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# SUPERCRITICAL MARKOV BRANCHING PROCESS WITH RANDOM INITIAL CONDITION

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**Abstract**

The supercritical Markov branching process $X(t)$ starting with one particle as initial condition has an extinction probability $0 \leq q < 1$. We study the influence of the random initial number of particles $X_0$ on the extinction probability, the survival probability by the time $t > 0$ and on the general behaviour of the number of particles $Y(t) = \sum_{i=1}^{X_0} X_i(t)$, where $X_i(t)$, $i = 1, 2, \dots$ are independent copies of $X(t)$. We consider the cases when the nonnegative integer-valued random variable $X_0$ is geometrically shifted (or non-shifted), Negative-Binomial or Pólya-Aeppli distributed. The branching mechanism in consideration is defined by a quadratic function. We prove that in these cases the random number of particles $Y(t)$ alive at time $t > 0$ follows the same probability law as the initial condition, with different parameters depending on time $t$.

**Key words:** branching process, birth-death process, cosmic rays cascades, Negative-Binomial distribution, Pólya-Aeppli distribution, extinction probability

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**1. Introduction.** Branching processes and especially birth-death ones are the fundamental models describing the cosmic rays cascades and nuclear fission chain as they were introduced by HARRIS [3]. Later on, the branching processes have been used to derive various distributions of the population with multiplication and the case of particle injection by an external source (immigration of entities), especially in neutron cascades by DOROGOV and CHISTYAKOV [2], and

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Pazsit and Pal [6]. The topic was reintroduced recently by Tchorbadjieff [10] with simulations of cosmic rays cascades. Two different continuous time models were demonstrated: Markov chain and age dependent branching process with several types of particles.

We introduce here the model of Branching Particle System with an additional randomness, represented by the initial condition. Our main interest is the invariance property of the initial distribution by the reproduction law of particles. If the initial number of particles given by the nonnegative integer-valued random variable $X_0$ is infinitely divisible on the "spatial" parameter $R$, considered as the radius of the particles flux, then for each branching process, the number of particles alive at the time $t > 0$ will be also infinitely divisible on this parameter. It is due to the property of independence of evolution of particles. It is well known that the nonnegative integer-valued and infinitely divisible distributions are compound Poisson with $P(X_0 = 0) > 0$, see [9]. It means that the branching reaction will not start with positive probability and the initial condition will strongly influence the extinction probability. But, there are many interesting distributions with $P(X_0 \geq 1) = 1$, such as shifted geometric distribution. In order to find these distributions in an explicit form, we chose the probability generating functions (p.g.f.) of the initial condition and branching mechanism to be expressed by the linear-fractional function.

We consider the cases when the random variable $X_0$ is geometrically shifted (or non-shifted), Negative-Binomial or Pólya-Aeppli distributed. The branching mechanism in consideration is defined by a quadratic function. We prove that in these cases the random number of particles $Y(t)$ alive at time $t > 0$ follows the same probability law as the integer-valued random variable $X_0$, with different parameters depending on time $t$. The selected distributions are interconnected because the sum of random number of geometrically distributed random variables takes part in the compound Poisson processes and subordinated Lévy processes. The Pólya-Aeppli process can be constructed by subordination (random time change) of the Negative-Binomial process to the Poisson process, see Mayster [4,5].

**2. Preliminaries.** Let $X(t)$ be a Markov branching process with branching mechanism of reproduction defined by the quadratic function $h(s) = ps^2 + 1 - p, 0 < p \leq 1$, starting with one particle as initial condition. If a given particle is alive at a certain time, its additional life length is a random variable exponentially distributed with parameter $K > 0$, see [1]. Then the infinitesimal p.g.f. $f(s) = K(ps^2 - s + 1 - p)$. It is known, that the p.g.f. of branching process $X(t)$ defined by $F(t, s) = \sum_{k=0}^{\infty} s^k P(X(t) = k|X(0) = 1)$ satisfies the **backward** Kolmogorov equation: $\frac{\partial}{\partial t}(F(t, s)) = f(F(t, s))$ and **forward** Kolmogorov equation: $\dfrac{\partial}{\partial t}\left(F(t,s)\right)=f(s)\dfrac{\partial}{\partial s}\left(F(t,s)\right)$, with the initial condition: $F(0,s)=s$. We denote by $m=K(2p-1)$ the mean of the infinitesimal offspring number, i.e. $\dfrac{df}{ds}(1)=m$. Then the mathematical expectation is $E[(X(t)]=e^{mt}$. A branching process $X(t)$ is classified as supercritical, critical or subcritical following respectively the inequalities: $m>0, m=0, m<0$. In this short communication we consider only the supercritical case, $m>0$.

The extinction probability, traditionally denoted by $q$, is the smallest nonnegative solution of the equation $h(s)=s$. For the given reproduction function $h(s)$ the extinction probability is given by $q=\dfrac{1-p}{p}$. The process $X(t)$ is supercritical for $\dfrac{1}{2}<p\leq 1$. It is convenient in this case to represent the infinitesimal p.g.f. $f(s)$ in the form: $f(s)=Kp(s-1)(s-q)$. When $p=1$ we have the Yule process with extinction probability $q=0$. These non-linear Kolmogorov equations have an explicit solution of the form:

$$(1)\qquad F(t,s)=\frac{s(q-e^{-mt})-q+qe^{-mt}}{s(1-e^{-mt})-1+qe^{-mt}},\quad q=\frac{1}{p}-1,\quad m=K(2p-1)>0.$$

Extinction probability $q=\lim\limits_{t\mapsto\infty}F(t,0)$ is a fixed point for $F(t,s)$: $F(t,q)=q$, see the classical books [1,3,8]. The extinction probability of $X(t)$ by time $t>0$ denoted by $A:=A(t)$ is given by:

$$(2)\qquad A:=F(t,0)=\frac{q(1-e^{-mt})}{1-qe^{-mt}}.$$

The p.g.f. $F(t,s)$ can be presented in the form:

$$(3)\qquad F(t,s)=A+(1-A)\frac{(1-\alpha)s}{1-\alpha s},\quad \alpha=\frac{1-e^{-mt}}{1-qe^{-mt}}.$$

The representation (3) signifies by itself that for any fixed $t>0$ the random variable $X(t)$ follows a zero-modified geometric distribution as follows:

$$P(X(t)=0)=A,\quad P(X(t)=1)=(1-A)(1-\alpha),$$
$$P(X(t)=k)=(1-A)(1-\alpha)\alpha^{k-1},\quad k=2,3,\ldots.$$

We have

$$1-A=\frac{1-q}{1-qe^{-mt}},\quad 1-\alpha=\frac{(1-q)e^{-mt}}{1-qe^{-mt}}.$$

Naturally, in this geometric sequence the probability of "success" $(1-\alpha)$ is proportional to the probability of non-extinction by the time $t>0$:

$$(1-\alpha)E[X(t)]=1-A.$$

We see that the individual probability of surviving $(1-\alpha)$ in the supercritical process tends to zero, but the population survives with positive probability $(1-q)$. In the subcritical case the situation is just the opposite.

As shown in SAGITOV [7], the linear-fractional p.g.f. plays an important role in the theory of branching processes and description of their properties.

**3. Markov branching process starting with random number of particles $\boldsymbol{X_0}$ located at one point.** Let $X_i(t), i = 1, 2, \dots$ be independent copies of the branching process $X(t)$ starting with one particle. Denote by $Y(t)$ the process starting with a random number of particles $X_0$ , i.e.

$$Y(t) = \sum_{i=1}^{X_0} X_i(t). \tag{4}$$

Suppose the p.g.f. of the initial number of particles $X_0$ is denoted by $U_0(s) = E[s^{X_0}]$. Then the conditional p.g.f. $E[s^{X(t)}|X(0) = X_0] = U_0(F(t, s))$. The p.g.f. of the process $Y(t)$, defined by

$$U(t, s) := E[s^{Y(t)}] = U_0(F(t, s))$$

satisfies only the following **forward** Kolmogorov equation: $\frac{\partial}{\partial t}(U(t,s)) = Kp(s-1)(s-q)\frac{\partial}{\partial s}(U(t,s))$ with the initial condition: $U(0, s) = U_0(s)$. The extinction probability of the process $Y(t)$ is defined by

$$Q := \lim_{t\to\infty} P(Y(t) = 0) = U_0(q).$$

The mathematical expectation is $E[Y(t)] = E[X_0]E[X(t)]$.

**Theorem 3.1.** *Let the random variable $X_0$ have a* ***shifted*** *geometric distribution of the form:*

$$P(X_0 = k) = (1-\varrho)\varrho^{k-1}, \quad 0 < \varrho < 1, \quad k = 1, 2, \dots.$$

*Then the p.g.f. $U(t, s) = E[s^{Y(t)}]$ of (4) can be presented in the form:*

$$U(t, s) = B + (1-B)\frac{(1-\beta)s}{1-\beta s},$$

*where*

$$B = \frac{Q(1-e^{-mt})}{1-Qe^{-mt}}, \quad \beta = \frac{q - Qe^{-mt}}{q(1-Qe^{-mt})}, \quad Q = \frac{(1-\varrho)q}{1-\varrho q}.$$

**Proof.** The p.g.f. $U_0(s) = \frac{(1-\varrho)s}{1-\varrho s}$ and $E[Y(t)] = \frac{e^{mt}}{1-\varrho}$. Using (1) one obtains the following expression:

$$1 - \varrho F(t, s) = \frac{e^{-mt}(q-s)(1-\varrho) + (s-1)(1-\varrho q)}{e^{-mt}(q-s) + s - 1}. \tag{5}$$

The explicit presentation for $U(t,s) = \dfrac{(1-\varrho)F(t,s)}{1-\varrho F(t,s)}$ is given by:

$$U(t,s) = \frac{(1-\varrho)[s(q-e^{-mt}) - q(1-e^{-mt})]}{s[1-\varrho q-(1-\varrho)e^{-mt}] - (1-\varrho q) + q(1-\varrho)e^{-mt}}. \tag{6}$$

The probability of extinction by the time $t < \infty$ for the process $Y(t)$:

$$B := U(t,0) = \frac{q(1-\varrho)(1-e^{-mt})}{1-\varrho q - q(1-\varrho)e^{-mt}}.$$

□

**Theorem 3.2.** *Suppose the random variable $\overline{X_0}$ follows the* ***non-shifted*** *geometric distribution with p.g.f. $\overline{U_0}(s) = \dfrac{(1-\varrho)}{1-\varrho s}$. Then for any fixed $t > 0$ the random variable*

$$Z(t) = \sum_{i=1}^{\overline{X_0}} X_i(t)$$

*has a zero-modified geometric distribution with p.g.f. $\overline{U}(t,s) = E[s^{Z(t)}]$ represented as follows:*

$$\overline{U}(t,s) = C + (1-C)\frac{(1-\beta)s}{1-\beta s}, \quad \beta = \frac{1-\varrho q-(1-\varrho)e^{-mt}}{1-\varrho q - q(1-\varrho)e^{-mt}},$$

*where*

$$C = \frac{\overline{Q}(1-qe^{-mt})}{1-\overline{Q}qe^{-mt}}, \quad \overline{Q} = \frac{1-\varrho}{1-\varrho q}.$$

**Proof.** Using (5) and comparing with (6) we present $\overline{U}(t,s)$ in the form convenient to take derivatives by the variable $s$:

$$\overline{U}(t,s) = \frac{(1-\varrho)[s(1-e^{-mt}) + qe^{-mt} - 1]}{s[1-\varrho q-(1-\varrho)e^{-mt}] - (1-\varrho q) + (1-\varrho)qe^{-mt}]}.$$

The value of $C = \overline{U}(t,0)$ and $1-C$ are obvious:

$$C = \frac{(1-\varrho)(1-qe^{-mt})}{1-\varrho q-(1-\varrho)qe^{-mt}}, \quad 1-C = \frac{\varrho(1-q)}{1-\varrho q-(1-\varrho)qe^{-mt}}.$$

The geometric sequence created in this theorem has the same probability of "success" $(1-\beta)$ as in the previous theorem. In this case, the branching reaction does not start with positive probability $P(X_0 = 0) = 1-\varrho > 0$. The extinction probability $\overline{Q}$ can be greater, equal or less than $q$ following $\varrho$. If $p = \varrho$, then $\overline{Q} = q$, if $\varrho < p < 1$, then $\overline{Q} > q$, if $\varrho > p > \dfrac{1}{2}$, then $\overline{Q} < q$. The following relations take place always: $CA = B$, $Q = \overline{Q}q$ , $Q < \overline{Q}$ and $Q < q$, where $A$ is given by (2). □

**4. Branching processes starting with random number of particles dispersed over the radius $R$ of the flux of particles.** Consider an ordered sequence $\{x_1, x_2, \ldots\}, 0 < x_1 < x_2 < \cdots$, on the real half-line $(0, \infty)$. Let the length of intervals $(0, x_1), \ldots, (x_i, x_{i+1}), i = 1, 2, \ldots$, be random independent identically distributed variables. When they follow exponential distribution with parameter $\theta > 0$ the number $N(R)$ of points $\{x_1, x_2, \ldots\}$ located in the interval $(0, R)$ is described by the Poisson distribution:

$$P(N(R) = k) = e^{-\theta R} \frac{(\theta R)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

**Theorem 4.1.** *Let the random variable $X_0 = N(R)$ with p.g.f. $U_0(s) = e^{-\theta R(1-s)}$. Let us define the process*

$$Z(t) = \sum_{i=1}^{N(R)} X_i(t).$$

*Then for any fixed time parameter $t$ the random variable $Z(t)$ follows the Pólya-Aeppli probability distribution with p.g.f. $V(R, t, s) = E[s^{Z(t)}]$ given by:*

$$V(R, t, s) = \exp\left\{-\Theta R \left(\frac{1-s}{1-\Upsilon s}\right)\right\},$$

*where* $\Upsilon = \dfrac{1 - e^{-mt}}{1 - q e^{-mt}}$ *and* $\Theta = \dfrac{\theta(1-q)}{1 - q e^{-mt}}$.

**Proof.** We remember that $V(R, t, s) = U_0(F(t, s))$ and from (1) it follows:

$$1 - F(t, s) = \frac{(1-s)(q-1)}{e^{-mt}(q-s) + s - 1}.$$

Obviously:

$$V(R, t, s) = e^{-\theta R(1 - F(t,s))} = \exp\left\{-\frac{\theta R(1-q)}{1 - q e^{-mt}} \left(\frac{1-s}{1-\Upsilon s}\right)\right\}.$$

□

The Pólya-Aeppli probability distribution is the infinitely divisible distribution with p.g.f. of the form: $U_0(s) = \exp\left\{-\theta R \dfrac{1-s}{1-\varrho s}\right\}$ which reduces to homogeneous Poisson distribution when $\varrho = 0$. Thus the proofs of following two theorems follow directly from this and the already obtained results.

**Theorem 4.2.** *Let the random variable $X_0$ be defined by the Pólya-Aeppli probability distribution:*

$$P(X_0 = k) = e^{-\theta R} \sum_{j=1}^{k} \binom{k-1}{j-1} \varrho^{k-j} \frac{(\theta R)^j (1-\varrho)^j}{j!}, \quad 0 < \varrho < 1, \quad k = 1, 2, \ldots.$$

*and*

$$P(X_0 = 0) = e^{-\theta R}.$$

*Let us define the process*

$$Z(t) = \sum_{i=1}^{X_0} X_i(t).$$

*Then the p.g.f.* $V(R,t,s) = E[s^{Z(t)}]$ *is given by:*

$$V(R,t,s) = \exp\left\{-\frac{\Theta R(1-s)}{1-\Upsilon s}\right\},$$

*where*

$$\Theta = \frac{\theta(1-q)}{(1-\varrho q)-(1-\varrho)qe^{-mt}}, \quad \Upsilon = \frac{(1-\varrho q)-(1-\varrho)e^{-mt}}{(1-\varrho q)-(1-\varrho)qe^{-mt}}.$$

The Negative-Binomial distribution is the infinitely divisible distribution with p.g.f.

$$U_0(s) = \left(\frac{1-\varrho}{1-\varrho s}\right)^R,$$

which for $R = 1$ is given by the non-shifted geometric distribution.

**Theorem 4.3.** *Let the random variable* $\overline{X_0}$ *be defined by the Negative-Binomial distribution:*

$$P(X_0 = k) = (1-\varrho)^R \varrho^k \frac{(k+R-1)!}{k!(R-1)!}, \quad 0 < \varrho < 1, \quad k = 0,1,2,\dots.$$

*Let us define the process*

$$Z(t) = \sum_{i=1}^{\overline{X_0}} X_i(t).$$

*Then the p.g.f.* $V(R,t,s) = E[s^{Z(t)}]$ *can be presented in the form:*

$$V(R,t,s) = \left(C + (1-C)\frac{(1-\beta)s}{1-\beta s}\right)^R,$$

*where the parameters* $C$ *and* $\beta$ *are the same as for the non-shifted geometric distribution considered in Theorem* 3.2.

**5. Conclusions.** Modelling and controlling supercritical processes is a very important task for many natural processes and engineering systems, especially for nuclear fission. There, the initial conditions are very important. With this work we considered some particular cases for Markov branching processes with initial discrete distribution. However, for completeness of proposed results the model will be extended to critical, subcritical cases and limit theorems.

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