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Extended interval arithmetics: new results and applications

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Abstract

In this paper we compare two well studied interval arithmetic structures, which are different extensions of familiar interval arithmetic; the first one is obtained by extending the concept of interval, the other by extending the set of basic arithmetic operations. Certain relations between these two structures have been formulated which outline a new direction of applications. By demonstrating that all theorems of both theories have analogues in each other we conclude that both structures can be equally well used in practical applications.

1. INTRODUCTION

Here we further develop a convenient method for the presentation of intervals by means of their end-points [7], [13]. Our method, which can be briefly referred as "plus-minus" method, allows to present the product of two zero-free intervals $A = [a^-, a^+]$, $B = [b^-, b^+]$ by means of their sign σ in the convenient form $A \times B = [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}]$. This simple form can be used when extending $A \times B$ for generalized intervals $A, B \in R^2$, which leads to a nice algebraic structure considered in [5], [6], [15]. We note that this extension is different from the one generated by the "min-max" expression $A \times B = [\min\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}, \max\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}]$.

We first briefly introduce the conventional interval arithmetic $\mathcal{S}$ using our "plus-minus" method of denoting intervals by their end-points. In section 2 we introduce the interval structure $\mathcal{K}$ which can be obtained from $\mathcal{S}$ by extending the concept of interval; in section 3 we introduce the structure $\mathcal{M}$ by extending the set of basic arithmetic operations in $\mathcal{S}$. The order of our presentation is of no importance: in [13] we follow an alternative order by introducing first $\mathcal{M}$ then $\mathcal{K}$. In section 4 we introduce the structure $\mathcal{KM}$ which is a generalization of both $\mathcal{K}$ and $\mathcal{M}$ and formulate relations between the operations in $\mathcal{KM}$. In section 5 we demonstrate that an interval arithmetic expression (or statement) in $\mathcal{K}$ can be reformulated in $\mathcal{M}$ and vice versa. As an example from the distributive law in $\mathcal{M}$ we obtain a new distributive law in $\mathcal{K}$. In our presentation we carefully distinguish the basic operations from the dependent (composite) ones; this can be of some help for

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future similar investigations in the setting of abstract algebraic structures. In this paper we restrict ourselves to the case of finite intervals; the situation involving infinite intervals is considered in [13].

A (proper) interval $[a,b], a \le b$, is a compact set on the real line R defined by $[a,b] = \{x \mid a \le x \le b\}$. The set $\{[a,b] \mid a,b \in R, a \le b\}$ of all intervals is denoted by IR. The left end-point of the interval $A \in IR$ is denoted by a^- or A^-, and the right end-point by a^+ or A^+, so that $A = [a^-, a^+] = [A^-, A^+]$. Hence, for $A \in IR$ the symbol a^s (or A^s), with $s \in \{+,-\}$, denotes certain end-point of A, which can be the left one or the right one depending on the value of s. We define the product st for $s,t \in \{+,-\}$ by setting $++ = -- = +,\ \ +- = -+ = -$, so that $a^{++} = a^{--} = a^+$ etc.

Denote the set of intervals containing zero by $Z = \{A \in IR \mid 0 \in A\} = \{A \mid a^- \le 0 \text{ and } a^+ \ge 0\}$; the set of intervals which do not contain zero is $IR \setminus Z = \{A \in IR \mid 0 \notin A\}$. Define a sign functional $\sigma : IR \setminus Z \to \{+,-\}$, by means of $\sigma(A) = \{+, \text{ if } a^- > 0;\ -, \text{ if } a^+ < 0\}$.

The interval arithmetic $\mathcal{S} = (IR, +, \times, /, \subseteq)$ [1, 14, 16, 17] consists of the set IR together with a relation for inclusion $\subseteq$ and the basic operations addition $+ : IR \bigotimes IR \to IR$, multiplication $\times : IR \bigotimes IR \to IR$ and inversion (reciprocal value) $/ : IR \setminus Z \to IR$, defined by

$$
\begin{aligned}
A \subseteq B \ &\Leftrightarrow\ (b^- \le a^-) \text{ and } (a^+ \le b^+), \quad \text{for } A, B \in IR\,, && (1)\\
A + B \ &=\ [a^- + b^-, a^+ + b^+], \quad \text{for } A, B \in IR\,, && (2)\\
A \times B \ &=\ \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}], & \text{for } A, B \in IR \setminus Z\,, \\ [a^{\delta} b^{-\delta}, a^{\delta} b^{\delta}],\ \ \delta = \sigma(A), & \text{for } A \in IR \setminus Z,\ B \in Z\,, \\ [a^{-\delta} b^{\delta}, a^{\delta} b^{\delta}],\ \ \delta = \sigma(B), & \text{for } A \in Z,\ B \in IR \setminus Z, \end{cases} && (3)\\
A \times B \ &=\ [\min\{a^- b^+, a^+ b^-\}\,,\ \max\{a^- b^-, a^+ b^+\}], \quad \text{for } A, B \in Z\,, && (4)\\
1\,/\,B \ &=\ [1/b^+, 1/b^-],\ B \in IR \setminus Z. && (5)
\end{aligned}
$$

In the special case when A is a degenerate interval of the form $A = [a,a] = a$, we have $A \times B = a \times B = [ab^{-\sigma(a)}, ab^{\sigma(a)}] = \{[ab^-, ab^+], \text{ if } a \ge 0;\ [ab^+, ab^-], \text{ if } a < 0\}$. For $a = -1$ we have $(-1) \times B = -B = -[b^-, b^+] = [-b^+, -b^-]$. The operations subtraction $A - B$ and division A/B are defined in $\mathcal{S}$ as composite operations by

$$
\begin{aligned}
A\ -\ B \ &=\ A + (-1) \times B = A + (-B) = [a^- - b^+, a^+ - b^-], \text{ for } A, B \in IR, && (6)\\
A/B \ &=\ A \times (1/B) = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A, B \in IR \setminus Z, \\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}],\ \ \delta = \sigma(B), & \text{for } A \in Z, B \in IR \setminus Z\,. \end{cases} && (7)
\end{aligned}
$$

We note that the operation inversion "$1/B$" in $\mathcal{S}$ can not be composed by means of the operations "$+$" and "$\times$" and therefore should be assumed as basic. The operations $+, -, \times, /$ in $\mathcal{S}$ defined by (2)–(6) satisfy the relations: $A * B = \{a * b \mid a \in A, b \in B\}$, $* \in \{+, -, \times, /\}$. The properties of $\mathcal{S} = (IR, +, \times, /, \subseteq)$ are well studied [1, 14, 16, 17]. We recall here two important properties which will be used as examples further on:
S1. The operations "$+$" and "$\times$" satisfy the following associative laws:
$(A + B) + C = A + (B + C), \qquad (A \times B) \times C = A \times (B \times C)$;
S2. The operations (2)–(6) in $\mathcal{S}$ (when well defined) are isotone w. r. t. "$\subseteq$":
$A \subseteq A_1,\ B \subseteq B_1 \Rightarrow A * B \subseteq A_1 * B_1, \ \ * \in \{+, -, \times, /\}$.

2. THE INTERVAL STRUCTURE $\mathcal{K} = (\mathcal{H}, +, \times, \subseteq)$

In this section the definition domains of interval-arithmetic relation (1) and of operations (2)–(4) are extended from IR into the set $\mathcal{H} = \{[a,b] \mid a,b \in R\} \cong R^2$ of all ordered couples of real numbers [5, 6, 15]. The first component of $A \in \mathcal{H}$ is denoted by $a^- = A^-$, the second one by $a^+ = A^+$, so that $A = [a^-, a^+] = [A^-, A^+]$. The elements of $\mathcal{H}$ are called generalized intervals; a generalized interval $A = [a^-, a^+] \in \mathcal{H}$ is a proper (regular) one if $a^- \leq a^+$, and improper one if $a^- \geq a^+$. The set of all elements of $\mathcal{H}$, which are proper intervals is equivalent to IR and is further denoted again by IR; the set of all improper intervals is denoted by $\overline{IR}$, so that $\mathcal{H} = IR \cup \overline{IR}$. Degenerate intervals of the form $A = [a,a]$ belong both to IR and $\overline{IR}$. Define $\overline{Z} = \{A \in \overline{IR} \mid a^+ \leq 0 \text{ and } a^- \geq 0\}$, $\mathcal{T} = Z \cup \overline{Z}$. Define $\sigma : \mathcal{H} \backslash \mathcal{T} \rightarrow \{+,-\}$, by $\sigma(A) = \{+, \text{ if } a^- > 0 \text{ and } a^+ > 0\ ;\ -, \text{ if } a^- < 0 \text{ and } a^+ < 0\}$.

The extended interval arithmetic $\mathcal{K} = (\mathcal{H}, +, \times, \subseteq)$ is obtained by extending the definition domains of $+, \times$ and $\subseteq$ as defined in $\mathcal{S}$ from IR to $\mathcal{H}$. A formal substitution of IR by $\mathcal{H}$ and of Z by $\mathcal{T}$ in (1)–(3) yields the definitions of $+$, $\times$ and $\subseteq$ in $\mathcal{H}$:

$$A \subseteq B \quad \Leftrightarrow \quad (b^- \leq a^-) \text{ and } (a^+ \leq b^+), \quad \text{for } A, B \in \mathcal{H}, \tag{8}$$

$$A + B \quad = \quad [a^- + b^-, a^+ + b^+], \quad \text{for } A, B \in \mathcal{H}\ , \tag{9}$$

$$A \times B \quad = \quad \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}], \quad for\ A, B \in \mathcal{H} \setminus \mathcal{T}, \\ [a^{\delta\tau(B)} b^{-\delta}, a^{\delta\tau(B)} b^{\delta}], \quad \delta = \sigma(A), \quad \text{for } A \in \mathcal{H} \setminus \mathcal{T},\ B \in \mathcal{T}, \\ [a^{-\delta} b^{\delta\tau(A)}, a^{\delta} b^{\delta\tau(A)}], \quad \delta = \sigma(B), \quad \text{for } A \in \mathcal{T},\ B \in \mathcal{H} \setminus \mathcal{T}. \end{cases} \tag{10}$$

For the extension of (4), which should accomplish the definition of $A \times B$ for the situation when both $A, B \in \mathcal{T}$, we regretfully cannot make use of the "plus-minus" technique; following E. Kaucher [5], [6] we set:

$$A \times B = \begin{cases} [\min\{a^- b^+, a^+ b^-\}, \max\{a^- b^-, a^+ b^+\}], \quad \text{for } A, B \in Z, \\ [\max\{a^- b^-, a^+ b^+\}, \min\{a^- b^+, a^+ b^-\}], \quad \text{for } A, B \in \overline{Z}, \\ 0\ , \quad \text{for } A \in Z,\ B \in \overline{Z} \quad \text{or} \quad A \in \overline{Z},\ B \in Z\ . \end{cases} \tag{11}$$

From (10) for $A = [a,a] = a$, $B \in \mathcal{H}$ we have $a \times B = [ab^{-\sigma(a)}, ab^{\sigma(a)}]$. Substituting $a = -1$ we obtain $(-1) \times B = -B = [-b^+, -b^-]$. The composite operation $A + (-1) \times B = A + (-B) = [a^- - b^+, a^+ - b^-]$, for $A, B \in \mathcal{H}$ is an extension of the $\mathcal{S}$-subtraction into $\mathcal{H}$ and will be further denoted $A - B$ as in (6).

The substructures $(\mathcal{H}, +, \subseteq)$ and $(\mathcal{H} \setminus \mathcal{T}, \times, \subseteq)$ of $\mathcal{K}$ are isotone groups [5]; there exist inverse elements with respect to operations (9) and (10). Denote the inverse additive element of $A \in \mathcal{H}$ by $-_h A$, and the inverse element of $A \in \mathcal{H} \setminus \mathcal{T}$ with respect to "$\times$" by $1/_h A$. For the inverse elements we obtain the end-point presentations $-_h A = [-a^-, -a^+]$, for $A \in \mathcal{H}$, and $1/_h A = [1/a^-, 1/a^+]$, for $A \in \mathcal{H} \setminus \mathcal{T}$.

The inverse additive element $-_h A$ should not be confused with the negative element $-A = (-1) \times A = [-a^+, -a^-]$. Using the monadic operators $-A = [-a^+, -a^-]$ and $-_h A = [-a^-, -a^+]$ the following (also monadic) operator can be composed:

$$\overline{A} = -_h(-A) = -(-_h A) = [a^+, a^-], \tag{12}$$

which is called in [5, 6] conjugation. Note that conjugation in $\mathcal{H}$ is a composite operation derived from the basic operations $+, \times$ and their inverse.

Equalities (12) suggest to look for a monadic operator $1/A$ in $\mathcal{H}\setminus\mathcal{T}$ which possibly satisfies the relations

$$1/_h(1/A) = 1/(1/_hA) = \overline{A}. \tag{13}$$

Indeed, such is the unique operator "inversion" $1/A = \overline{1/_hA} = 1/_h\overline{A} = [1/a^+, 1/a^-]$, for $A \in \mathcal{H}\setminus\mathcal{T}$; opposite to the situation in $\mathcal{S}$ it is a composite operator which is an extension of (5). We now compose the operation $A\times(1/B)$ for $A \in \mathcal{H}$, $B \in \mathcal{H}\setminus\mathcal{T}$. This operation, which will be further denoted by A/B, is an extension in $\mathcal{H}$ of the $\mathcal{S}$-operation A/B defined by (7); we have

$$A/B = A\times(1/B) = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A, B \in \mathcal{H}\setminus\mathcal{T}, \\ [a^{-\delta}/b^{-\delta\tau(A)}, a^{\delta}/b^{-\delta\tau(A)}], \ \delta = \sigma(B), & \text{for } A \in \mathcal{T},\ B \in \mathcal{H}\setminus\mathcal{T}. \end{cases}$$

From $\overline{A} = -(-_hA)$ and $\overline{A} = 1/(1/_hA)$ (see (12) and (13)) we obtain the following expressions for the inverse elements: $-_hA = -\overline{A}$, $1/_hA = 1/\overline{A}$. The inverse elements $-_hA$, $1/_hA$ generate operations $A + (-_hB) = A + (-\overline{B}) = A - \overline{B}$, $A\times(1/_hB) = A\times(1/\overline{B}) = A/\overline{B}$ which are inverse to the operations $A+B$ and $A\times B$, respectively. Denoting these two operations by $A -_h B$ and $A/_hB$, resp., we have

$$A -_h B = A + (-_hB) = A - \overline{B} = [a^- - b^-, a^+ - b^+], \quad \text{for } A, B \in \mathcal{H},$$
$$A /_h B = A\times(1/_hB) = A/\overline{B} = \begin{cases} [a^{-\sigma(B)}/b^{-\sigma(A)}, a^{\sigma(B)}/b^{\sigma(A)}], & \text{for } A, B \in \mathcal{H}\setminus\mathcal{T}, \\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}], \ \delta = \sigma(B), & \text{for } A \in \mathcal{T},\ B \in \mathcal{H}\setminus\mathcal{T}. \end{cases}$$

From the last equality we obtain $A/B = A/_h\overline{B} = A/_h(-_h((-1)\times B))$, showing that the operation for division "/" can be composed by means of the basic operation "$\times$" and the inverse operations $-_h$ and $/_h$ (alternatively to the situation in $\mathcal{S}$ where "/" is an independent operation). Since division "/" in $\mathcal{H}$ can be derived from operations (9)–(11) and their inverse the symbol "/" may not be necessarily included in the notation for the algebraic structure $\mathcal{K} = (\mathcal{H}, +, \times, \subseteq)$. We see that, the interval structure $\mathcal{K} = (\mathcal{H}, +, \times, \subseteq)$ involves the operations subtraction "$-$", division "/", conjugation, the inverse operations $A - \overline{B}$, $A/\overline{B}$ and their conjugated $\overline{A} - B$ and $\overline{A}/B$. Other useful compound operations are $A + \overline{B}$, $A\times\overline{B}$ and their conjugated $\overline{A} + B$, $\overline{A}\times B$; their end-point presentations are resp. (note that $\sigma(B) = \sigma(\overline{B})$):

$$\begin{aligned}
A + \overline{B} &= [a^- + b^+, a^+ + b^-], \quad \text{for } A, B \in \mathcal{H}; \\
\overline{A} + B &= [a^+ + b^-, a^- + b^+], \quad \text{for } A, B \in \mathcal{H}; \\
A\times\overline{B} &= \begin{cases} [a^{-\sigma(B)}b^{\sigma(A)}, a^{\sigma(B)}b^{-\sigma(A)}], & \text{for } A, B \in \mathcal{H}\setminus\mathcal{T}, \\ [a^{\delta}b^{\delta}, a^{\delta}b^{-\delta}], \ \delta = \sigma(A), & \text{for } A \in \mathcal{H}\setminus\mathcal{T},\ B \in \mathcal{T}, \\ [a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}], \ \delta = \sigma(B), & \text{for } A \in \mathcal{T},\ B \in \mathcal{H}\setminus\mathcal{T}; \end{cases} \\
\overline{A}\times B &= \begin{cases} [a^{\sigma(B)}b^{-\sigma(A)}, a^{-\sigma(B)}b^{\sigma(A)}], & \text{for } A, B \in \mathcal{H}\setminus\mathcal{T}, \\ [a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}], \ \delta = \sigma(A), & \text{for } A \in \mathcal{H}\setminus\mathcal{T},\ B \in \mathcal{T}, \\ [a^{\delta}b^{\delta}, a^{-\delta}b^{\delta}], \ \delta = \sigma(B), & \text{for } A \in \mathcal{T},\ B \in \mathcal{H}\setminus\mathcal{T}. \end{cases}
\end{aligned}$$

By analogy with $A -_h B = A - \overline{B}$, $A/_hB = A/\overline{B}$ we may denote $A +_h B = A + \overline{B}$, $A \times_h B = A\times\overline{B}$; then $B +_h A = B + \overline{A} = \overline{A +_h B}$, $B \times_h A = B\times\overline{A} = \overline{A \times_h B}$, showing that "$+_h$" and "$\times_h$" are not commutative operations.

It can be shown [5], [6], [15] that $\subseteq, +, -, \times, /$ preserve many of their properties in IR under their extension (8)–(11) from IR to $\mathcal{H}$. In particular the association property **S1** and the inclusion isotonicity property **S2** remain true in $\mathcal{K}$ in the same forms as formulated in $\mathcal{S}$.

We shall next consider a possibility to reformulate an interval arithmetic expression in $\mathcal{K}$ into an expression in $\mathcal{S}$. To this end we shall further use the notations $A_- = \overline{A}$, $A_+ = A$. Define the operator "type of a generalized interval" $\tau : \mathcal{H} \to \{+, -\}$ by $\tau(A) = \{+,\ \text{if } a^+ \geq a^-;\ -,\ \text{if } a^+ < a^-\}$. Further, assign to $A \in \mathcal{H}$ the interval $(A)_p = \{A,\ \text{if } a^+ \geq a^-;\ \overline{A},\ \text{if } a^+ < a^-\}$. We have $(A)_p = (\overline{A})_p = A_{\tau(A)} \in IR$, that is the intervals A and $(A)_p$ have same endpoints and $(A)_p$ is a proper interval.

Proposition 1. For $A, B \in \mathcal{H}$ and $\tau(A) = \tau(B) = \tau$ we have $\tau(A+B) = \tau(A \times B) = \tau$ and

$$\begin{aligned}
(A+B)_p &= (A)_p + (B)_p = \left\{A + B,\ \text{if } A, B \in IR;\ \overline{A} + \overline{B},\ \text{if } A, B \in \overline{IR}\right\}, \\
(A \times B)_p &= (A)_p \times (B)_p = \left\{A \times B,\ \text{if } A, B \in IR;\ \overline{A} \times \overline{B},\ \text{if } A, B \in \overline{IR}\right\}, \\
A \subseteq B &\Leftrightarrow \left\{A \subseteq B,\ \ \text{if } A, B \in IR; \quad \overline{A} \supseteq \overline{B},\ \ \text{if } A, B \in \overline{IR}\right\}.
\end{aligned}$$

Proposition 1 shows that one can substitute an arbitrary expression in $\mathcal{K}$ involving $\mathcal{H}$-intervals *of equal type* by an expression in $\mathcal{S}$ involving proper intervals. If the $\mathcal{H}$-intervals involved are *of different type* we cannot do this; the algebraic system $\mathcal{S}= (IR, +, \times, /, \subseteq)$ does not provide suitable tools for this purpose. In the next section we extend the system $\mathcal{S}$ by means of two nonstandard interval arithmetic operations "$+^-$" and "$\times^-$" obtaining thus a system $\mathcal{M} = (IR, +, +^-, \times, \times^-, \subseteq)$ possessing such tools. We note that the interval structures $\mathcal{K}$ and $\mathcal{M}$ are obtained in two completely different ways. Recall that $\mathcal{K}$ is obtained by: i) a generalization of the concept of interval (i.e. by an extension of the support IR of $\mathcal{S}$ into the set $\mathcal{H}$), and ii) by an extension of the definition domains of the operations for addition and multiplication and of the relation for inclusion from IR into $\mathcal{H}$. On the other side $\mathcal{M}$ is obtained by introducing two new interval-arithmetic operations in $\mathcal{S}$, using thereby the usual concept of interval (i.e. element of IR).

3. THE INTERVAL STRUCTURE $\mathcal{M} = (IR, +, +^-, \times, \times^-, \subseteq)$

In this section we use all notations from the introduction referring to the space $\mathcal{S}$. The support of $\mathcal{M}$ is the set IR. The interval-arithmetic structure $\mathcal{M}$ is introduced as an extension of $\mathcal{S} = (IR, +, \times, /, \subseteq)$ by means of two (nonstandard) operations $+^-, \times^-$ defined by (cf. [2, 3, 4], [7]–[13]):

$$A +^- B = [a^{-\alpha} + b^{\alpha}, a^{\alpha} + b^{-\alpha}], \quad \text{for } A, B \in IR, \tag{14}$$

$$A \times^- B = \begin{cases} [a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon} b^{\sigma(A)\varepsilon}], & \text{for } A, B \in IR \setminus Z, \\ [a^{-\delta} b^{-\delta}, a^{-\delta} b^{\delta}],\ \ \delta = \sigma(A), & \text{for } A \in IR \setminus Z,\ B \in Z, \\ [a^{-\delta} b^{-\delta}, a^{\delta} b^{-\delta}],\ \ \delta = \sigma(B), & \text{for } A \in Z,\ B \in IR \setminus Z, \\ [\ \max\{a^- b^+, a^+ b^-\},\ \min\{a^- b^-, a^+ b^+\}], & \text{for } A, B \in Z, \end{cases} \tag{15}$$

where the sign operators $\alpha, \varepsilon \in \{+, -\}$ are chosen in such a way that the intervals involved in the right hand sides are elements of IR, that is $a^{-\alpha} + b^{\alpha} \leq a^{\alpha} + b^{-\alpha}$, $a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon} \leq$

$a^{-\sigma(B)\varepsilon}b^{\sigma(A)\varepsilon}$. We can easily determine α, ε as follows. Define

$$\begin{aligned}
\omega(A) &= \mid a^+ - a^- \mid, \quad \text{for } A \in IR, \\
\chi(A) &= a^{-\sigma(A)}/a^{\sigma(A)} = \{a^-/a^+ \ \text{ if } \sigma(A) = +; \quad a^+/a^- \ \text{ if } \sigma(A) = -\}, \quad \text{for } A \in IR \setminus Z.
\end{aligned}$$

We now define the sign operators $\phi : IR \bigotimes IR \to \{+,-\}$ and $\psi : (IR\backslash Z) \bigotimes (IR\backslash Z) \to \{+,-\}$ by means of

$$\begin{aligned}
\phi(A,B) &= \operatorname{sign}(\omega(A) - \omega(B)) = \{+, \ \text{ if } \omega(A) \ge \omega(B); \quad -, \ \text{ otherwise}\}, \\
\psi(A,B) &= \operatorname{sign}(\chi(A) - \chi(B)) = \{+, \ \text{ if } \chi(A) \ge \chi(B); \quad -, \ \text{ otherwise}\}.
\end{aligned}$$

Then α, ε in (14), (15) can be defined as $\alpha = \phi(A,B), \ \ \varepsilon = \psi(A,B)$.

Using the traditional min-max notations we can write

$$\begin{aligned}
A +^- B &= [\min\{a^- + b^+, a^+ + b^-\}, \max\{a^- + b^+, a^+ + b^-\}], \ \text{ for } A, B \in IR, \\
A \times^- B &= [\min\{a^{\sigma(B)}b^{-\sigma(A)}, a^{-\sigma(B)}b^{\sigma(A)}\}, \max\{a^{\sigma(B)}b^{-\sigma(A)}, a^{-\sigma(B)}b^{\sigma(A)}\}], \ A, B \in IR \setminus Z,
\end{aligned}$$

instead of the relevant expressions (14), (15) involving α, ε. However, when embedding $\mathcal{M}$ in $\mathcal{H} \cong R^2$ formulae (14), (15) are more useful (see section 4).

The elements $-A = [-a^+, -a^-]$ and $1/A = [1/a^+, 1/a^-]$ are inverse with respect to the operations $+^-$ and $\times^-$, resp. (note that $A +^- (-A) = 0, \quad A \times^- (1/A) = 1$). We introduce below four useful composite operations in $\mathcal{M}$:

$$A - B = A + (-B) = [a^- - b^+, a^+ - b^-], \ \text{ for } A, B \in IR, \tag{16}$$

$$A -^- B = A +^- (-B) = [a^{-\alpha} - b^{-\alpha}, a^{\alpha} - b^{\alpha}], \ \text{ for } A, B \in IR, \tag{17}$$

$$A/B = A \times (1/B) = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A, B \in IR \setminus Z, \\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}], \ \ \delta = \sigma(B), & \text{for } A \in Z, \ B \in IR \setminus Z, \end{cases} \tag{18}$$

$$A/^-B = A \times^- (1/B) = \begin{cases} [a^{\sigma(B)\varepsilon}/b^{\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}/b^{-\sigma(A)\varepsilon}], & \text{for } A, B \in IR \setminus Z, \\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}], \ \ \delta = \sigma(B), & \text{for } A \in Z, \ B \in IR \setminus Z. \end{cases} \tag{19}$$

Formulae (2)–(4), (14), (15) together with (16)–(19) summarize eight arithmetic operations in $\mathcal{M}$ (four basic and four composite). Operations (16) and (18) coincide resp. with the interval arithmetic operations (6) and (9) in $\mathcal{S}$.

Being inverse with respect to "$\times^-$" inversion $1/A$ is not a basic operation in $\mathcal{M}$. The interval-arithmetic operation (18) for division "/" is composed by means of the operation "$\times$" and the inverse element "$1/A$". The operation (16) for subtraction is also a composite operation. Hence we can denote the algebraic system $\mathcal{M}$ as $(IR, +, +^-, \times, \times^-, \subseteq)$, excluding thereby inversion (and resp. division) from the set of independent (basic) operations of $\mathcal{M}$. Recall that in $\mathcal{S}$ the operation "$1/A$" has to be assumed as basic.

Remark. We can consider as basic the operations $\{+, -^-, \times, /^-\}$; the operations $\{+^-, -, \times^-, /\}$ are then expressed as composite operations by means of: $A +^- B = A -^- (-B)$, $A - B = A + (-B)$, $A \times^- B = A/^-(1/B)$, $A/B = A \times (1/B)$. The operations "$-^-$" and "$/^-$" have been chosen as basic in [2]–[4], [7]–[12] and have been denoted by "$-$" and "/" respectively; the four composite operations $\{+^-, -, \times^-, /\}$ have been denoted by $\{\oplus, \ominus, \otimes, \oslash\}$ respectively. These notations seem to be natural but they are regretfully in confusion with the notations of the interval arithmetic operations for subtraction and division in $\mathcal{S}$ as adopted in [1], [14], [16], [17].

The algebraic properties of the interval structure $\mathcal{M} = (IR, +, +^-, \times, \times^-, \subseteq)$ are well studied (see [2],[3],[7]–[13]); they incorporate the properties of $\mathcal{S}$. We give below two properties concerning the associative and inclusion monotonicity rules.

M1. The operation "+" satisfies **S1**. In addition, for $A, B, C \in IR$:

$$
\begin{aligned}
(A+B)+^- C &= \begin{cases} A+(B+^- C) & \text{if } \omega(B) \ge \omega(C), \\ A+^-(B+^- C) & \text{if } \omega(B) < \omega(C); \end{cases} \\
(A+^- B)+C &= \begin{cases} A+(B+^- C) & \text{if } \omega(A) \ge \omega(B),\ \omega(B) < \omega(C), \\ A+^-(B+^- C) & \text{if } \omega(A) \ge \omega(B),\ \omega(B) \ge \omega(C), \\ A+^-(B+C) & \text{if } \omega(A) < \omega(B); \end{cases} \\
(A+^- B)+^- C &= \begin{cases} A+(B+^- C) & \text{if } \omega(A) < \omega(B),\ \omega(B) < \omega(C), \\ A+^-(B+^- C) & \text{if } \omega(A) < \omega(B),\ \omega(B) \ge \omega(C), \\ A+^-(B+C) & \text{if } \omega(A) \ge \omega(B). \end{cases}
\end{aligned}
$$

The operation "$\times$" satisfies **S1**. Further, for $A, B, C \in IR \setminus Z$:

$$
\begin{aligned}
(A\times B)\times^- C &= \begin{cases} A\times(B\times^- C) & \text{if } \chi(B) \le \chi(C), \\ A\times^-(B\times^- C) & \text{if } \chi(B) > \chi(C); \end{cases} \\
(A\times^- B)\times C &= \begin{cases} A\times^-(B\times^- C) & \text{if } \chi(A) \le \chi(B),\ \chi(B) \le \chi(C), \\ A\times(B\times^- C) & \text{if } \chi(A) \le \chi(B),\ \chi(B) > \chi(C), \\ A\times^-(B\times C) & \text{if } \chi(A) > \chi(B); \end{cases} \\
(A\times^- B)\times^- C &= \begin{cases} A\times^-(B\times^- C) & \text{if } \chi(A) \ge \chi(B),\ \chi(B) \le \chi(C), \\ A\times(B\times^- C) & \text{if } \chi(A) \ge \chi(B),\ \chi(B) > \chi(C), \\ A\times^-(B\times C) & \text{if } \chi(A) < \chi(B); \end{cases}
\end{aligned}
$$

M2. The operations $+, -, \times, /$ satisfy **S2**. The operations $+^-, -^-, \times^-, /^-$ satisfy the following inclusion isotonicity rules:
For $X, X_1, Y, Y_1 \in IR$, $\ * \in \{+^-, -^-\}$, we have:

$$
\begin{aligned}
&\text{If } X \supseteq X_1, Y \subseteq Y_1 \text{ and } \omega(X) \le \omega(Y) \text{ then } X * Y \subseteq X_1 * Y_1, \\
&\text{if } X \supseteq X_1, Y \subseteq Y_1 \text{ and } \omega(X_1) \ge \omega(Y_1) \text{ then } X * Y \supseteq X_1 * Y_1.
\end{aligned}
$$

For $X, X_1, Y, Y_1 \in IR \setminus Z$, $\ * \in \{\times^-, /^-\}$, we have:

$$
\begin{aligned}
&\text{If } X \supseteq X_1, Y \subseteq Y_1 \text{ and } \min\{\chi(X), \chi(X_1)\} \ge \max\{\chi(Y), \chi(Y_1)\} \text{ then } X * Y \subseteq X_1 * Y_1, \\
&\text{if } X \supseteq X_1, Y \subseteq Y_1 \text{ and } \max\{\chi(X), \chi(X_1)\} \le \min\{\chi(Y), \chi(Y_1)\} \text{ then } X * Y \supseteq X_1 * Y_1.
\end{aligned}
$$

4. RELATIONS BETWEEN $\mathcal{K}$ AND $\mathcal{M}$.

In this section we shall make use of all concepts from section 2 concerning the structure $\mathcal{K}$ and shall extend all concepts used in section 3 into $\mathcal{H}$.

We extend the definition of $\omega(A) = |a^+ - a^-|$, for $A \in \mathcal{H}$ and of $\chi(A) = a^{-\sigma(A)}/a^{\sigma(A)} = \{a^-/a^+, \text{ if } \sigma(A) = +; \quad a^+/a^-, \text{ if } \sigma(A) = -\}$, for $A \in \mathcal{H} \setminus \mathcal{T}$. We extend the definitions of the sign functionals $\phi : \mathcal{H} \otimes \mathcal{H} \to \{+, -\}$, $\psi : (\mathcal{H} \setminus \mathcal{T}) \otimes (\mathcal{H} \setminus \mathcal{T}) \to \{+, -\}$ by setting:

$$
\begin{aligned}
\phi(A, B) &= \operatorname{sign}(\omega(A) - \omega(B)) = \{+,\ \omega(A) \ge \omega(B);\quad -,\ \text{otherwise}\},\ A, B \in \mathcal{H}, \\
\psi(A, B) &= \operatorname{sign}(\chi(A) - \chi(B)) = \{+,\ \chi(A) \ge \chi(B);\quad -,\ \text{otherwise}\},\ A, B \in \mathcal{H} \setminus \mathcal{T}.
\end{aligned}
$$

The $\mathcal{M}$-operations $+^-$ and $\times^-$ can be extended in $\mathcal{H}$ by formally substituting in (14), (15) the set IR by $\mathcal{H}$ and Z by $\mathcal{T}$. Setting $\alpha = \phi(A, B)$ and $\varepsilon = \psi(A, B)$ we thus define

$$A +^- B \;=\; [a^{-\alpha} + b^{\alpha}, a^{\alpha} + b^{-\alpha}], \quad \text{for } A, B \in \mathcal{H}, \tag{20}$$

$$A \times^- B \;=\; \begin{cases} [a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon} b^{\sigma(A)\varepsilon}], & \text{for } A, B \in \mathcal{H} \setminus \mathcal{T}, \\ [a^{-\delta} b^{-\delta}, a^{-\delta} b^{\delta}], \ \ \delta = \sigma(A), & \text{for } A \in \mathcal{H} \setminus \mathcal{T},\ B \in \mathcal{T}, \\ [a^{-\delta} b^{-\delta}, a^{\delta} b^{-\delta}], \ \ \delta = \sigma(B), & \text{for } A \in \mathcal{T},\ B \in \mathcal{H} \setminus \mathcal{T}, \\ [\ \max\{a^- b^+, a^+ b^-\},\ \min\{a^- b^-, a^+ b^+\}] & \text{for } A, B \in Z, \\ [\ \min\{a^- b^+, a^+ b^-\},\ \max\{a^- b^-, a^+ b^+\}], & \text{for } A, B \in \overline{Z}, \\ 0, \quad \text{for } A \in Z, B \in \overline{Z} \ \text{ or } \ A \in \overline{Z}, B \in Z. \end{cases} \tag{21}$$

The algebraic structure $\mathcal{KM} = (\mathcal{H}, +, \times, +^-, \times^-, \subseteq)$ defined by (8)–(11), (20), (21) is obviously a generalization of both $\mathcal{K}$ and $\mathcal{M}$. It involves all operations in $\mathcal{K}$ and the extensions of all operations in $\mathcal{M}$ like $A -^- B = A +^- (-B)$, $A/^- B = A \times^- (1/B)$ etc.

The following proposition accomplishes Proposition 1 and shows how any computation in $\mathcal{K}$ can be performed in $\mathcal{M}$. Below we make use of the notations $+^{++} = +^{--} = +^{+} = +$, $+^{+-} = +^{-+} = +^-$, $\times^{++} = \times^{--} = \times^{+} = \times$, $\times^{+-} = \times^{-+} = \times^-$ in accordance with the definitions of the "product" st for $s, t \in \{+, -\}$.

Proposition 2. For $A, B \in \mathcal{H}$, we have

$$(A + B)_p = (A)_p +^{\tau(A)\tau(B)} (B)_p; \qquad \tau(A + B) = \begin{cases} \tau(A), & \text{if } \omega(A) \geq \omega(B), \\ \tau(B), & \text{if } \omega(A) < \omega(B); \end{cases}$$

for $A, B \in \mathcal{H} \setminus \mathcal{T}$ we have

$$(A \times B)_p \;=\; (A)_p \times^{\tau(A)\tau(B)} (B)_p; \qquad \tau(A \times B) = \begin{cases} \tau(A), & \text{if } \chi(A) \leq \chi(B), \\ \tau(B), & \text{if } \chi(A) > \chi(B); \end{cases}$$

for $A, B \in \mathcal{H}$, the $\mathcal{K}$-relation $A \subseteq B$ is equivalent to one of the following $\mathcal{M}$-relations:

i) $A \subseteq B$, if $A, B \in IR$;
ii) $\overline{A} \supseteq \overline{B}$, if $A, B \in \overline{IR}$;
iii) $(B \subseteq \overline{A})$ or $(\overline{A} \subseteq B)$ or $(\overline{A} \leq B, b^- \in \overline{A})$ or $(\overline{A} \geq B, b^+ \in \overline{A})$, if $A \in \overline{I(R)}, B \in I(R)$.

Remark. The situation $A \in IR$, $B \in \overline{IR}$, $A \neq B$, contradicts to the assumption $A \subseteq B$ and hence is not possible.

Proposition 3. For $A, B \in \mathcal{H}$, we have

$$A +^- B = \{A + \overline{B}, \ \text{if } \omega(A) \geq \omega(B); \ \ \overline{A} + B, \ \text{if } \omega(A) < \omega(B)\}, \ A, B \in \mathcal{H};$$
$$A -^- B = \{A - \overline{B}, \ \text{if } \omega(A) \geq \omega(B); \ \ \overline{A} - B, \ \text{if } \omega(A) < \omega(B)\}, \ A, B \in \mathcal{H};$$

for $A, B \in \mathcal{H} \setminus \mathcal{T}$ we have

$$A \times^- B \;=\; \{A \times \overline{B}, \ \text{if } \chi(A) \geq \chi(B); \ \ \overline{A} \times B, \ \text{if } \chi(A) < \chi(B)\},$$
$$A/^- B \;=\; \{A/\overline{B}, \ \text{if } \chi(A) \geq \chi(B); \ \ \overline{A}/B, \ \text{if } \chi(A) < \chi(B)\}.$$

Proposition 3 shows that the (nonstandard) operations $+^-$, $\times^-$ (and $-^-$, $/^-$) in $\mathcal{KM}$ can be expressed by means of the (standard) operations addition, multiplication and conjugation using the operators ω and χ. Since conjugation is a composite operation we see that the operations $+^-$, $\times^-$ in $\mathcal{KM}$ can be expressed by means of the operations $+$ and $\times$ in each one of the situations $\omega(A) \geq \omega(B)$, $\omega(A) < \omega(B)$, resp. $\chi(A) \geq \chi(B)$, $\chi(A) < \chi(B)$. However, the nonstandard operations $+^-$, $\times^-$ are not compositions of the operations $+$ and $\times$. Using Proposition 3 we can "translate" any expression in $\mathcal{M}$ into a corresponding expression in $\mathcal{K}$.

5. NOTES ON APPLICATIONS

Proposition 2 shows that any interval arithmetic expression (statement) in $\mathcal{K}$ can be reformulated in $\mathcal{M}$, i.e. by separately computing the type τ of the intermediate results and their endpoints, computing thereby only with proper intervals. This shows that $\mathcal{K}$ and $\mathcal{M}$ can be equally well used in practical applications. We demonstrate below how a statement in $\mathcal{K}$ can be reformulated in $\mathcal{M}$ and vice versa.

Example 1. Let us find the $\mathcal{M}$-equivalent of the $\mathcal{K}$-proposition: "$(A + B) + C = A + (B + C)$ for $A, B, C \in \mathcal{H}$". If $\tau(A) = \tau(B) = \tau(C)$ using Proposition 2 we obtain $(A_p + B_p) + C_p = A_p + (B_p + C_p)$ which corresponds to the $\mathcal{M}$-assertion $(A + B) + C = A + (B + C)$ for $A, B, C \in IR$. Consider a situation when A, B, C are not of the same type, e.g. $\tau(A) = \tau(B) = -\tau(C)$. According to Proposition 2 this is equivalent to

$$(A_p + B_p) +^- C_p = \begin{cases} A_p + (B_p +^- C_p), & \text{if } \omega(B) \geq \omega(C), \\ A_p +^- (B_p +^- C_p), & \text{if } \omega(B) < \omega(C), \end{cases}$$

which is exactly the second equality in **M1** (see section 3). Considering all possible cases we arrive to the associative laws as formulated in **M1**.

Example 2. Let us transform the $\mathcal{K}$-assertion: "For $X, X_1, Y, Y_1 \in \mathcal{H}$, $X \subseteq X_1$ and $Y \subseteq Y_1 \Rightarrow X + Y \subseteq X_1 + Y_1$" into an $\mathcal{M}$-assertion using Proposition 2. Consider for instance the subcase $X, X_1 \in IR$, $Y, Y_1 \in \overline{IR}$; then our assertion reads: "For $X, X_1, Y_-, Y_{1-} \in IR$, $X \subseteq X_1$ and $Y_- \supseteq Y_{1-} \Rightarrow \{X +^- Y_- \subseteq X_1 +^- Y_{1-}$ if $\omega(X) \geq \omega(Y_-)$; $X +^- Y_- \supseteq X_1 +^- Y_{1-}$ if $\omega(X_1) \leq \omega(Y_{1-})\}$", taking into account that $\omega(X) \geq \omega(Y_-) \Rightarrow \omega(X_1) \geq \omega(Y_{1-})$ and $\omega(X_1) \leq \omega(Y_{1-}) \Rightarrow \omega(X) \leq \omega(Y_-)$. We can proceed in a similar way in the rest of the cases. Summarizing all subcases we arrive to the corresponding part of proposition **M2** (resp., **S2**) concerning the operations "$+$" and "$+^-$".

The next example shows that by means of Proposition 3 we can reformulate an $\mathcal{M}$-theorem into a corresponding $\mathcal{K}$-theorem.

Example 3. From the distributive laws in $\mathcal{M}$ using Proposition 3 we obtain the following general distributive relation in $\mathcal{K}$ (see [13] for more details): For $A, B, C, A+B \in \mathcal{H} \setminus \mathcal{T}$,

$$(A + B) \times C = \begin{cases} (A \times C) + (B \times C), & \text{if } \sigma(A) = \sigma(B) \ \ (= \sigma(A + B)), \\ (A \times C) + (B \times \overline{C}), & \text{if } \sigma(A) = -\sigma(B) = \sigma(A + B), \\ (A \times \overline{C}) + (B \times C), & \text{if } \sigma(A) = -\sigma(B) = -\sigma(A + B). \end{cases}$$

The above examples show that any result in $\mathcal{K}$ can be reformulated into a corresponding result in $\mathcal{M}$ and vice versa. Another direction of potential applications is discussed in [13], where the concept of "directed range" of a function is proposed and related to $\mathcal{K}$.

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