gamma <^{\tau(X)} 0;$$

The parametric solutions above have the following characterization: $X'_k, X''_k \in \mathcal{S}_k$, $\mu(X'_k) = \mu(X''_k)$, $k = 1, 2, 3, 4$.

*Example 3.* The algebraic solutions $X \in \mathcal{D} \setminus \mathcal{T}$ to equation (1) are

$$X_1 = [2, 3], \text{ and } X_2 = [-15, -43].$$

### 4.2 Solutions to equation $A \times X = B$

The equation $A \times X = B$, where $A, B \in \mathcal{D}$, $A \neq 0$ has the following solutions:

- $B/\overline{A}$, if $A \in \mathcal{D} \setminus \mathcal{T}$, $B \neq 0$;
- $0$, if $A \in \mathcal{D} \setminus \mathcal{T}$, $B = 0$, $a^-a^+ \neq 0$;
- $[b^{-\nu(A)\tau(A)}/a^{\nu(A)},\ b^{\nu(A)\tau(A)}/a^{\nu(A)}]$, if $A, B \in \mathcal{T}$, $\tau(A) = \tau(B)$, $\chi(B) \le \chi(A)$;
- $[\eta,\ -\zeta]_{\tau(A)}$, if $A \in \mathcal{D} \setminus \mathcal{T}$, $B = 0$, $a^-a^+ = 0$;
- $[\alpha,\ -\beta]_{\tau(A)}$ and $0$, if $A \in \mathcal{T}$, $B = 0$;
- $[\nu(A)\nu(B)\alpha,\ b^{-\nu(B)}/a^{-\nu(A)}]_{\tau(A)\nu(A)\nu(B)}$ and $[-\nu(A)\nu(B)\delta,\ b^{-\nu(B)}/a^{-\nu(A)}]_{\tau(A)\nu(A)\nu(B)}$, if $A, B \in \mathcal{T}$, $\tau(A) = \tau(B)$, $\chi(B) = \chi(A)$;

  wherein $\alpha, \beta > 0$, $\eta, \zeta \ge 0$ and $0 < \delta \le \left|b^{-\nu(B)}/a^{-\nu(A)}\right|$

- no algebraic solution, if either $A \in \mathcal{T}$, $B \in \mathcal{D} \setminus \mathcal{T}$, or $A, B \in \mathcal{T}$, $\tau(A) \neq \tau(B)$, or $A, B \in \mathcal{T}$, $\tau(A) = \tau(B)$, $\chi(B) > \chi(A)$.

The first three cases specify an unique numeric algebraic solution to the equation $A \times X = B$ and the conditions for its existence. Next three cases specify all parametric algebraic solutions to the equation under consideration. Note, that for $A, B \in \mathcal{T}$, there exist only zero involving algebraic solutions, while for $A \in \mathcal{D} \setminus \mathcal{T}$, the parametric solutions are positive, negative and involving zero.

If the equation $A \times X = B$ has no algebraic solution in $\mathcal{D}$, latter can be sought in the space of inner and outer extended intervals, obtained by division of intervals containing/contained in zero [Kaucher 1973], [Popova 1994]. In this case, the united solution set $X_{\exists\exists}$ of the equation $A \times X = B$ is

$$\begin{aligned} X_{\exists\exists} &= \{x \in \mathbb{R} \mid (\exists a \in A)(\exists b \in B)(a.x = b)\} \\ &= \mathrm{pro}(B/\overline{A}) \end{aligned}$$

where the division operation is well defined [Kaucher 1973], [Popova 1994] as:

$$B/\overline{A} = \begin{cases} [t^-, \ -\tau(T)\infty], \quad [\tau(T)\infty, \ t^+], & \text{if } B \in \mathcal{D} \setminus \mathcal{T} \\ [-\infty, \ \infty]_{\tau(B)}, & \text{if } B \in \mathcal{T}, \ \tau(A) = + \\ [b^+/a^-, \ b^-/a^+], \quad [b^-/a^-, \ b^+/a^+], & \text{if } B \in \mathcal{T}, \ \tau(A) = -, \end{cases}$$

where $A \in \mathcal{T}$ and $T = [t^-, \ t^+] = [b^{\tau(A)}/a^{\sigma(B)}, \ b^{\tau(A)}/a^{-\sigma(B)}]$.

## 5 Concluding Remarks

The extended interval arithmetic over $\mathcal{D}$ possesses better algebraic properties than conventional interval arithmetic and allows explicit algebraic solution of certain interval problems embedded there. We have demonstrated at the beginning of Section 4 how, by applying successive algebraic transformations, an interval equation can be transformed into a number of pseudo-linear interval equations in one variable, which algebraic solutions were presented explicitly in the paper.

Theorem 4, specifying the general normal form of a pseudo-linear interval expression, is of basic importance for all theoretical investigations based on elementary algebraic transformations. Applying this theorem to the left-hand side of the equation (7) we proved, without knowing the exact values of the coefficients, that left-hand side of this equation involves not more than eight additive terms.

We have demonstrated a technique for finding all algebraic solutions to a general pseudo-linear interval equation in one variable by solving corresponding number of linear equation systems in $\mathbb{R}^2$. All numeric and parametric algebraic solutions to some basic types pseudo-linear interval equations in one variable are presented explicitly. The way we have presented the algebraic solutions to a pseudo-linear interval equation in one variable — by classes of extended intervals having particular characteristic and the corresponding conditions for their existence — facilitates the solution of problems subjected to constrains. For example, many practical problems are interested in positive algebraic solutions and Theorem 5 specifies exactly which conditions have to be checked in this case. When solving tolerance problems by algebraic solution of a pseudo-linear interval equation we need only proper algebraic solutions and Theorem 5 says exactly which conditions have to be checked, or which systems in $\mathbb{R}^2$ have to be solved.

The applicability of all theoretical results, presented here, will be extremely facilitated if implemented in a computer algebra system supporting extended interval arithmetic. We think that computer algebra systems provide the best environment for exploiting the algebraic properties of the arithmetic over $\mathcal{D}$. A *Mathematica* package [Popova and Ullrich 1996] for extended interval arithmetic provides some functionalities that cannot be obtained by conventional interval arithmetic. By now this package contains facilities for:

- inwardly and outwardly rounded numerical computations with extended intervals providing that interval operations handle mathematical constants, exact singletons, integer (or rational) numbers exactly, when combined with inexact numbers;
- obtaining inner inclusions only by outwardly rounded operations and the corresponding dual of the input interval expression [Gardeñes and Trepat 1980];
- tight range computation for monotone rational interval functions reducing the dependency problem by an extended interval-arithmetic technique [Gardeñes and Trepat 1980];
- elementary algebraic transformations based on algebraic identities (3);
- numerical solution to certain interval equations in one variable;
- automatic simplification, based on Theorem 2, of symbolic-numerical interval expressions [Popova and Ullrich 1997].

A single function delivering all numerical and/or parametric algebraic solutions to a pseudo-linear interval equation in one variable is designed and its implementation is forthcoming. We believe that utilizing this function together with the other facilities of the package will increase the efficiency of interval applications.

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