Lithuanian Mathematical Journal

Wright function in the solution to the Kolmogorov equation of the Markov branching process with geometric reproduction of particles*

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Received February 25, 2022; revised March 18, 2023

Abstract. The topic of this work is the supercritical geometric reproduction of particles in the model of a Markov branching process. The solution to the Kolmogorov equation is expressed by the Wright function. The series expansion of this representation is obtained by the Lagrange inversion method. The asymptotic behavior is described by using two different equivalent forms for the Laplace transform. They include the computation of the limit distribution and its moments. The exact formula for the asymptotic density is written in terms of the reduced Wright function. In particular, when the ultimate extinction probability $q = 1/2$, the density of the limit random variable is given by the incomplete gamma function.

MSC: 60J80,11B83, 33C20, 33E20

Keywords: asymptotic behavior, geometric reproduction of particles, incomplete gamma function, Lagrange inversion method, supercritical MBP, Wright function

1 Introduction

The special functions provide a family of power series probability distributions [2, 9], especially in the representation of stable distributions [22]. Many particular cases of the Wright function ${}_1\Psi_1(\alpha, a; \beta, b; z)$ arising in probability theory are considered in [17]. In the model of the Markov branching process (MBP) with geometric reproduction of particles, the Lambert-W and Wright functions are part of solutions to the Kolmogorov equations [20]. Another special function, the Fox-H function defines the limiting behavior of MBP with reproduction given by the Siguya distribution [12].

Usually, the general special functions take part in fractional calculus [11]. Initially, Wright [24] defined the function $\phi(\beta, b; z)$ only for $\beta > 0$ and then extended its definition for $\beta > -1$ in [25]. The Wright function is classified into the first and second kinds when $\beta > 0$ and $\beta > -1$, respectively. The Wright function of the second kind is considered in the survey paper [14]. It is noted there that the first-kind Wright function is of exponential order, but that of the second kind is not, and, naturally, they have different asymptotic behaviors. The analytical properties and applications of the Wright function are developed in [5].

* This research has been partially supported by the Bulgarian National Science Fund, grant No. KP-06-H22/3.

Branching processes describe the systems of particles with the phenomena of extinction and multiplication [1, 6, 18]. Let $X(t)$, $t > 0$, be a time-homogeneous MBP starting with one particle as the initial condition. The possessed Markov property is due to the assumption of the exponentially distributed lifetime of particles with constant parameter $K > 0$. Thus, for a unit number of particles, $X(0) = 1$, the time interval to the next splitting time moment is exponentially distributed with density $K\mathrm{e}^{-Kt}$. Then, at any time $t > 0$ when the number of particles in the system is $X(t) = n$, $n = 2, 3, \dots$, the time interval to the next splitting time moment is exponentially distributed with density $nK\mathrm{e}^{-nKt}$ as the minimum of n independent exponentially distributed random variables.

Another important part in addition to the time inflation of the MBP is the reproduction law. To create a generalized model, any known probabilistic distribution can be assumed as a reproduction model. In this text the offspring number is defined by a geometrically distributed integer-valued random variable η. The main parameter of the MBP is the mean $m = \mathbf{E}[\eta]$. In this parameterization the probability mass function of the reproduction is given by

$$\mathbf{P}(\eta = k) = \frac{m^k}{(1+m)^{k+1}}, \quad k = 0, 1, \dots, \ m > 0. \tag{1.1}$$

A possible practical candidate for implementation is statistical physics, where under appropriate conditions, the geometric distribution is also associated with the Bose–Einstein distribution [7, 19].

The model of the time-homogeneous Markov branching process with geometric reproduction of particles was introduced in [20]. The obtained solutions were for the probability generating function (p.g.f.) $F(t, s) := \mathbf{E}[s^{X(t)}]$ in critical and subcritical processes. In both cases the results consist of special functions. In the critical case the p.g.f. $F(t, s)$ is defined by the composition of Lambert-W and linear-fractional functions. The probability mass function (p.m.f.) of $X(t)$, $t > 0$, is expressed by the values of the Lambert-W function at the point $x = \mathrm{e}^{Kt+1}$. The continuity of the p.g.f. $F(t, s)$ in the neighborhood of the point $s = 1$ is studied in [21].

The p.g.f. $F(t, s)$ in the subcritical case is expressed as a composition of Gauss hypergeometric ${}_2F_1(a, b; c; z)$ [9] and Wright ${}_1\Psi_1(\alpha, a; \beta, b; z)$ functions [16, 17]. The conditional limit distribution is defined in explicit form. It is a new unimodal integer-valued distribution supported by $\{1, 2, \dots\}$. Its index of dispersion depends on the solution of a transcendental equation.

The supercritical MBP has the remarkable behavior: the mathematical expectation $\mathbf{E}[X(t)]$ increases exponentially to infinity, but the ultimate extinction probability is still positive and less than one,

$$q := \lim_{t\to\infty} \mathbf{P}\big(X(t) = 0\big) = \frac{1}{m}, \quad 0 < q < 1, \ m > 1. \tag{1.2}$$

The normalized number of particles $Z(t) = X(t)/\mathbf{E}[X(t)]$, being a martingale, converges to a random variable $W \geqslant 0$ [1, 6, 18]. In this work, we obtain an explicit form of its Laplace transform $\varphi(\lambda)$ and of the absolutely continuous part $w(x)$ of its probability distribution.

Section 2 is devoted to the solution to the backward Kolmogorov equation. The probability mass function $\mathbf{P}(X(t) = n)$, $n = 0, 1, \dots$, is defined precisely by the Fáa Di Bruno formula [10] applied to the implicit solution. The explicit solution is defined by applying the Lagrange inversion method in Section 3. Then in Section 4, we deliver the Laplace transform of the limiting random variable (r.v.) $W \geqslant 0$. It is expressed by the Wright function showing clearly that $\mathbf{P}(W = 0) = q$. The absolutely continuous part $w(x)$ of its probability distribution is given by the reduced Wright function $\phi(\beta, b; z)$ of the second kind [5, 14]. Finally, in Section 5, we apply the second method to define the Laplace transform $\varphi(\lambda)$ in the form of power series over the λ^n. The coefficients in front of λ^n give exactly the moments of the limiting random variable $W \geqslant 0$. Section 6 is devoted to the applications and explicit solutions for the particular cases $q = 1/3, 1/2, 2/3$. When $q = 1/2$, the density function $w(x)$ can be computed applying the incomplete gamma function or, equivalently, by the error function $\mathrm{erfc}(x)$.

The main convenience of the obtained results based on special functions is the computation simplicity using numerical evaluation [13] and mathematical software tools. The Lagrange inversion method makes the usage of enumerative combinatorics methods indispensable in this study. The survey of the Lagrange inversion formula with applications to combinatorial and formal power series identities is given in [4] and [15].

2 Supercritical geometric branching

The probability generating function of the reproduction law (1.1) is a linear-fractional function,

$$h(s) = \frac{1}{1+m-ms}, \qquad h'(s) = \frac{m}{(1+m-ms)^2}, \qquad h''(s) = \frac{2m^2}{(1+m-ms)^3}.$$

The equation $h(s) = s$ has two solutions, $s_1 = 1/m$ and $s_2 = 1$, where $m = h'(1)$. The value $s_1 = 1/m$ is a fixed point for the p.g.f. $h(s)$ and its first derivative $h'(s)$. The branching mechanism is classified as subcritical if $0 < m < 1$, critical if $m = 1$, and supercritical if $m > 1$.

This classification is in accordance with the notion of extinction probability. The subcritical and critical MBPs extinct certainly. In the supercritical case the ultimate extinction probability is $0 < q < 1$ (1.2). In this case the p.g.f. $h(s)$ with $q = 1/m$ can be rewritten as

$$h(s) = \frac{q}{1+q-s}, \qquad \mathbf{P}(\eta = k) = \frac{q}{(1+q)^{k+1}}, \quad k = 0, 1, \dots .$$

The infinitesimal generating function present in the Kolmogorov equations

$$f(s) = K(h(s) - s) = \frac{K(1-s)(q-s)}{1+q-s}$$

has the following derivatives:

$$f'(s) = \frac{K(-s^2 + 2s(1+q) + q - (1+q)^2)}{(1+q-s)^2}, \quad f^{(n)}(s) = \frac{Kqn!}{(1+q-s)^{n+1}}, \quad s \neq 1+q, \ n = 2, 3, \dots .$$

In particular, for the first derivative, we have the following inequalities:

$$f'(0) = \frac{-K(1+q+q^2)}{(1+q)^2} < 0, \qquad f'(q) = -K(1-q) < 0, \qquad f'(1) = \frac{K(1-q)}{q} > 0. \tag{2.1}$$

The p.g.f. of the branching process $X(t)$, $t > 0$, defined by

$$F(t,s) := \sum_{k=0}^{\infty} s^k \mathbf{P}\big(X(t) = k \bigm| X(0) = 1\big) \tag{2.2}$$

yields the nonlinear backward Kolmogorov equation with initial condition

$$\frac{\partial}{\partial t}\big(F(t,s)\big) = f\big(F(t,s)\big), \qquad F(0,s) = s. \tag{2.3}$$

This equation is solved by the method of separate variables in the form

$$\frac{\mathrm{d}(F(t,s))}{f(F(t,s))} = \mathrm{d}t, \qquad F(0,s) = s. \tag{2.4}$$

We remark the following decomposition using (2.1):

$$\frac{f'(q)}{f(s)} = \frac{q}{1-s} + \frac{1}{s-q}, \quad f'(q) = -K(1-q) < 0, \quad s \neq q, \ s \neq 1.$$

Then the implicit solution to Eqs. (2.4) and (2.3) is given by

$$\frac{F(t,s)-q}{(1-F(t,s))^q} = \frac{\mathrm{e}^{-K(1-q)t}(s-q)}{(1-s)^q}.$$

The explicit solution is written using the function

$$D(s) = \frac{s-q}{(1-s)^q}, \quad s<1, \quad D(q)=0, \quad D(0)=-q, \quad 0<q<1, \tag{2.5}$$

and its composite inverse function $D^{-1}(x)$. To define the inverse function, the most convenient monotony interval must be selected. The first derivative is calculated as

$$D'(s) = \frac{(1-q)(1+q-s)}{(1-s)^{q+1}} > 0, \qquad D'(q) = \frac{1}{(1-q)^q} > 0, \qquad D'(0) = (1-q)(1+q) > 0.$$

Moreover,

$$\frac{D'(s)}{D(s)} = \frac{f'(q)}{f(s)}, \qquad f'(q) = -K(1-q).$$

The second derivative is given by

$$D''(s) = \frac{q(1-q)(2+q-s)}{(1-s)^{q+2}} > 0.$$

The function $D(s)$ is increasing and convex in the domain $s<1$. The function $D^{-1}(x)$ is increasing and concave in $-\infty < x < \infty$. We obtain the explicit solution for (2.2) as

$$D\big(F(t,s)\big) = \mathrm{e}^{f'(q)t} D(s), \qquad F(t,s) = D^{-1}\big(\mathrm{e}^{f'(q)t} D(s)\big), \qquad f'(q) = -K(1-q) < 0. \tag{2.6}$$

The point $s=q$, $0<q<1$, is fixed for the p.g.f. $F(t,s)$, $|s|<1$, because

$$F(t,q) = D^{-1}\big(\mathrm{e}^{f'(q)t} D(q)\big) = D^{-1}(0) = q.$$

The extinction probability at time $t>0$ is denoted by $\mathbf{P}(X(t)=0) = Q(t)$. It is derived directly from (2.6) in terms of the inverse function:

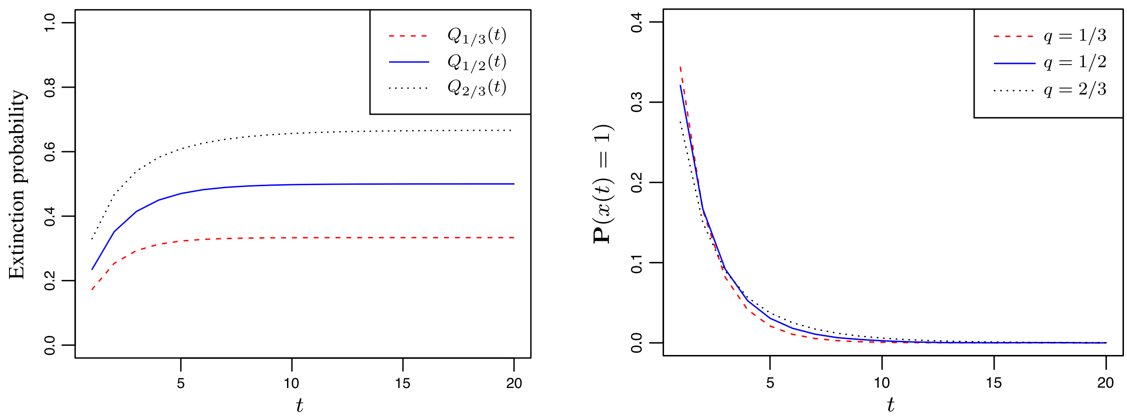
$$Q(t) = F(t,0) = D^{-1}\big(-q\mathrm{e}^{f'(q)t}\big) = D^{-1}\big(-q\mathrm{e}^{-K(1-q)t}\big), \qquad D(0) = -q. \tag{2.7}$$

The graph representation in Fig. 1(a) shows the quick convergence of $Q(t)$ to $0<q<1$ as $t\to\infty$.

The probability mass function is given by the derivatives of the p.g.f. at zero:

$$\mathbf{P}\big(X(t)=k\big) = \frac{1}{k!}\frac{\partial^k (F(t,0)}{\partial s^k}, \quad k=1,2,3,\ldots .$$

It can be obtained from (2.5) and (2.6) by the consecutive derivatives of the composite function $D(F(t,s))$. Some results are shown in Fig. 1(b).



(a) Extinction probabilities (b) Probabilities for $\mathbf{P}(X(t)=1)$

Figure 1. Plots of the p.m.f. of the supercritical geometric branching process for $q = 1/3, 1/2, 2/3$.

For further use, we introduce the falling and rising factorials:

$$[x]_{n\downarrow} = x(x-1)\dots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}, \tag{2.8}$$

$$[x]_{n\uparrow} = x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{2.9}$$

Now, using (2.9), we can write the derivatives of the function $D(s)$ (2.5) in the following form:

$$D^{(k)}(s) = \frac{(q+k-s)(1-q)[q]_{(k-1)\uparrow}}{(1-s)^{q+k}}, \quad \frac{D^{(k)}(s)}{D'(s)} = \frac{(q+k-s)[q]_{(k-1)\uparrow}}{(q+1-s)(1-s)^{k-1}}, \quad s<1. \tag{2.10}$$

The precise values at the points $s = 0$ and $s = q$ are

$$D^{(k)}(0) = (1-q)[q]_{(k-1)\uparrow}(q+k), \qquad D^{(k)}(q) = \frac{k[q]_{(k-1)\uparrow}}{(1-q)^{q+k-1}}.$$

Further, using the Fáa di Bruno formulas [10] and partial Bell polynomials $B_{k,j}$ [23] allows us to express the kth derivative of the composite function as follows:

$$\frac{\partial^k}{\partial s^k} D\bigl(F(t,s)\bigr) = \sum_{j=1}^{k} D^{(j)}\bigl(F(t,s)\bigr) B_{k,j}(F_\bullet), \qquad B_{k,1}(F_\bullet) = F_k, \qquad B_{k,k}(F_\bullet) = (F_1)^k,$$

where the sequence $F_\bullet = (F_1, F_2, \dots)$ is given by the derivatives $F_k = \partial^k F(t,s)/\partial s^k$. This way, we obtain the following recurrent relation for derivatives of the p.g.f. at zero and respectively for the p.m.f.:

$$F_s^{(k)}(t,0) = \frac{\mathrm{e}^{-K(1-q)t}(q+k)(1-Q(t))^{q+1}[q]_{(k-1)\uparrow}}{1+q-Q(t)} - \sum_{j=2}^{k} \frac{B_{k,j}(F_\bullet)(q+j-Q(t))[q]_{(j-1)\uparrow}}{(q+1-Q(t))(1-Q(t))^{j-1}}. \tag{2.11}$$

For example, replacing $k=1$ and $k=2$ in the previous relation (2.11), we calculate

$$F'_s(t,0) = \frac{\mathrm{e}^{-K(1-q)t}(q+1)(1-Q(t))^{q+1}}{1+q-Q(t)}, \tag{2.12}$$

$$F''_s(t,0) = \frac{\mathrm{e}^{-K(1-q)t}q(q+2)(1-Q(t))^{q+1}}{1+q-Q(t)} - \frac{(F'_s(t,0))^2(q+2-Q(t))q}{(q+1-Q(t))(1-Q(t))},$$

and so on for $k=3,4,\dots$.

3 Lagrange inversion method

The function $D(s)$ in (2.5) is an analytic function in the interval $s<1$. All its derivatives (2.10) are strictly positive there. The inverse function $D^{-1}(x)$ defines the solution to the backward Kolmogorov equation and all characteristics of $X(t), t>0$, especially the asymptotic behavior. It is derived as a solution to the following equation:

$$D(s)=x, \quad s<1, \quad D^{-1}(x)=s, \quad -\infty<x<\infty. \tag{3.1}$$

The Taylor series expansion of this inverse function is given by the Lagrange inversion method [4, 15]. The theorem of Lagrange states that the series expansion has a nonzero radius of convergence, that is, $D^{-1}(x)$ represents an analytic function of x in a neighborhood of the point $x=D(s)$. For computational convenience, we introduce the new variable $z=s-q$ and consider the function $D_0(z)=D(s)$. Since

$$D(s) = \frac{s-q}{(1-s)^q} = (s-q)\sum_{n=0}^{\infty}\frac{s^n[q]_{n\uparrow}}{n!},$$

taking $s=q+z$, we obtain

$$D_0(z) = \frac{z}{(1-q)^q(1-\frac{z}{1-q})^q} = \frac{z}{(1-q)^q}\sum_{n=0}^{\infty}\frac{[q]_{n\uparrow}}{n!}\left(\frac{z}{1-q}\right)^n = \sum_{k=1}^{\infty}\frac{k[q]_{(k-1)\uparrow}}{(1-q)^{q+k-1}}\left(\frac{z^k}{k!}\right).$$

Obviously, D_0 is a particular case of the Gauss hypergeometric function [2, 9]. All derivatives of the functions $D(s)$ and $D_0(z)$ are related to each other as follows:

$$D(s)=D_0(s-q), \qquad D^{(k)}(s)=D_0^{(k)}(s-q), \qquad D^{(k)}(q+z)=D_0^{(k)}(z).$$

Finally, the solution of $D_0^{-1}(x)$ as a series expansion is found and proved in the following theorem using the Lagrange inversion method.

Theorem 1. *Let the solution to the backward Kolmogorov equation for a supercritical branching process $X(t)$ induced by geometric probability following* (2.6) *be given by the function*

$$D(s)=\frac{s-q}{(1-s)^q}, \quad s<1.$$

Then the series expansion of the inverse function is

$$D^{-1}(x) = q + \sum_{k=1}^{\infty}\frac{b_k x^k}{k!}, \quad |x|<m^q,\ m>1,$$

where

$$b_k = (1-q)^{qk}\left(\frac{-1}{1-q}\right)^{k-1}[qk]_{(k-1)\downarrow}. \tag{3.2}$$

Proof. Following definition (3.1) and applying the substitution $s = q + z$ lead to

$$x = D(s) = D_0(s-q), \qquad D_0^{-1}(x) = s - q = D^{-1}(x) - q,$$

and therefore we have the following expression:

$$D^{-1}(x) = q + D_0^{-1}(x). \tag{3.3}$$

Thus it is confirmed that

$$D^{-1}(D(s)) = q + D_0^{-1}\big(D(s)\big) = q + D_0^{-1}\big(D_0(s-q)\big) = q + s - q = s.$$

The function $D_0(z)$ is an exponential generating function with coefficients $a_k = D_0^{(k)}(0) = D^{(k)}(q)$ given as

$$D_0(z) = \sum_{k=1}^{\infty}\frac{a_k}{k!}z^k, \quad a_k = \frac{k[q]_{(k-1)\uparrow}}{(1-q)^{k+q-1}}.$$

Let the Taylor series expansion of the inverse function in a neighborhood of $x = 0$ be

$$D_0^{-1}(x) = \sum_{k=1}^{\infty}\frac{b_k x^k}{k!}.$$

Next, we apply the Lagrange inversion method in the following form (see [2, 9]):

$$D_0(z) = \frac{z}{g(z)}, \qquad g(z) = (1-q)^q\left(1 - \frac{z}{1-q}\right)^q, \qquad g(0) = (1-q)^q > 0.$$

The coefficients b_k are given by the derivatives of the function $(g(z))^k$ at the point $z = 0$ as follows:

$$b_k = \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\big[\big(g(z)\big)^k\big]_{z=0}.$$

Now the Taylor series expansion of the function $(g(z))^k$ can be given by binomial coefficients as

$$\big(g(z)\big)^k = (1-q)^{qk}\sum_{j=0}^{\infty}\left(\frac{z}{1-q}\right)^j\frac{(-1)^j[qk]_{j\downarrow}}{j!}.$$

The jth derivative of this function at the point $z = 0$ is

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j}\big[\big(g(z)\big)^k\big]_{z=0} = (1-q)^{qk}\left(\frac{-1}{1-q}\right)^j[qk]_{j\downarrow}.$$

To obtain the coefficient b_k in (3.2), it suffices to take $j = k - 1$. Then

$$D_0^{-1}(x) = \sum_{k=1}^{\infty}(1-q)^{qk}\left(\frac{-1}{1-q}\right)^{k-1}[qk]_{(k-1)\downarrow}\left(\frac{x^k}{k!}\right), \qquad b_1 = (1-q)^q < 1.$$

Applying the definition of decreasing factorials by gamma function (2.8),

$$[qk]_{(k-1)\downarrow} = \frac{\Gamma(qk+1)}{\Gamma(qk+1-k+1)} = \frac{qk\Gamma(qk)}{\Gamma((q-1)k+2)} = \frac{qk\Gamma(q(k-1)+q)}{\Gamma((q-1)(k-1)+q+1)}, \tag{3.4}$$

leads to

$$D_0^{-1}(x) = q(1-q)^q x \sum_{k=1}^{\infty} \left(\frac{1}{(1-q)^{1-q}}\right)^{k-1} \frac{\Gamma(q(k-1)+q)}{\Gamma((q-1)(k-1)+q+1)} \frac{(-x)^{k-1}}{(k-1)!}. \tag{3.5}$$

Note that the coefficients in (3.2) are either positive, zero, or negative due to the decreasing factorials. It is confirmed by applying the reflection formulas for the gamma function,

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z\sin(\pi z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \dots,$$

to representation (3.4) with the notation $z = (1-q)k > 0$:

$$[qk]_{(k-1)\downarrow} = \frac{\Gamma(qk+1)}{\Gamma(1+(1-(1-q)k))} = \frac{\Gamma(qk+1)\Gamma((1-q)k)\sin(\pi(1-q)k)}{\pi(1-(1-q)k)}. \tag{3.6}$$

The radius of convergence of the series expansion to $D^{-1}(x)$ is calculated with a root test based on

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|b_k|}{k!}} = \left(\frac{1}{(1-q)^{1-q}}\right) \limsup_{k\to\infty} \sqrt[k]{(1-q)\frac{|[qk]_{(k-1)\downarrow}|}{k!}}$$

and Stirling's formula for the gamma function given as follows:

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}\left(\frac{z}{\mathrm{e}}\right)^z, \quad \Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{\mathrm{e}}\right)^z, \quad z \to \infty.$$

As $k \to \infty$, for the factors in (3.6), we have the following equivalence:

$$\Gamma(qk+1) \sim \sqrt{2\pi qk}\left(\frac{qk}{\mathrm{e}}\right)^{qk}, \qquad \Gamma((1-q)k) \sim \frac{\sqrt{2\pi(1-q)k}}{(1-q)k}\left(\frac{(1-q)k}{\mathrm{e}}\right)^{(1-q)k}.$$

Then from the equivalence for (3.6) and for $\Gamma(k+1) = k!$ in the denominator, we derive the following multiple:

$$\left\{\left(\frac{qk}{\mathrm{e}}\right)^q \left(\frac{(1-q)k}{\mathrm{e}}\right)^{(1-q)} \frac{\mathrm{e}}{k}\right\}^k = \left\{q^q(1-q)^{1-q}\right\}^k.$$

The last multiple of (3.6) converges to 1, that is,

$$\limsup_{k\to\infty} \sqrt[2k]{\frac{2\pi(1-q)q}{k}} = 1, \qquad \limsup_{k\to\infty} \sqrt[k]{\left|\sin\big(\pi(1-q)k\big)\right|} = 1.$$

As a result,

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|b_k|}{k!}} = \left(\frac{1}{(1-q)^{1-q}}\right)\left\{q^q(1-q)^{1-q}\right\} = q^q.$$

Finally, the radius of convergence of the inverse function $D^{-1}(x)$ series expansion is obtained in dependence on the parameter q as follows:

$$R\bigl(D^{-1}\bigr)(q) = \frac{1}{q^q} = m^q > 1, \quad 0 < q < 1. \qquad \square \tag{3.7}$$

It is worth noting that the inverse function $D^{-1}(x)$ can be expressed by the Wright function, defined as (see [17])

$$ {}_1\Psi_1(\alpha, a; \beta, b; z) := \sum_{n=0}^{\infty} \frac{\Gamma(\alpha n + a)}{\Gamma(\beta n + b)} \frac{z^n}{n!}.$$

This can be proved after changing the variable $k - 1 = j$ in (3.5), which leads to the following representation:

$$D_0^{-1}(x) = {}_1\Psi_1\left(q, q;\, q-1,\, q+1; -\frac{x}{(1-q)^{1-q}}\right) q(1-q)^q x. \tag{3.8}$$

4 Asymptotic behavior

The family of random variables $Z(t)$, $t > 0$, defined by

$$Z(t) := \frac{X(t)}{\mathbf{E}[X(t)]}, \qquad \mathbf{E}\bigl[X(t)\bigr] = \mathrm{e}^{f'(1)t}, \qquad f'(1) = K(m-1) > 0, \tag{4.1}$$

is a nonnegative martingale with respect to the natural filtration [1, 6, 18], and hence the following limit exists:

$$\lim_{t\to\infty} Z(t) = W, \qquad \mathbf{E}[W] = 1, \qquad \mathbf{P}(W > 0) = 1 - q, \qquad \mathbf{P}(W = 0) = q. \tag{4.2}$$

There are two methods to define the Laplace transform of the random variable $W \geqslant 0$ in this model. The first one is given by the limiting behavior of the process $Z(t)$, $t > 0$, as $t \to \infty$, defined by its p.g.f.

$$\mathbf{E}\bigl[\mathrm{e}^{-\lambda Z(t)}\bigr] = F(t, s), \quad s = \mathrm{e}^{-\lambda y}, \quad y := y(t) = \mathrm{e}^{-f'(1)t}, \quad t > 0.$$

Consider the Laplace transform of the limiting random variable W:

$$\varphi(\lambda) := \mathbf{E}\bigl[\mathrm{e}^{-\lambda W}\bigr] = \lim_{t\to\infty} F\bigl(t, \mathrm{e}^{-\lambda y}\bigr), \qquad \lim_{t\to\infty} y(t) = 0.$$

Thereby from (2.5) and (3.3) it follows that

$$\lim_{t\to\infty} F\bigl(t, \mathrm{e}^{-\lambda y}\bigr) = \lim_{t\to\infty} D^{-1}\left(\frac{\mathrm{e}^{f'(q)t}(s-q)}{(1-s)^q}\right), \quad s = \mathrm{e}^{-\lambda y(t)}.$$

Knowing that D^{-1} is a continuous function and applying the equivalence

$$1 - s = 1 - \mathrm{e}^{-\lambda y(t)} \sim \lambda y(t) = \lambda \mathrm{e}^{-f'(1)t}, \quad t \to \infty,$$

in addition to the relation

$$f'(q) + q f'(1) = \frac{-K(m-1)}{m} + \frac{K(m-1)}{m} = 0,$$

we can derive the following limit:

$$\lim_{t\to\infty}\frac{\mathrm{e}^{f'(q)t}(s-q)}{(1-s)^q}=\lim_{t\to\infty}\frac{\mathrm{e}^{f'(q)t}(s-q)}{(\lambda\mathrm{e}^{-f'(1)t})^q}=\frac{1-q}{\lambda^q}.$$

Finally,

$$\varphi(\lambda):=\mathbf{E}\big[\mathrm{e}^{-\lambda W}\big]=D^{-1}\left(\frac{1-q}{\lambda^q}\right)=q+\sum_{k=1}^{\infty}\frac{b_k}{k!}\left(\frac{1-q}{\lambda^q}\right)^k.$$

We summarize all these reflections (3.3), (3.5), and (3.8) in the following theorems.

Theorem 2. *The Laplace transform* $\varphi(\lambda)=\mathbf{E}[\mathrm{e}^{-\lambda W}]$ *of the limiting random variable* W *defined in* (4.1) *and* (4.2) *is given by the series expansion*:

$$\varphi(\lambda)=D^{-1}\left(\frac{1-q}{\lambda^q}\right)=q+\frac{(1-q)^{q+1}}{\lambda^q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}[qk]_{(k-1)\downarrow}}{k!}\left(\frac{(1-q)^q}{\lambda^q}\right)^{k-1}$$

and expressed by the Wright function as follows:

$$\varphi(\lambda)=q+{}_1\Psi_1\left(q,q;\,q-1,\,q+1;-\left(\frac{1-q}{\lambda}\right)^q\right)\frac{q(1-q)^{q+1}}{\lambda^q},\quad \lambda>q(1-q)^m. \tag{4.3}$$

A direct application of this theorem is the computation of probability density function $w(x)$ expressed by series expansion. The proof and precise definition are summarized in the next theorem.

Theorem 3. *The probability distribution of the limiting random variable* W *has an atom at the point* $x=0$ *with probability* $\mathbf{P}(W=0)=q$ *and absolutely continuous part with probability density function* $w(x)$ *expressed by the series expansion*

$$w(x)=\frac{q(1-q)^{1+q}}{x^{1-q}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{(x(1-q))^{qj}}{\Gamma((q-1)j+q+1)},\quad x>0. \tag{4.4}$$

Proof. The Laplace transform of the random variable W, using (3.4), is given by

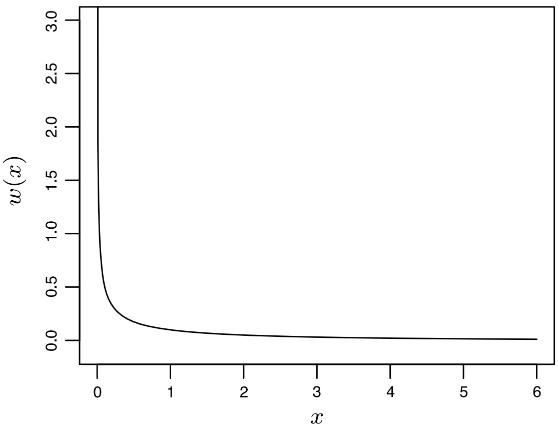
$$\varphi(\lambda)=q+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\left(\frac{1-q}{\lambda^q}\right)^k\frac{(1-q)^{qk}}{(1-q)^{k-1}}\frac{\Gamma(kq+1)}{\Gamma((q-1)k+2)},\quad 0<q<1.$$

The inversion of the Laplace transform $\varphi(\lambda)$ is done by inverting term by term [5] the series expansion using the well-known formula

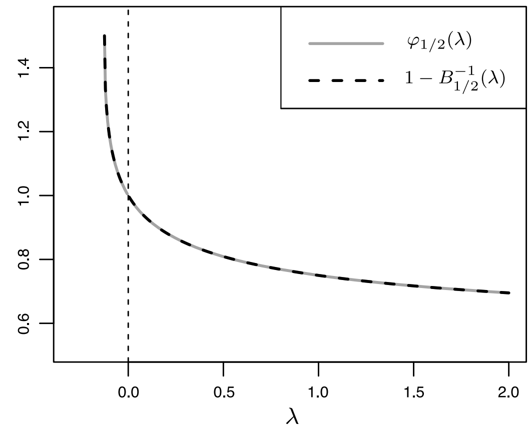
$$L\big[x^{kq-1}\big]=\frac{\Gamma(kq)}{\lambda^{kq}}.$$

The density part of the distribution for the limiting random variable $W\geqslant 0$ is

$$\begin{aligned}w(x)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\frac{qk(1-q)(1-q)^{qk}x^{qk-1}}{\Gamma((q-1)k+2)}\\&=\frac{q(1-q)((1-q)x)^q}{x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}\frac{(x(1-q))^{q(k-1)}}{\Gamma((q-1)(k-1)+q+1)}.\end{aligned}$$



(a) Probability density $w(x)$ computed using incomplete gamma function after applying Eq. (4.5)



(b) Laplace transform $\varphi_{1/2}(\lambda)$ for $-1/8 < \lambda < 2$ and the equivalent solution by $B^{-1}(\lambda) = 1 - B_0^{-1}(\lambda)$ and $D^{-1}(1/2\sqrt{\lambda})$.

Figure 2. Process properties for $q = 1/2$.

The change of the summation variable $k - 1 = j$ leads to

$$w(x) = {}_1\Psi_1\big(0, 1;\, q-1,\, 1+q; -\big(x(1-q)\big)^q\big)\left(\frac{q(1-q)^{1+q}}{x^{1-q}}\right), \quad x > 0.$$

Example 1. Let the ultimate extinction probability $q = 1/2$. It is the most simple case. Then the exact solution for the Laplace transform, temporarily denoted by $\varphi(\lambda) := \varphi_{1/2}(\lambda)$, reads

$$\varphi_{1/2}(\lambda) = \frac{1}{2} + \frac{1}{1+\sqrt{1+8\lambda}}, \quad \lambda > -\frac{1}{8}, \qquad \varphi_{1/2}\left(-\frac{1}{8}\right) = \frac{3}{2}.$$

The elementarily properties of the Laplace transform $\varphi_{1/2}(\lambda)$ and formula (83) in [3, p. 652] give

$$\mathbf{P}(W = 0) = \frac{1}{2}, \qquad w(x) = \frac{\mathrm{e}^{-x/8}}{\sqrt{8\pi x}} - \frac{\operatorname{erfc}(\sqrt{x}/2\sqrt{2})}{8}.$$

The density $w(x)$ can be represented by the incomplete gamma function $\gamma(\alpha, x) = \int_{z=0}^{x} z^{\alpha-1}\mathrm{e}^{-z}\,\mathrm{d}z$ [8]:

$$w(x) = \frac{\mathrm{e}^{-x/8}}{\sqrt{8\pi x}} + \frac{1}{8\sqrt{\pi}}\gamma\left(\frac{1}{2}, \frac{x}{8}\right) - \frac{1}{8}. \tag{4.5}$$

The availability of many different software implementations of the incomplete gamma functions enables easy computations. The derived probability mass function is shown in Fig. 2(a).

The equivalence of obtained solution for $w(x)$ and $\varphi_{1/2}(\lambda)$ is shown in Fig. 2(b). The numerical evaluation of the density $w(x)$ for any $0 < q < 1$ can be calculated directly by the series expansion (4.4).

Remark 1. Traditionally, the reduced Wright function [14] is denoted as follows:

$$\phi(\beta, b; z) = {}_1\Psi_1(0, 1; \beta, b; z).$$

When $-1<\beta<0$, the reduced Wright function is known as that of the second kind. The recurrence relation for the reduced Wright function $\phi(\beta,b;z)$ reads

$$\beta z\phi(\beta,\,\beta+b;z)=\phi(\beta,\,b-1;z)+(1-b)\phi(\beta,b;z),\qquad \frac{\mathrm{d}}{\mathrm{d}z}\phi(\beta,b;z)=\phi(\beta,\,\beta+b;z).$$

It means that when $\beta=q-1$, $b=2$, $\beta+b=q-1+2=q+1$, and $z=-((1-q)x)^q<0$. The p.m.f.

$$w(x)=\frac{q}{x}(q-1)z\phi(q-1,\,q-1+2;z),\quad z=-((1-q)x)^q<0,$$

is given by the difference of two entire functions divided by $x>0$:

$$w(x)=\frac{q}{x}\big\{\phi\big(q-1,\,1;-\big((1-q)x\big)^q\big)-\phi\big(q-1,\,2,-\big((1-q)x\big)^q\big)\big\}.$$

The reduced Wright function with $-1<\beta<0$ is an entire function of order p greater than one and type σ, as it is specified in [5]:

$$p=\frac{1}{\beta+1}=\frac{1}{q},\quad \sigma=\frac{1+\beta}{|\beta|^{\beta/(\beta+1)}}=q(1-q)^{(1-q)/q},\quad 0<q<1.$$

5 Laplace transform

The second method to define the Laplace transform $\varphi(\lambda)$ (see [1,6,18]) is based on the differential equation

$$\varphi'(\lambda)=\frac{f(\varphi(\lambda))}{\lambda f'(1)},\qquad \varphi(0)=1.$$

The solution of this equation is given by the function $B(x)$ defined as follows:

$$\varphi(\lambda)=x,\qquad B(x):=\varphi^{-1}(x)=\lambda,\qquad \varphi(\lambda)=B^{-1}(\lambda).$$

It is given by the integral

$$B(x)=(1-x)\exp\Bigg\{\int\limits_1^x\bigg(\frac{h'(1)-1}{h(s)-s}+\frac{1}{1-s}\bigg)\,\mathrm{d}s\Bigg\},\quad q<x<1.$$

We have by definition of the geometric reproduction (1.1) and (1.2) the following representation:

$$h(s)-s=\frac{(s-q)(s-1)}{1+q-s},\qquad h'(1)-1=\frac{1}{q}-1=\frac{1-q}{q}.$$

Thus the integral can be calculated as

$$\int\limits_1^x\bigg(\frac{(1-q)(1+q-s)}{q(s-q)(s-1)}+\frac{1}{1-s}\bigg)\,\mathrm{d}s=\frac{-1}{q}\int\limits_1^x\frac{\mathrm{d}s}{s-q}.$$

Applying it to the function $B(x)$ gives the following result:

$$B(x)=(1-x)\exp\bigg\{\frac{-1}{q}\log\frac{x-q}{1-q}\bigg\}=(1-x)\bigg(\frac{1-q}{x-q}\bigg)^{1/q},\quad q<x.$$

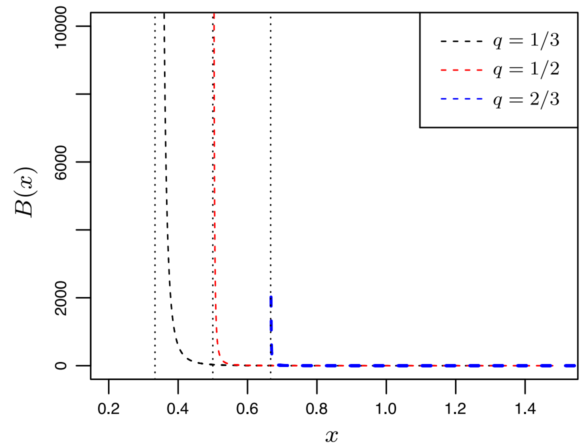


Figure 3. Plot of $B_q(x)$ for $q = 1/3, 1/2, 2/3$ by resolution of 0.001 for x. The vertical lines show the asymptotic values.

It is convenient to use the parameter $m = 1/q > 1$ and to rewrite our solution as follows:

$$B(x) = \frac{(1-x)(m-1)^m}{(mx-1)^m}, \qquad B'(x) = -\frac{(m-1)^{m+1}(m+1-mx)}{(mx-1)^{m+1}}.$$

To define the inverse function $B^{-1}(x)$, we must choose the most convenient monotony interval again. The first derivative

$$B'(x) = -\left(\frac{1-q}{x-q}\right)^{1+m} \frac{(q+1-x)}{q}, \quad m = \frac{1}{q}, \qquad B'(1) = -1,$$

is negative in the interval $q < x < q+1$ and positive for $x > q+1$. It means that the function $B(x)$, having a vertical asymptote at the point $x = q$, is decreasing in the interval $q < x < 1+q$ and has a minimum at the point $x = q+1$. Only the interval $q < x < q+1$ is convenient to define the inverse function. The minimal value being negative,

$$\lambda_* := B(1+q) = -q(1-q)^m < 0 \tag{5.1}$$

will be considered as the Laplace abscise for the inverse function,

$$\varphi(\lambda) = B^{-1}(\lambda), \quad -q(1-q)^m < \lambda < \infty, \qquad \varphi(\lambda_*) = q+1, \qquad \varphi(0) = 1, \qquad B(1) = 0.$$

The vertical asymptote of $B(x)$ is symmetric to the horizontal asymptote for $\varphi(\lambda)$. We remark that $\varphi(\lambda)$ is decreasing in the interval $\lambda_* < \lambda < \infty$ and has the horizontal asymptote q, $\varphi(\lambda) > q$; see Fig. 3.

The higher-order derivatives of the function $B(x)$, $q < x < 1+q$, are

$$B^{(k)}(x) = \frac{(-1)^k (m-1)^{m+1} m^{k-1} [m]_{(k-1)\uparrow} (m+k-mx)}{(mx-1)^{m+k}}, \quad x > \frac{1}{m}.$$

where at the point $x = 1$, we have

$$B^{(k)}(1) = (-1)^k \frac{k[m]_{(k-1)\uparrow}}{(1-q)^{k-1}} = (-1)^k \frac{k\Gamma(m+k-1)}{(1-q)^{k-1}\Gamma(m)}, \quad m > 1.$$

The Taylor series expansion of $B(x)$ in a neighborhood of the point $x = 1$ reads

$$B(x) = \sum_{n=1}^{\infty} \frac{(-1)^n c_n (x-1)^n}{n!}, \quad c_n = \frac{n[m]_{(n-1)\uparrow}}{(1-q)^{n-1}} = \frac{n\Gamma(m+n-1)}{(1-q)^{n-1}\Gamma(m)} > 0, \quad m > 1. \tag{5.2}$$

Respectively, we define the Taylor series expansion of $\varphi(\lambda)$ in a neighborhood of the point $\lambda = 0$ as follows:

$$\varphi(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)\lambda^n}{n!}.$$

To define the series expansion of the Laplace transform $\varphi(\lambda)$, we apply the Lagrange inversion method again. First of all, we define the function $B_0(z) = B(1-x)$ for $z < 1-q$ as

$$B_0(z) = \frac{z}{(1-\frac{z}{1-q})^m}, \qquad B(x) = B_0(1-x), \qquad z = 1-x, \qquad B_0\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right). \tag{5.3}$$

Then using the composite inverse of (5.3), we obtain the inverse function B^{-1} defining the Laplace transform in the form of power series over λ^n:

$$B^{-1}(x) = 1 - B_0^{-1}(x), \qquad \varphi(\lambda) = B^{-1}(\lambda), \qquad \varphi(\lambda) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k [mk]_{(k-1)\downarrow}}{(1-q)^{k-1}} \frac{\lambda^k}{k!}. \tag{5.4}$$

We remark that all derivatives

$$d_k := \varphi^{(k)}(0) = (-1)^k \frac{[mk]_{(k-1)\downarrow}}{(1-q)^{k-1}}$$

are alternating because $[mk]_{(k-1)\downarrow} > 0$ for $m > 1$. The radius of convergence in a neighborhood of zero is calculated by the root test as previously for (3.7), but we do not need to apply the reflection formula in the denominator. It is in agreement with the Laplace abscise,

$$\lambda > -q(1-q)^m, \qquad |\lambda| < q(1-q)^m.$$

The agreement between the coefficients c_n (5.2) and d_n (5.4) can be confirmed (verified) by the Fáa di Bruno formula [10] applied to the composition relation $\varphi(B(x)) = x$. The computational results are listed in Table 1. Based on the result of the Lagrange inversion method (5.4), we formulate the following theorem.

Table 1. Derivatives $B^{(k)}(1)$ and $\varphi^{(k)}(0)$

Derivatives $B^{(k)}(1)$	Derivatives $\varphi^{(k)}(0)$
$B^{(1)}(1) = -1$	$\varphi^{(1)}(0) = -1$
$B^{(2)}(1) = \frac{2m}{1-q}$	$\varphi^{(2)}(0) = \frac{2m}{1-q}$
$B^{(3)}(1) = \frac{-3m(m+1)}{(1-q)^2}$	$\varphi^{(3)}(0) = \frac{-3m(3m-1)}{(1-q)^2}$
$B^{(4)}(1) = \frac{4m(m+1)(m+2)}{(1-q)^3}$	$\varphi^{(4)}(0) = \frac{4m(4m-1)(4m-2)}{(1-q)^3}$
$B^{(k)}(1) = \frac{(-1)^k k[m]_{(k-1)\uparrow}}{(1-q)^{k-1}}$	$\varphi^{(k)}(0) = (-1)^k \frac{[mk]_{(k-1)\downarrow}}{(1-q)^{k-1}}$

Theorem 4. *Let the limiting random variable* W *be defined by* (4.1) *and* (4.2)*. Then its Laplace transform* $\varphi(\lambda) := \mathbf{E}[\mathrm{e}^{-\lambda W}]$ *is given by the power series on* λ^n *as follows*:

$$\varphi(\lambda) = 1 - \lambda \sum_{k=1}^{\infty} \left(\frac{-\lambda}{1-q}\right)^{k-1} \frac{[mk]_{(k-1)\downarrow}}{k!}, \quad |\lambda| < q(1-q)^m,$$

and its presentation expressed by the Wright function is

$$\varphi(\lambda) = 1 - {}_1\Psi_1\left(m, m; m-1,\, m+1; -\frac{\lambda}{1-q}\right) m\lambda.$$

Remark 2. The correspondence between two representations (4.3) and (5.4) is based on their definition and the following relations: $B(x) = \lambda$,

$$\left(\frac{1-q}{x-q}\right)^m = \frac{\lambda}{1-x}, \qquad (1-x)^q \left(\frac{1-q}{x-q}\right)^{qm} = \lambda^q, \qquad B(x) = \left(\frac{1-q}{D(x)}\right)^m, \qquad D(x) = \frac{1-q}{(B(x))^q}.$$

6 Applications

A direct application of the previous theorem is the computation of moments and statistical inferences. The important mth moments are computed for $m > 1$. They can be yielded using already known $\varphi^{(k)}(0)$, available in Table 1. We compute $\mathbf{E}[W] = 1$ and

$$\mathbf{E}\big[W^2\big] = \frac{2m}{1-q} = \frac{2m^2}{m-1}, \qquad \mathbf{E}\big[W^3\big] = \frac{3m(3m-1)}{(1-q)^2}, \qquad \mathbf{E}\big[W^4\big] = \frac{4m(4m-1)(4m-2)}{(1-q)^3}.$$

Then, for the central moments, we obtain

$$\mathbf{E}\big[(W-1)^2\big] = \frac{2m^2 - m + 1}{m-1} = 2m + \frac{m+1}{m-1}, \qquad \mathbf{E}\big[(W-1)^3\big] = \frac{9m^2 - 9m + 6}{(1-q)^2} + 2 > 2.$$

The index of dispersion, being equal to the variance, increases linearly with respect to the mean of reproduction. The measure of the asymmetry is positive for all $m > 1$. The skewness

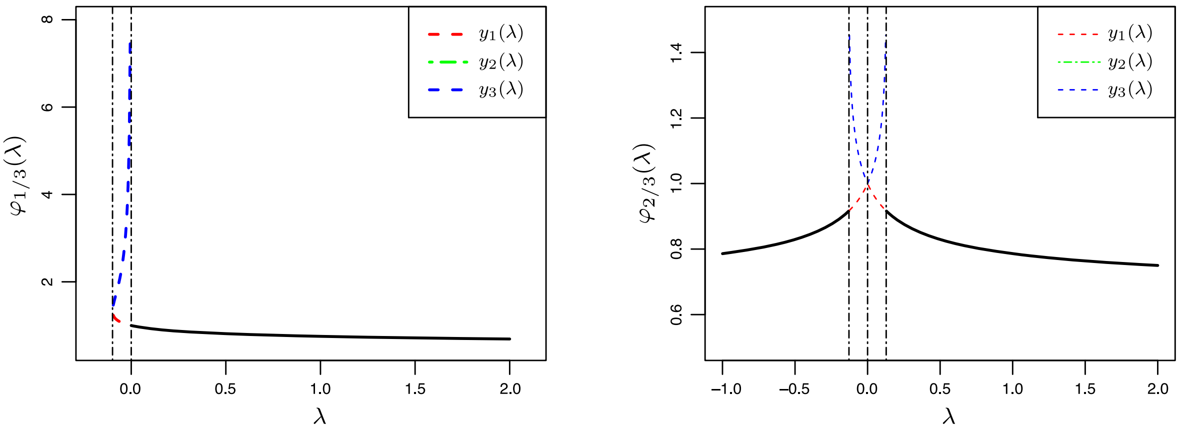
$$\mathrm{Skew}[W] := \frac{\mathbf{E}[(W-1)^3]}{(\mathbf{E}[(W-1)^2])^{3/2}} = \frac{9m^4 - 9m^3 + 8m^2 - 4m + 2}{(2m^2 - m + 1)\sqrt{(2m^2 - m + 1)(m-1)}} \sim \sqrt{m}, \quad m \to \infty.$$

The graph of the density $w(x)$ approaches the vertical axis (but it is integrable), and $\mathbf{E}[W] = 1$ always.

Another interesting result is the process evolution through the time. The development in time of the branching process $X(t), t > 0$, is described by the probabilities of the events concerning the number of particles alive at time $t > 0$, that is, $\mathbf{P}(X(t) = n)$, $n = 0, 1, 2, \dots$, according (2.11).

It is well known that when the parameter $0 < q < 1$ is a rational number, the inverse function D^{-1} (as given by the Wright function) is a solution to the algebraic equation; see [16]. Let us study several distinct cases where $q = 1/3, 1/2, 2/3$, respectively, $m = 3$, $m = 2$, $m = 3/2$. For the graph design, we indicate (as an index) only the parameter q in both representations of the Laplace transform:

$$\varphi(\lambda) := \varphi_q(\lambda) = D^{-1}\left(\frac{1-q}{\lambda^q}\right) = B_q^{-1}(\lambda), \qquad B_q(x) = \frac{(1-x)(m-1)^m}{(mx-1)^m} = (1-x)\left(\frac{1-q}{x-q}\right)^{1/q}.$$



(a) Laplace transform $\varphi_{1/3}(\lambda)$ for $-8/81 < \lambda < 2$. The three real solutions are enclosed between the dashed vertical lines from $-8/81 < \lambda < 0$. Only two of them are visible: $y_1(\lambda)$ and $y_3(\lambda)$. The solution of $y_2(\lambda)$ is in the negative range, outside of the figure scope.

(b) Laplace transform $\varphi_{2/3}(\lambda)$ for $-1 < \lambda < 2$. The three real solutions are enclosed between the dashed vertical lines from $|\lambda| < 2/(9\sqrt{3})$. Only two of them are visible: $y_1(\lambda)$ and $y_3(\lambda)$. The solution of $y_2(\lambda)$ is in the negative range, outside of the figure scope.

Figure 4. $\varphi_q(\lambda)$ for $q = 1/3$ (a) and $q = 2/3$ (b).

1. The graph representations of the extinction probability $Q(t)$ to the time $t > 0$ (2.7) and the values of the probabilities $\mathbf{P}(X(t) = 1)$, $t > 0$ (2.12), are based on the corresponding explicit form of D^{-1} presented in Figs. 1(a) and 1(b).

2. The graph representation of the functions $B(x) := B_q(x)$ is based on the following explicit forms:

$$B_{1/3}(x) = (1-x)\left(\frac{2}{3x-1}\right)^3, \qquad B_{1/2}(x) = (1-x)\left(\frac{1}{2x-1}\right)^2, \qquad B_{2/3}(x) = \frac{1-x}{(3x-2)\sqrt{3x-2}},$$

The results are plotted in Fig. 3. The numerical evaluation of $B_q(x)$ is not difficult and useful to confirm the symmetric values $\varphi_q(\lambda) = B_q^{-1}(\lambda)$ of the Laplace transform. The very strong decreasing behavior of $B_q(x)$ corresponds to the slow branch of $\varphi_q(\lambda)$ on the corresponding intervals.

3. The Laplace transform obtained by both methods yields equal results. It is especially demonstrated in the case of $q = 1/2$; see Fig. 2(b). For the other two cases of $q = 1/3$ and $q = 2/3$, the solutions are derived from cubic polynomials. In the interval $\lambda > 0$ for $\varphi_{1/3}(\lambda)$ and in the interval $\lambda > 2/(9\sqrt{3})$ for $\varphi_{2/3}(\lambda)$, the corresponding unique real solutions are

$$\varphi_{1/3}(\lambda) = \frac{1}{3} + \frac{2}{1 + \sqrt[3]{1 + \frac{81\lambda}{4} + \frac{9}{4}\sqrt{\lambda(8+81\lambda)}} + \sqrt[3]{1 + \frac{81\lambda}{4} - \frac{9}{4}\sqrt{\lambda(8+81\lambda)}}}, \quad \lambda > 0,$$

and

$$\varphi_{2/3}(\lambda) = \frac{2}{3} + \frac{1}{2 + \sqrt[3]{-1 + \frac{3}{2}(9\lambda)^2 + \frac{27\lambda}{2}\sqrt{(9\lambda)^2 - \frac{4}{3}}} + \sqrt[3]{-1 + \frac{3}{2}(9\lambda)^2 - \frac{27\lambda}{2}\sqrt{(9\lambda)^2 - \frac{4}{3}}}}, \quad \lambda > \frac{2}{9\sqrt{3}}.$$

There are one real and two complex solutions, respectively, for $\varphi_{1/3}(\lambda)$ and $\varphi_{2/3}(\lambda)$ when discriminants are positive. However, in the neighborhood of $\lambda = 0$, where λ is greater than the negative Laplace abscise (5.1), that is, $\lambda > \lambda_* = -q(1-q)^m < 0$, the corresponding discriminants are negative, and thus the cubic equations have three real roots. The real solutions are obtained by using periodic functions $y(\lambda)$ with $2\pi/3$ shift. The results for $q = 1/3$ and $q = 2/3$ are demonstrated in Fig. 4. Obviously, the Laplace transform is designed by the corresponding decreasing branch ensuring the continuity.

7 Conclusions

The probabilistic aspects of the branching reproduction model are described by the explicit solution to the Kolmogorov equation. The Laplace transform and the probability density function of the asymptotic behavior are expressed by the Wright function. The numerical evaluation is given by the graph representation.

It is worth noting that a strictly supercritical MBP occurs when $q = 0$, called also immortal. Conditioned on nonextinction, any supercritical MBP becomes immortal. However, in this case, another parameterization has to be applied. The main topics are the properties of compound geometric distribution for $X(t)$, $t > 0$, and compound exponential distribution for the limiting random variable $W > 0$. There are quite many problems for further study.

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