R E V I E W

on a dissertation work for the acquisition of the scientific degree "Doctor of Science"

in the field of higher education 4. Natural sciences, mathematics and informatics,
professional field 4.5. Mathematics,
scientific specialty: "Mathematical analysis"

Dissertation title: ***Subordination principle for generalized fractional evolution equations***

Author of the dissertation: ***Dr. Emilia Grigorova Bazhlekova***, Associate Professor at the Institute of Mathematics and Informatics (IMI) of the BAS

The review was prepared by: ***Dr. Maria Dimitrova Datcheva***, professor at the Institute of Mechanics at BAS, member of the Scientific Jury (SJ), approved by the Scientific Council of IMI-BAS on 15.07.2022 (protocol No. 7) and determined by Order No. 216/ 20.07.2022 of the Director of IMI-BAS, for conducting the procedure for awarding the scientific degree "Doctor of Science" to Emilia Grigorova Bazhlekova and according to the decision of the SJ taken at its first meeting on 21.07.2022 with protocol 1.

Submitted materials for the procedure: the administratively required documents, including the order and protocol 1/28.06.2022 of the pre-defense of the dissertation work in the Scientific-Structural Unit (SSU) "Analysis, Geometry and Topology" of IMI-BAN, curriculum vitae, list of 11 publications on the dissertation and copies of them, a reference to the observed independent citations of these publications, an abstract in Bulgarian and English and the dissertation in English. I was also provided with a comparative table for the fulfillment of the minimum requirements for the acquisition of the scientific degree "Doctor of Sciences" according to the Regulations of IMI-BAS for the development of the academic staff. After a careful check of the data in this table, it was found that the candidate meets, and in some of the indicators exceeds the minimum requirements, as well as the specific requirements for IMI-BAN defined in Art. 3 paragraph 1 item 1 of this regulation. This gave reason to the SJ to admit the dissertation work to evaluation and defense (decision taken at its first meeting, protocol 1/21.07.2022).

The author of the dissertation, Assoc. Prof. Dr. Emilia Bazhlekova, has a master's degree in Mathematics (Complex analysis) from the Faculty of Mathematics and Informatics (FMI) of the Sofia University (SU) "St. Kliment Ohridski", Sofia. In 2001, she defended her PhD thesis on the topic of "Fractional evolution equations in Banach spaces" at the Applied Analysis group, Eindhoven University of Technology, Eindhoven, The Netherlands, under the supervision of Prof. de Graaf and Prof. Clement. In the period from 1989 until now, with short interruptions, she was part of the scientific collegium of IMI-BAS - first as a PhD student (supervisor Prof. Ivan Dimovski), then as a mathematician, assistant professor, and from 2014 as an associate professor at the research unit “Analysis, geometry and topology/ Complex analysis”. Assoc. Prof. Bazhlekova is an established scientist in the field of the subject of the dissertation - according to Scopus data, she has a Hirsch index without self-citations of all co-authors - 9, 37 publications and over 300 independent citations, of which 259 are in publications from the last 5 years.

Objectives and relevance of the dissertation work

The main objective of the dissertation work is the study of the subordination principle and the development of a methodology for its application to evolution equations with fractional

derivatives in order to classify, find solutions and analyze these solutions. The topic is important, as it is related to the so-called fractional calculus - an area that is currently actively being worked on, as well as because of the importance of evolution equations with fractional derivatives for modeling processes in complex, disordered and percolation systems, hierarchical environment and media with memory.

Structure and content of the dissertation, Abstract – the author's summary

The dissertation is written in English and is the result of the candidate's scientific research reflected in 11 publications in renowned scientific journals and published in the period 2015-2021. The total volume of the dissertation is 200 pages and is structured in 8 chapters, a short preface, an introduction containing the objectives and the motivation for the conducted research, a conclusion summarizing the scientific contributions in the dissertation work, a list of references used (110 titles in total) and an alphabetical index.

Chapter 1 is entitled "Fractional calculus operators and special functions". It has a volume of 10 pages and in 5 sections the basic definitions and concepts are given, as well as the main properties of the operators and functions used in the dissertation work. *Chapter 2* is entitled "Introduction to subordination principle", where the first two sections are devoted to Bernstein functions and Volterra integral equations. The second two sections discuss the subordination principle for fractional evolution equations. Two general subordination theorems (given as 2.4 and 2.5 in the thesis and 2.1 and 2.2 in the *Abstract* – the author's summary) are proved, one reducing the question of subordination to a question whether a characteristic function of the equation is a Bernstein function, and the second theorem considers the case where the subordinate solution operator is bounded analytic and gives the sector of analyticity. These theorems are an important basis for obtaining the results in the following chapters, and their proof has not been published so far. *Chapter 3* "Space-time fractional evolution equations" is one of two voluminous chapters (30 pages) and this is so because it is devoted to a detailed study of the subordination principle for the more general case of linear homogeneous evolution equations with fractional derivatives in time as well as in spatial coordinates. The subordination formula for the solution operator of an abstract Cauchy problem for space-time fractional evolution equations is obtained, and it is shown that the subordination kernel coincides with the fundamental solution of a related one-dimensional problem [B5]. The existence of an analytic extension of the subordinating kernel is investigated. Numerical experiments are presented that prove consistency of analytical results obtained in the thesis and known existing exact solutions [B9].

In the rest of the dissertation, evolution equations with fractional derivatives only in time are considered. *Chapter 4* "Transition from diffusion to wave propagation" deals with the heat conduction equation with fractional Jeffreys' type constitutive law. Part of the results in this chapter are published in [B10]. Using the Bernstein function approach, two fundamentally different regimes of behavior are identified here depending on the ratio of the relaxation times for the heat flux and the temperature gradient, namely, diffusion and wave propagation. *Chapter 5* "Generalized subdiffusion equations" is the other voluminous chapter in the thesis (32 pages) and applies the subordination principle to an interesting class of equations involved in describing anomalous diffusion, which is characteristic of but not restricted to disordered and percolation systems and media. The results in this chapter are published in [B1], [B2] and [B3]. *Chapter 6* "Multinomial Mittag-Leffer type functions" is devoted to the solution and study of relaxation equations involving time derivatives of different fractional order, and part of the results are published in [B6].

Chapter 7 "Distributed-order diffusion-wave equations" and *Chapter 8* "Wave propagation in linear viscoelastic media" consider equations describing phenomena intermediate between diffusion and wave propagation. Interesting results have been obtained here, which have implications for the modeling using such equations of real physical processes. These two chapters are based on publications [B4] and [B7] (Chapter 7) and [B4], [B8] and [B11] (Chapter 8). Regarding the solution of a one-dimensional Cauchy problem for diffusion-wave equations (a generalization of the wave equation for describing wave propagation in complex media), in *Chapter 7* the principle of subordination is established, which implies the existence of a unique solution and enables an integral representation (separation) of the solution operator by a probability density function and the solution operator of the corresponding problem for the second-order equation. The analytical results are supported by numerical analysis. In *Chapter 8*, for the description of the hereditary properties of a linear viscoelastic medium, several examples are considered in which the models use completely monotonic relaxation kernels. These kernels are expressed in terms of Mittag-Leffler-type functions, including the recently introduced multinomial Prabhakar-type function. Some characteristics of the propagation functions, such as non-negativity, monotonicity, propagation speed, presence/absence of a wavefront, are discussed and an integral representation of the solution in explicit form is derived.

The Abstract is 34 pages long and contains the main results obtained in the dissertation. It adequately reflects the content and main results of the dissertation, as well as an analysis of the scientometric indicators of the publications on the dissertation, and the personal contribution of the candidate in each of them is declared. The text is written equally clearly in both Bulgarian and English.

Scientific contributions (according to the author's statement of contributions)

I fully accept the scientific contributions as stated in the author reference (part V of the Abstract, the statement of contributions and the conclusion to the dissertation under the title "Main scientific contributions"). The contributions are related to the development and application of the principle of subordination for the study, classification and solution of integrodifferential equations where the integrals are of convolutional type. Specifically, they consist of: developing a methodology for establishing subordination between a linear evolution equation of a general form and a linear evolution equation of fractional or integer order by establishing whether a given characteristic function belongs to the class of complete Bernstein functions; establishment of subordination relations for equations with time-fractional derivatives found in the literature, by means of which the solution is divided into a subordination kernel and a solution of a simpler equation of integer or fractional order; the principle of subordination was used for equations with fractional derivatives in time and space, and various representations of the subordination kernel were obtained, as well as its properties were investigated, the sector of the complex plane was determined, in which the subordinate solution is bounded analytic. Regarding the classification of integrodifferential equations based on the principle of subordination, two main classes of generalized fractional evolution equations are defined, namely - equations describing subdiffusion and diffusion wave equations.

The remaining contributions are related to specific equations from these two categories, also relevant for the description of physical processes, in which the fractional derivatives are only with respect to the time variable. The specific scientific contributions in this category are: on the example of the evolution equation (heat conduction equation) with Jeffreys’ fractional

constitutive law, the connection between the principle of subordination and the physical meaning of an evolution equation is shown; the type of kernels in diffusion and wave propagation regimes is established. The subordination relations for the generalized equation describing anomalous diffusion have been established, and an explicit approximation formula for the solution has been obtained, which is a generalization of the exponential formula for C_0-semigroups. Applying the subordination formula, a useful two-sided estimate for the solution of the generalized relaxation equation is obtained, and this estimate is used to study an inverse problem. A multinomial function of Prabhakar type was introduced and studied, and the conditions were found under which the function is completely monotonic. Based on this property of the introduced function, a model was defined, which generalizes known relaxation models. A partial solution to the question concerning the conditions under which the one-dimensional fundamental solution of the distributed-order diffusion-wave equation is a probability density has been obtained. The principle of subordination for diffusion-wave equations with several time derivatives of different orders was studied, and an integral representation of the subordination kernel was obtained. The cases of finite and infinite wave propagation speed were considered. Interesting and original results concerning the relaxation moduli of generalized fractional viscoelastic models have been obtained. For the Maxwell, Jeffreys' and Zener models it has been proved that a necessary and sufficient condition for them to have physical meaning is that the corresponding relaxation moduli are completely monotonic functions.

Originality and significance of the results: I am fully confident to consider the results to be new and original, i.e. not previously published by other authors. The significant number of citations (over 90) can be taken as evidence of the importance of the candidate's achievements and interest in her work by the scientific community - for example, the article [B4] from 2018 has been cited in independent sources at the time of this review 21 times (according to the data in Scopus).

Evaluation of the publications submitted to the dissertation work

The dissertation is based on 11 articles published in the period from 2015 to 2021 and were not used by the candidate in earlier procedures under the ZRASRB (the last procedure ended in 2014). According to the presented list of citations of these publications, the observed independent citations are 94, of which 77 are in journals with an impact factor. I assume that 9 of the publications are in journals with an impact factor because the author of the dissertation listed the article in Mathematics (MDPI), [B2], as having no impact factor, which reflects the situation at the date of publication, but to date Mathematics (MDPI) is in the top 10% among journals in the JCR Category "Mathematics", which is also contributed by the citations of [B2]. Of these publications, 6 fall into the first quartile (WoS). Two publications are in another journal in the top 10% in the same category (WoS, JCR Category "Mathematics") - Fractional Calculus and Applied Analysis. Of all the publications, 6 are independent, and the rest are with one co-author, assuming the role and contribution of the candidate as defined in the author's summary (no need of a distribution protocol). The presence of joint articles speaks of a teamwork attitude, and from the declared personal contributions in the joint publications it is clear that a combination of competences was also achieved and the scope of the applications of the results was expanded.

Approbation of the results

In the period from 2016 to 2020, the author presented her research on the topic of the dissertation at a number of international forums (over 10, in Serbia, Greece) including both

narrowly specialized and forums with the participation of specialists from various fields, for example, applied mathematics, mechanics, numerical analysis. Within IMI-BAS, current results on the subject have been regularly reported by the candidate at various seminars and annual scientific sessions.

Comments and recommendations:

I have no critical notes or recommendations. The text of the dissertation is written extremely clearly, the presentation of the topic of the research and the obtained results is well structured and given in a logical sequence, the conclusions are substantiated, the formulations and proofs are clear and are given in a volume completely sufficient to follow their content in essence. The dissertation has a finished look at the level of a monographic work. I am completely confident that the candidate has a good knowledge of both the older and the more recent scientific literature in the field, which is evident from the dissertation's bibliography.

Conclusion: The presented dissertation work for obtaining the scientific degree "Doctor of Science" is in a current field. It is very well written and the results in it are at a high scientific level. The dissertation and its author satisfy all the criteria for the acquisition of the scientific degree "Doctor of Sciences" according to the ZRASRB, its Regulations and the Regulations for the implementation of the ZRASRB of BAS and IMI.

I strongly recommend to the honorable Scientific Jury to award Assoc. Prof. Dr. Emilia Grigorova Bazhlekova the scientific degree "Doctor of Sciences" in the field of higher education 4. "Natural Sciences, Mathematics and Informatics", professional field 4.5 "Mathematics", scientific specialty "Mathematical analysis'.

13.10.2022 Maria Datcheva
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