Stability properties of the Repeated Richardson Extrapolation combined with some explicit Runge-Kutta methods

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The implementation and the stability properties of the Repeated Richardson Extrapolation when it is applied together with Explicit Runge–Kutta Methods is studied in this paper.
The initial–value problem for first–order non–linear systems of ordinary differential equations (ODEs) in the time interval $[a, b]$ is considered. Let us assume that it is solved approximately by some one–step numerical method of order **p** on the equidistant grid–points. Assuming furthermore that the calculations at the points $t_1, t_2, \ldots, t_(n-1); t_k = t_{k-1} + h; h = \frac{b-a}{n}$ has been completed and that the calculations at point t_n have to be carried out. This means that the approximation $y_(n-1)$ is available and must be used to calculate the next approximation y_n. When the Repeated Richardson Extrapolation is to be used, this can be done by performing successively in four steps:

- *Step 1*: Compute an approximation $z_n^{[1]}$ of the solution of at the point $t = t_n$ by using the selected numerical method with a large stepsize h (i.e. one step is to be performed during this process).

- *Step 2*: Compute an approximation $z_n^{[2]}$ of the solution at the point $t = t_n$ by using the selected numerical method with a medium stepsize $h/2$ (i.e. two steps are to be performed during this process).

- *Step 3*: Compute an approximation $z_n^{[3]}$ of the solution at the point $t = t_n$ by using the selected numerical method with a small stepsize $h/4$. (i.e. four steps are to be performed during this process).

- *Step 4*: Compute an approximation y_n of the solution at the point $t = t_n$ by using the approximations $z_n^{[1]}$, $z_n^{[2]}$ and $z_n^{[3]}$.

The following theorem related to the accuracy of the Repeated Richardson Extrapolation holds and it is proven:

Theorem: *Consider the solution of the system of ODEs and assume that the underlying one-step numerical method is of order of accuracy* $\boldsymbol{p}$*. Then the order of accuracy of the Repeated Richardson Extrapolation is at least* $\boldsymbol{p+2}$ *when the right-hand-side* $f(t, y)$ *is* $\boldsymbol{p+2}$ *times continuously differentiable.*

This theorem shows that the order of accuracy can be increased significantly when the Repeated Richardson Extrapolation is used, but it is necessary to pay some price (to carry out seven steps instead of only one) for achieving high accuracy. It is demonstrated that the high accuracy of the Repeated Richardson Extrapolation is sometimes allowing us to increase the stepsize and, solving the problem with a sufficiently large stepsize.

Two more theorems were proven. The first of them shows that the stability function of the Repeated Richardson Extrapolation is expressed by the stability function of the underlined one-step method, but it is different. This means that the two numerical methods, the underlying one-step method and the Repeated Richardson Extrapolation, will in general have different stability properties. This result is a motivation for studying the stability properties of the Repeated Richardson Extrapolation. The second one was proved graphically and concerns the absolute stability regions of numerical methods that are combinations of explicit Runge-Kutta methods and the Repeated Richardson Extrapolation. It wsa proved that these stability regions are always larger than the absolute stability regions of the underlying explicit Runge-Kutta methods when the conditions $\mathbf{p} = \mathbf{m}$ and $\mathbf{m}$=1, 2, 3, 4 are satisfied.

The following book and papers were used as a base for the reported research:

1. G. Dahlquist: *A special stability problem for linear multistep methods*, BIT, **Vol. 3** (1963), pp. 27–43.

2. J. D. Lambert: *Numerical Methods for Ordinary Differential Equations: The Initial Values Problem*, Wiley, New York, 1991.

3. L. F. Richardson: *The Deferred Approach to the Limit, ISingle Lattice*, Philosophical Transactions of the Royal Society of London, Series A, **Vol. 226** (1927), pp. 299–349.

4. Z. Zlatev, K. Georgiev and I. Dimov: *Studying absolute stability properties of the Richardson Extrapolation combined with explicit Runge-Kutta methods*, Computers and Mathematics with Applications, **Vol 67, No. 12** (2014), pp.2294–2307.