# ENERGY SUBCRITICAL NLS ON $\mathbf{R}^N \times \mathbf{R}/2\pi\mathbf{Z}$

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## 1. Resume

The main object of the talk is to present some results concerning the nonlinear Schrödinger equation (NLS) on the product space $\mathbf{R}^n \times \mathbf{T}$, with $n \geq 1$ and $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ is the one-dimensional standard torus. More precisely we consider the following family of Cauchy problems:

$$\begin{cases} i\partial_t u + \Delta_{x,y} u + \lambda |u|^\alpha u = 0, & (t,x,y) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{T} \\ u(0,x,y) = f(x,y) \in H^1_{x,y}, \end{cases} \tag{1.1}$$

where

$$\Delta_{x,y} = \sum_{l=1}^{n} \partial^2_{x_l} + \partial^2_y,$$

$\lambda$ is a real number, the nonlinearity parameter $\alpha$ satisfies the assumption

$$\alpha < \alpha^*(n), \qquad \alpha^*(n) = \begin{cases} \frac{4}{n-1} & \text{if} \quad n \geq 2, \\ +\infty & \text{if} \quad n = 1. \end{cases}$$

and

$$H^1_{x,y} = (1 - \Delta_{x,y})^{-\frac{1}{2}} L^2_{x,y}, \text{ where } L^2_{x,y} = L^2(\mathbf{R}^n \times \mathbf{T}).$$

We underline that the power nonlinearity $\alpha^*(n)$ corresponds to the $H^1$-critical NLS in $\mathbf{R}^{n+1}$ as well as to the $H^{1/2}$-critical nonlinearity in $\mathbf{R}^n$.
In this regime we are able to obtain the following:

(1) for any initial datum $f \in H^1_{x,y}$, the problem (1.1) has a unique local solution

$$u(t,x,y) \in C((-T,T); H^1_{x,y}),$$

where $T = T(\|f\|_{H^1_{x,y}}) > 0$;

(2) there exists $\varepsilon_0 > 0$ such that the flow

$$B_{\varepsilon_0}(0) \cap H^1_{x,y} \ni f \to u(t,x,y) \in C((-T,T); H^1_{x,y}),$$

is Lipschitz continuous.

Moreover, by using a conservation laws argument, one can see that the local solutions above are global in time in the following cases:

- $\lambda < 0$ (i.e. (1.1) is defocusing);
- $\lambda \in \mathbf{R}$ and $0 < \alpha < 4/(n+1)$ (i.e. (1.1) is $L^2$-subcritical);

- $\lambda > 0$, $4/(n+1) \le \lambda < 4/(n-1)$ and initial data $f$ small enough in $H^1_{x,y}$ (i.e. (1.1) is energy-subcritical, $L^2$-supercritical, focusing and the initial data are small).

We recall that the key ingredient used in the literature in order to get (at least) local well-posedeness for (1.1) is a suitable version of Strichartz estimates of the type

$$\|e^{it\Delta_{x,y}} f\|_{L^q_t L^r_x L^2_y} \le C\|f\|_{L^2_{x,y}},$$

where $(q, r)$ are such that

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad q \ge 2, \quad (n, q) \neq (2, 2). \tag{1.2}$$

On the other hand it is well-known (by classical Euclidean theory) that the best nonlinearity that can be reached with this technique is the $L^2$-critical nonlinearitiy in $\mathbf{R}^n$, i.e. $0 < \alpha < 4/n$. Our main contribution is the analysis of (1.1) in the case $4/n \le \alpha < \alpha^*(n)$ (i.e. nonlinearites which are $L^2$- supercritical and $H^{1/2}$-subcritical in $\mathbf{R}^n$). The main difficulty in the transposition of the analysis above in this larger regime of nonlinearities is that, in analogy with the theory of $L^2$-supercritical NLS in $\mathbf{R}^n$, it seems to be necessary to work with Strichartz estimates involving derivatives w.r.t. $x$ variable.
To overcome this obstacle we exploit a class of inhomogeneous Strichartz estimates with respect the $x$ variable true in a range of Lebesgue exponents larger than the one given in the usual homogeneous estimates context. This allow us to perform a fixed point argument by considering in our partially periodic setting the same numerology involved in the analysis of NLS posed in the euclidean space $\mathbf{R}^n$ via admissible Strichartz norms $L^q_t L^r_x$, with $(q, r)$ as in (1.2).